Symplectic Geometry

Nicholas Proudfoot
Department of Mathematics, University of Oregon, Eugene, OR 97403

These notes are written for a ten week graduate class on symplectic geometry. Most of the material here is included in Michèle Audin's book *Torus actions on symplectic manifolds*, which I used heavily in preparing these notes. There are a few topics that we cover that are not in Audin's book, such as polygon spaces, cohomology of flag manifolds, and CS/GKM theory. More significant, however, are the omissions; Audin's book contains too much material for a course of this length, and I have included just enough to accomplish my main goals, which are as follows:

- Introduce interesting classes of examples of symplectic manifolds, including partial flag manifolds, polygon spaces, and toric varieties.
- Prove Delzant's theorem, which says that the constructive definition of a toric variety (start with a polyhedron and build a space out of it) is equivalent to the abstract definition.
- Use Hamiltonian torus actions to understand Betti numbers (via Morse theory) and cohomology rings (via various equivariant tricks) of symplectic manifolds.

I have also been strongly influenced by Ana Cannas da Silva's *Lectures on symplectic geometry* [CdS] and by conversations with Allen Knutson, from whom I learned the subject.

What is all of this about?

A symplectic form on a manifold X is a closed, nondegenerate 2-form. A nondegenerate 2-form is a smoothly varying perfect pairing on the tangent spaces of X that is skew-symmetric (I'll remind you what "closed" means in a little while). For example, let $X := S^2 = \{p \in \mathbb{R}^3 \mid |p| = 1\}$. Then for all $p \in X$, $T_pX = \{q \in \mathbb{R}^3 \mid p \cdot q = 0\}$. The 2-form ω given by the formula $\omega_p(q, r) = p \cdot (q \times r)$ is symplectic. (Since X is a 2-manifold, all 2-forms are automatically closed.) If you are looking down at the sphere at the point p, this means that any unit tangent vector at p pairs to 1 with the vector obtained by rotating the original vector 90 degrees counter-clockwise, but it pairs trivially with itself (this is always the case, by skew-symmetry).

It is interesting to compare this symplectic form to the usual Riemannian metric. A Riemannian metric is a smoothly varying perfect pairing on the tangent spaces of X that is symmetric. In the case of the standard Riemannian metric on S^2, any unit tangent vector pairs to 1 with itself, but pairs trivially with vectors that differ from it by a 90 degree rotation.

One interesting thing that one can do with a Riemannian metric is define the gradient of a function. Given a smooth function $f : X \to \mathbb{R}$, its derivative df is a 1-form. That means that, for any $p \in X$, we obtain a linear function on T_pX taking a tangent vector v to $df_p(v)$, the directional derivative of f at p along v. This linear function must be given by pairing with some vector, and that vector is called the **gradient** of f at p. For example, if f is the height function on S^2, then its gradient always points in the direction of the north pole. At the equator it has norm 1, and it shrinks as the latitude gets higher or lower, vanishing at the two poles.

One can also define the **symplectic gradient** of f, using the symplectic form rather than the Riemannian metric. That is, the symplectic gradient v_f is characterized by the condition that $\omega_p(v_f(p), v) = df_p(v)$ for all $v \in T_pX$. (It was important for me to write it out to specify the order, which is something I didn't have to worry about in the Riemannian case!) The symplectic gradient is obtained from the Riemannian gradient by rotating 90 degrees clockwise.

When you have a vector field on X, you can consider the family of diffeomorphisms of X given by flowing along that vector field. In our example, flowing along the Riemannian gradient may be described as "reverse oozing". On the other hand, flowing along the symplectic gradient is spinning (counter-clockwise if you are looking down at the north pole). This is a nicer flow in many respects. For example, it preserves level sets of the function, and it does not "deform" the sphere. These two phenomena are general facts about symplectic gradients, as we shall later see. Another nice property is that the flow is periodic; this does not hold in general, but it happens a lot, and we will be particularly interested in functions whose symplectic gradients induce periodic flows. Such functions are called ***U*(1) moment maps**.

One of the main goals of the class will be to use $U(1)$ moment maps to study the cohomology of X. For example, we will prove that a $U(1)$ moment map is always a "perfect Morse-Bott function", which means that the Betti numbers of X can be read off from the critical sets of the function along with the behavior of the flow near those critical points. (In the case of S^2, the south pole contributes to $H^0(S^2)$ because the flow is clockwise around the south pole, while the north pole contributes to $H^2(S^2)$ because the flow is counter-clockwise.) We will then use equivariant cohomology to show that products of cohomology classes can also be computed in terms of local data at the fixed points, and use this to understand the cohomology ring of X.

The previous two paragraphs are about $U(1)$ symmetries of symplectic manifolds, but there is no need to stop there. The more symmetries that we have, the more we can say about the topology of our manifold. Throughout the text, the two main sets of examples on which we will focus are flag manifolds and toric varieties. These are, in two different senses, the "most symmetric" symplectic manifolds: a flag manifolds is a homogeneous space for the unitary group $U(n)$, while toric varieties have the largest possible *abelian* symmetry group for their dimension. These two classes of symplectic manifolds also provide strong connections to other fields of mathematics. The geometry of the flag manifold is central to the study of representation theory of $U(n)$, while toric varieties interact richly with with the combinatorics of polytopes. We will see how Stanely used toric varieties to prove half of the McMullen conjecture on f-vectors of polytopes. If we have time, we will also use the Kirwan-Ness theorem to translate a problem about tensor product multiplicities of $U(n)$ representations into symplectic geometry.

Contents

1 Differential topology

We'll begin with a quick review of some basic definitions in differential topology and Lie theory.

1.1 Manifolds

I'll start by listing a few things that I will **not** review, because you know them so well already.

- The definition of a manifold and of a smooth map between two manifolds.
- If X is a manifold and $p \in X$, then we have a real vector space T_pX, called **the tangent space to X at p**.
- If X is a manifold, then we have a manifold TX, equipped with a smooth map $TX \to X$, such that the preimage of p is canonically identified with T_pX. The manifold TX is called **the tangent bundle of X**.
- If $f : X \to Y$ is a smooth map and $p \in X$, there is an induced linear map $df_p : T_pX \to T_{f(p)}Y$. These linear maps fit together into a smooth map $df : TX \to TY$.
- If $f : X \to Y$ is a smooth map, a point $q \in Y$ is called a **regular value** if $df_p : T_pX \to T_qY$ is surjective for all $p \in f^{-1}(q)$. In this case, $f^{-1}(q)$ is a manifold, and $T_p f^{-1}(q) = \ker df_p$.

1.2 Differential forms

Let X be a manifold.

Definition 1.1. A **vector field** on X is a smooth section of the map $TX \to X$. Intuitively, this means that it is a smoothly varying choice of tangent vector at every point. Thus, if v is a vector field on X and $p \in X$, then $v(p) \in T_pX \subset TX$. We will sometimes write v_p instead of $v(p)$.

We will denote the set of vector fields on X by $\mathrm{VF}(X)$. Note that $\mathrm{VF}(X)$ is a module over $C^\infty(X)$ via pointwise multiplication.

Definition 1.2. For any natural number k, a **k-form** is a smooth section of the vector bundle $\wedge^k T^*X$. Concretely, this means that a k-form ω consists of, for every point $p \in X$, a multilinear function

$$\omega_p : (T_pX)^k \to \mathbb{R}$$

satisfying the following two properties:

- permuting the k inputs changes the output by the sign of the permutation
- if $v_1, \ldots, v_k$ are vector fields, then $p \mapsto \omega_p\big(v_1(p), \ldots, v_k(p)\big)$ is a smooth function on X.

We will denote the set of k-forms on X by $\Omega^k(X)$; this is also a module over $C^\infty(X)$ via pointwise multiplication. Furthermore, we have $\Omega^0(X) \cong C^\infty(X)$. We write

$$\Omega^\bullet(X) := \sum_{k \geq 0} \Omega^k(X).$$

Note that, if $\dim X < k$, then $\wedge^k T_p^* X = 0$ for all $p \in X$, so all k-forms are identically zero.

Definition 1.3. If $\omega \in \Omega^k(X)$ and $\theta \in \Omega^\ell(X)$, then $\omega \wedge \theta \in \Omega^{k+\ell}(X)$ is the $(k+\ell)$-form whose value at p is $\omega_p \wedge \theta_p$. For example, if $k = \ell = 1$, we have

$$(\omega \wedge \theta)(v_1, v_2) = \omega(v_1)\theta(v_2) - \omega(v_2)\theta(v_1).$$[1]

More generally, we start by taking the multilinear map given by plugging the first k vector fields into ω and the last ℓ into θ, and then multiplying the resulting functions. This is not alternating, so we make it alternating by antisymmetrizing.

Proposition 1.4. *The above product makes $\Omega^\bullet(X)$ into a graded-commutative $C^\infty(X)$-algebra.*

The next piece of structure to introduce is the exterior derivative of a differential form. Rather than actually constructing it in coordinates, I will simply assert everything that you need to know about it.

Proposition 1.5. *There is a unique $\mathbb{R}$-linear map $d : \Omega^\bullet(X) \to \Omega^\bullet(X)$ with the following properties:*

- *d raises degree by 1*
- $d^2 = 0$
- *if ω is a k-form and θ is a form of any degree, then $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$*
- *if $f \in C^\infty(X) \cong \Omega^0(X)$, then $df \in \Omega^1(X)$ is what you think it is.*

In fancier (but equivalent) language, there is a unique way to extend the differential on functions to all forms in such a way as to make $(\Omega^\bullet(X), d)$ into a differential graded algebra.

Differential forms can be pulled back along smooth maps, and the pullback enjoys all of the functoriality properties that you want it to. Make sure that you understand all the properties about pullbacks that are being asserted with the following statement.

Proposition 1.6. *The assignment $X \mapsto (\Omega^\bullet(X), d)$ gives a contravariant functor from the category of manifolds to the category of differential graded algebras.*

Definition 1.7. A differential form ω is called **closed** if $d\omega = 0$. For example, a function is closed if and only if it is locally constant; if $\dim X = n$, then all n-forms are closed.

There are two natural ways to combine a vector field with a differential form and obtain a new differential form. The first is to plug the vector field into the form, which lowers degree by 1. The second is to take the derivative of the form along the vector field (generalizing the notion of the directional derivative of a function), which preserves degree. We'll unpack these two operations below.

[1] There are two conventions here, one of which involves dividing by 2 in this equation.

Definition 1.8. If $v \in \mathrm{VF}(X)$ and $\omega \in \Omega^k(X)$, then $i_v\omega \in \Omega^{k-1}(X)$ is defined by the formula

$$(i_v\omega)(v_1, \ldots, v_k) := \omega(v, v_1, \ldots, v_k).$$

Definition 1.9. If $v \in \mathrm{VF}(X)$ and $\omega \in \Omega^k(X)$, then the **Lie derivative** $L_v\omega \in \Omega^k(X)$ is defined by the formula

$$L_v\omega := i_v d\omega + d i_v \omega.$$

Note that, if $f \in C^\infty(X)$, then $i_v f = 0$, so we have $L_v f = i_v df$, which is simply the directional derivative of f along v.

The motivation for this definition is as follows. For $t \in \mathbb{R}$, let $\phi_t : X \to X$ be the automorphism defined by "flowing along v" for time t. Basic results in the theory of differential equations say that ϕ_t is well defined when t is sufficiently close to zero. (If X is compact, then it is well defined for all $t \in \mathbb{R}$.) Then we have

$$L_v\omega = \lim_{t \to 0} \frac{\phi_t^*\omega - \omega}{t}.$$

This fact is known as **Cartan's magic formula**.

Exercise 1.10. *Prove Cartan's magic formula (all you need is Proposition 1.5).*

Suppose now that X is an n-manifold. A non-vanishing element of $\Omega^n(X)$ is called a **volume form**. An **orientation** of X is a choice of volume form up to an equivalence relation, where $\omega \sim \theta$ if and only if they differ by multiplication by a positive function. Thus, if X is orientable with r connected components, then it has exactly 2^r different orientations.

If X is compact and oriented, then we have a linear map

$$\int_X : \Omega^n(X) \to \mathbb{R}.$$

I will not spell out the full definition here; it involves locally identifying X with $\mathbb{R}^n$ and patching together the resulting integrals on Euclidean space. The integral of a volume form on a non-empty compact manifold is always positive.

1.3 Lie groups and Lie algebras

Definition 1.11. A **Lie group** is a manifold endowed with a group structure such that the multiplication and inversion maps are both smooth.

Example 1.12. The group $U(1)$ of unit complex numbers is a Lie group. The underlying manifold is just a circle.

Example 1.13. For any natural number n, let $T^n := U(1)^n$. This is called the n-torus, and will be the most common example of a Lie group in these notes. Any compact, connected, abelian, Lie group is isomorphic to T^n for some n.

Example 1.14. For any natural number n, let $U(n)$ be the group of unitary $n \times n$ complex matrices. That is, the set of matrices M such that $M^*M = \mathrm{Id}$, where M^* is the complex conjugate transpose of M. You should convince yourself that $U(n)$ is a submanifold of $\mathbb{C}^{n^2}$, that it is closed under matrix multiplication and inversion, and that both of these operations are smooth.

We say that $H \subset G$ is a **Lie subgroup** if it is simultaneously a subgroup and a closed submanifold. For example, the diagonal elements of $U(n)$ form a Lie subgroup that is isomorphic to the n-torus T^n. A theorem of Cartan states that any closed subgroup is a Lie subgroup; that is, the condition of being a submanifold automatic.

Definition 1.15. A **Lie algebra** is a vector space $\mathfrak{g}$ along with an antisymmetric, bilinear operation

$$[-\,,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity: $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ for all $a, b, c \in \mathfrak{g}$.

The definition of a Lie algebra is motivated by the following construction. Let G be a Lie group, and let $\mathfrak{g} = \mathrm{Lie}(G) := T_eG$, where $e \in G$ is the identity element. For each $g \in G$, we have the map $\Psi_g : G \to G$ given by $\Psi_g(h) := ghg^{-1}$. Differentiating at the identity gives us a linear map $\mathrm{Ad}_g := d_e\Psi_g : \mathfrak{g} \to \mathfrak{g}$. The assignment $g \mapsto \mathrm{Ad}_g$ is a smooth map from G to $\mathrm{End}(\mathfrak{g})$, the vector space of linear transformations from $\mathfrak{g}$ to itself.

Now take our smooth map $G \to \mathrm{End}(\mathfrak{g})$ and differentiate this at the identity; we get a linear map

$$\mathfrak{g} = T_eG \to T_{\mathrm{Id}} \, \mathrm{End}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g}).$$

For any $a \in \mathfrak{g}$, let $\mathrm{ad}_a \in \mathrm{End}(\mathfrak{g})$ denote its image. Now, for $a, b \in \mathfrak{g}$, define $[a, b] := \mathrm{ad}_a(b)$.

Proposition 1.16. *The bracket defined above makes $\mathfrak{g}$ into a Lie algebra. Furthermore, every finite dimensional Lie algebra comes from a Lie group in this way.*

Exercise 1.17. *Prove the first half of Proposition 1.16. That is, prove that the bracket we have defined is antisymmetric and satisfies the Jacobi identity.*

Example 1.18. The Lie algebra $\mathfrak{t}^n$ of T^n is pretty boring; the bracket is identically zero! This is because T^n is commutative, so the adjoint map defined above is constant.

Example 1.19. Let us now compute the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ along with its bracket. I will do this calculation in somewhat informal terms, which I hope will illuminate the important ideas. If you have never studied Lie algebras before, you should work carefully through this example!

Roughly speaking, the space $\mathfrak{u}(n) := T_{\mathrm{Id}}U(n)$ is the set of $n \times n$ matrices A such that, for a small real number ϵ, the matrix $\mathrm{Id} + \epsilon A$ "almost" lies in $U(n)$. That is, A lies in $\mathfrak{u}(n)$ if and only if $(\mathrm{Id} + \epsilon A)^*(\mathrm{Id} + \epsilon A) - \mathrm{Id}$ has order ϵ^2 or smaller. This is equivalent to the condition that $A^* + A = 0$, i.e. that A is **skew-Hermitian**.

We can compute the bracket in a similar manner. For $M \in U(n)$ and $B \in \mathfrak{u}(n)$, we have $M(\operatorname{Id} + \epsilon B)M^{-1} = \operatorname{Id} + \epsilon MBM^{-1}$, so our smooth map $U(n) \to \operatorname{End}\big(\mathfrak{u}(n)\big)$ takes the unitary matrix M to the endomorphism of $\mathfrak{u}(n)$ given by conjugation by M. This means that we have

$$(\operatorname{Id} + \epsilon A)B(\operatorname{Id} + \epsilon A)^{-1} = \operatorname{Id} + \epsilon[A, B] + O(\epsilon^2).$$

When ϵ is small enough, $\operatorname{Id} + \epsilon A$ is invertible, with inverse $\operatorname{Id} - \epsilon A + O(\epsilon^2)$. This tells us that $[A, B] = AB - BA$.

1.4 Lie group actions on vector spaces

Let V be a real vector space and G a Lie group.

Definition 1.20. A **linear action** of G on V is a smooth homomorphism $G \to GL(V)$. Equivalently, it is a smooth map $G \times V \to V$, $(g, v) \mapsto g \cdot v$, satisfying the conditions that $e \cdot v = v$ and $g \cdot (h \cdot v) = (gh) \cdot v$, along with the extra condition that $g \cdot v$ depends linearly on v. If V is equipped with a linear action of G, we call V a **representation** of G.

Example 1.21. The group $U(n)$ acts on $\mathbb{C}^n$ (which we may regard as a real vector space) via the inclusion $U(n) \to GL(\mathbb{C}^n)$. That is, for all $M \in U(n)$ and $v \in \mathbb{C}^n$, $M \cdot v = Mv$.

Example 1.22. Any Lie group G acts on $\mathfrak{g}$ via the **adjoint representation**. That is, for all $g \in G$ and $a \in \mathfrak{g}$, $g \cdot a = \operatorname{Ad}_g(a)$. If $G = T^n$, then $\operatorname{Ad}_g(a) = a$. If $G = U(n)$, then we've shown that $\mathfrak{u}(n)$ can be identified with the space of $n \times n$ skew-Hermitian matrices, and we have $\operatorname{Ad}_M(A) = MAM^{-1}$ (Example 1.19).

Let V be a representation of G. We define the **dual** action of G on V^* by the formula

$$\langle g \cdot \sigma, v \rangle = \langle \sigma, g^{-1} \cdot v \rangle$$

for all $g \in G$, $v \in V$, and $\sigma \in V^*$.

Exercise 1.23. *Show that this is a linear action. In so doing, convince yourself that the inverse needs to be there!*

Proposition 1.24. *Suppose that V and W are representations of G, and that there exists a perfect bilinear pairing $\langle -, - \rangle : V \times W \to \mathbb{R}$ such that $\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle$ for all $v \in V$, $w \in W$, and $g \in G$. Then W is isomorphic to the dual of V.*

Proof. The map $w \mapsto \langle -, w \rangle$ is a linear isomorphism from W to V^*; we only need to check that it is G-equivariant. Indeed, we have $g \cdot w \mapsto \langle -, g \cdot w \rangle = \langle g^{-1} \cdot -, w \rangle = g \cdot \langle -, w \rangle$. □

Suppose that V is a representation of G. Differentiating the map $G \to GL(V)$ at the identity, we obtain a linear map $\mathfrak{g} \to T_{\operatorname{Id}} GL(V) = \operatorname{End}(V)$. For all $a \in \mathfrak{g}$ and $v \in V$, we write $a \cdot v$ for the vector obtained by hitting v with the image of a. We have already discussed this map when

$V = \mathfrak{g}$ is the adjoint representation. In this case, we defined $[a, b] := \mathrm{ad}_a(b) := a \cdot b$. We will also wish to consider the **coadjoint representation** $\mathfrak{g}^*$. In this case, we define $\mathrm{Ad}^*_g(\xi) := g \cdot \xi$ and $\mathrm{ad}^*_a(\xi) := a \cdot \xi$ for all $g \in G$, $a \in \mathfrak{g}$, and $\xi \in \mathfrak{g}^*$.

Exercise 1.25. *Let V be a representation of G. Show that, for all $a \in \mathfrak{g}, v \in V, \sigma \in V^*$, we have*

$$\langle a \cdot \sigma, v\rangle = -\langle \sigma, a \cdot v\rangle.$$

(The minus sign is related to the inverse in the definition of the dual action.) In particular, if V is the adjoint representation, we have

$$\langle \mathrm{ad}^*_a \xi, b\rangle = -\langle \xi, [a, b]\rangle.$$

Let's consider the coadjoint representation of $\mathfrak{u}(n)$.

Proposition 1.26. *The coadjoint action of $U(n)$ on $\mathfrak{u}(n)^*$ is isomorphic to the conjugation action on the space of $n \times n$ Hermitian matrices (that is, the set of A such that $A = A^*$). With this identification, we have $\mathrm{ad}^*_A(\Sigma) = [A, \Sigma]$ for any skew-Hermitian A and Hermitian Σ.*

Proof. To prove the first statement, we need to find a perfect pairing

$$\{n \times n \text{ Hermitian matrices}\} \times \{n \times n \text{ skew-Hermitian matrices}\} \to \mathbb{R}$$

that is invariant under conjugating both inputs by an element of $U(n)$. Such a pairing is given by $\langle \Sigma, A\rangle := -\operatorname{tr}(i\Sigma A)$.[2] For the second statement, we need to check that $[A, \Sigma]$ is Hermitian, and that $-\operatorname{tr}(i[A, \Sigma]B) = \operatorname{tr}(i\Sigma[A, B])$. □

Exercise 1.27. *Fill in the details of the proof of Proposition 1.26. That is, show that $-\operatorname{tr}(i\Sigma A)$ is real-valued, conjugation invariant, and non-degenerate, and show that $-\operatorname{tr}(i[A, \Sigma]B) = \operatorname{tr}(i\Sigma[A, B])$.*

Exercise 1.28. *Consider the group $\mathrm{SO}(3)$ of 3×3 real matrices M such that $M^t M = \mathrm{Id}$ and $\det M = 1$. Show that both the adjoint and coadjoint representation of $\mathrm{SO}(3)$ are isomorphic to the standard representation $\mathbb{R}^3$.*

1.5 Lie group actions on manifolds

Let X be a manifold and G a Lie group.

Definition 1.29. An **action** of G on X is a smooth map $G \times X \to X$, $(g, x) \mapsto g \cdot x$, satisfying the conditions that $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$.

Definition 1.30. For any $p \in X$, the **stabilizer** of p is the subgroup $G_p := \{g \in G \mid g \cdot p = p\}$. If $G_p = \{e\}$ for all $p \in X$, we say that the action of G on X is **free**. If G_p is discrete for all $p \in X$, we say that the action of G on X is **locally free**.

[2] We include the minus sign so that, when $n = 1$, we get the obvious pairing between $\mathbb{R}$ and $i\mathbb{R}$.

Example 1.31. Consider the rotation action of T^1 on S^2. If we think of S^2 as the unit sphere in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, then the action is given by the formula $t \cdot (z, r) = (tz, r)$. Each of the two poles has stabilizer T^1, and T^1 acts freely away from the poles.

Example 1.32. Next, consider the Hopf action of T^1 on S^3. If we think of S^3 as the unit sphere in $\mathbb{C}^2$, then the action is given by the formula $t \cdot (z, w) = (tz, tw)$. This action is free.

Example 1.33. Finally, consider the action of T^1 on S^3 by the formula $t \cdot (z, w) = (tz, t^2 w)$. The stabilizer of (z, w) is trivial if $z \neq 0$ and it is $\{\pm 1\}$ if $z = 0$. Thus the action is locally free, and it is free away from the submanifold $S^1 \subset S^3$ defined by the vanishing of z.

For any $p \in X$ and $g \in G_p$, g defines an automorphism of X that takes p to itself, and therefore (by differentiating) a linear automorphism of T_pX. In particular, if

$$p \in X^G := \{p \in X \mid G_p = G\},$$

we obtain a linear action of G on T_pX.

Proposition 1.34. *If G is compact, then the fixed point set $X^G \subset X$ is a closed submanifold.*

Proof. Choosing a Riemannian metric on X, we can find a local diffeomorphism from T_pX to X taking 0 to p. Since G is compact, we can use an averaging trick to make our metric (and therefore our local diffeomorphism) G-equivariant. Thus, in a neighborhood of p, the inclusion of X^G into X looks like the inclusion of $(T_pX)^G$ into T_pX. Since this is just an inclusion of vector spaces, it is a closed embedding. □

If G acts on X, then every element of $\mathfrak{g}$ induces a vector field on X as follows. Differentiating the action map at (e, p), we obtain a map $TG \times TX \to TX$. We have $\mathfrak{g} = T_eG \subset TG$ and $X \subset TX$, therefore we may restrict to a map $\mathfrak{g} \times X \to TX$. For any $a \in \mathfrak{g}$, the restriction

$$\hat{a} : X \cong \{a\} \times X \to TX$$

is a vector field. More intuitively, for each $p \in X$, consider the smooth map $G \to X$ taking g to $g \cdot p$. Differentiating at the identity gives a linear map $\mathfrak{g} \to T_pX$. For each $a \in \mathfrak{g}$, $\hat{a}_p$ is equal to the image of a along this map.

Exercise 1.35. *Check that, if G acts linearly on V, then for all $a \in \mathfrak{g}$ and $v \in V$, we have*

$$\hat{a}_v = a \cdot v.$$

Remark 1.36. The space $\mathrm{VF}(X)$ may be identified with the Lie algebra of the infinite-dimensional Lie group $\mathrm{Diff}(X)$. The action map defines a group homomorphism $G \to \mathrm{Diff}(X)$, and the map $\mathfrak{g} \to \mathrm{VF}(X)$ is its derivative. Justin Hilburn points out that the Lie bracket on $\mathrm{VF}(X)$ induced by the observation that it is the Lie algebra of $\mathrm{Diff}(X)$ is the opposite of the usual Lie bracket;

this will be important in the proof of Proposition 2.18. The point is that we think of the group as acting on the space but we think of the vector field as acting on functions; when we go from spaces to functions, we pick up a minus sign.

The following proposition can be deduced as a corollary of the "slice theorem" [Au, I.2.1].

Proposition 1.37. *Suppose that G is a compact Lie group acting freely on a manifold X. Then the quotient space X/G is a manifold, and the projection $\pi : X \to X/G$ is smooth. Furthermore, for all $p \in X$, we have a short exact sequence*

$$0 \to \mathfrak{g} \to T_pX \to T_{\pi(p)}X/G \to 0,$$

where the first map takes a to $\hat{a}_p$ and the second map is $d\pi_p$.

Example 1.38. For the free action of Example 1.32, the quotient space is diffeomorphic to S^2.

Example 1.39. Suppose that G is a Lie group and $H \subset G$ is a Lie subgroup. Then H acts freely on G by (inverse) right multiplication, so the coset space G/H is a manifold. The exact sequence of Proposition 1.37 at the point $e \in G$ is $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$.

Note that G acts on G/H by left multiplication. The stabilizer of a point gH is equal to the subgroup $gHg^{-1} \subset G$.

Remark 1.40. If the action is only locally free, then X/G will be an orbifold rather than a manifold.

Definition 1.41. An action of G on X is called **transitive** if it consists of a single orbit.

The action of G on G/H from Example 1.39 is clearly transitive. In fact, the following proposition says that all transitive actions are of this form.

Proposition 1.42. *Suppose that a Lie group G acts transitively on X. Then for any $p \in X$, X is G-equivariantly diffeomorphic to G/G_p.*

Proof. Consider the map from G/G_p to X sending gG_p to $g \cdot p$; this is clearly an equivariant, smooth, bijection. We leave it as an exercise to show that the inverse is smooth. □

Example 1.43. Let

$$X := \Big\{(L_1, \ldots, L_n) \;\Big|\; \dim L_i = 1\ \forall i,\ L_i \perp L_j\ \forall i \neq j\Big\}$$

be the manifold of orthogonal frames in $\mathbb{C}^n$ with respect to the Hermitian inner product, and let $G = U(n)$. The action is clearly transitive, and the stabilizer of the standard frame is T^n, thus Proposition 1.42 says that X is G-equivariantly diffeomorphic to $U(n)/T^n$. Note that X is also diffeomorphic to the manifold of flags in $\mathbb{C}^n$:

$$X \cong \Big\{(F_1, \ldots, F_n) \;\Big|\; \dim F_i = i\ \forall i,\ F_i \subset F_{i+1}\ \forall i\Big\}.$$

Definition 1.44. The **Lie algebra stabilizer** of p is the Lie sub-algebra $\mathfrak{g}_p := \{a \in \mathfrak{g} \mid \hat{a}_p = 0\}$.

Proposition 1.45. *For all $p \in X$, $\mathfrak{g}_p = \mathrm{Lie}(G_p)$. In particular, the action of G is locally free if and only if $\mathfrak{g}_p = 0$ for all $p \in X$.*

Proof. Consider the map from G to X taking g to $g \cdot p$; by definition, $\mathfrak{g}_p$ is the kernel of the derivative of this map. We can factor this map as follows:

$$G \to G/G_p \cong G \cdot p \subset X,$$

where the isomorphism $G/G_p \cong G \cdot p$ follows from Proposition 1.42. Differentiating, we get

$$\mathfrak{g} \to \mathfrak{g}/\,\mathrm{Lie}(G_p) \cong T_p G \cdot p \subset T_p X.$$

The kernel of this composition is clearly $\mathrm{Lie}(G_p)$, and the proposition follows. □

Proposition 1.46. *Suppose that a Lie group G acts transitively on X. For all $p \in X$, we have a canonical isomorphism of vector spaces $T_p X \cong \mathfrak{g}/\mathfrak{g}_p$.*

Proof. By transitivity, the map $\mathfrak{g} \to T_p X$ taking a to $\hat{a}_p$ is surjective, and its kernel is $\mathfrak{g}_p$. (Alternatively, it follows from Example 1.39, Proposition 1.42, and Proposition 1.45.) □

2 Symplectic manifolds

This section will contain the definition, basic properties, and first examples of symplectic manifolds. We will also prove Darboux's Theorem, which says that symplectic manifolds (like manifolds but unlike Riemannian manifolds) have no local invariants other than dimension.

2.1 Definition and basic properties

Definition 2.1. A 2-form $\omega \in \Omega^2(X)$ is **symplectic** if it is is closed and non-degenerate. This second condition means that, for all $p \in X$, the function $v \mapsto \omega_p(v, -)$ should be an isomorphism from $T_p X$ to $T_p^* X$. A **symplectic manifold** is a pair (X, ω), where ω is a symplectic form.

It is interesting to compare this definition to the definition of a Riemannian manifold. A Riemannian metric on X is a way to pair two tangent vectors at a point and get a number, such that the pairing at every point is non-degenerate and symmetric. A symplectic form is the same except that it is antisymmetric, plus there is this extra condition that the form should be closed. The relevance of these differences becomes clear when we think about the symplectic gradient. (See Remark 2.29 for a different philosophical discussion of the closedness condition.)

Definition 2.2. Let (X, ω) be a symplectic manifold, and let $f \in C^\infty(X)$ be a smooth function. Then the non-degeneracy of ω implies that there is a unique vector field v_f with the property that $i_{v_f}\omega = df$. The vector field v_f is called the **symplectic gradient** of f. Note that the analogous construction with a Riemannian metric would give the usual gradient.

Proposition 2.3. *Flowing along the vector field v_f preserves both ω and f.*

Proof. Recall that the Lie derivative (Definition 1.9) measures the infinitesimal change of a differential form along a vector field, so we only need to show that $L_{v_f}\omega = 0$ and $L_{v_f} f = 0$. By Cartan's magic formula, $L_{v_f} = i_{v_f} d + d i_{v_f}$, so

$$L_{v_f}\omega = i_{v_f} d\omega + d i_{v_f}\omega = 0 + ddf = 0$$

and

$$L_{v_f} f = i_{v_f} df + d i_{v_f} f = i_{v_f} i_{v_f}\omega + 0 = \omega(v_f, v_f) = 0.$$

(Note that the first calculation uses closedness, and the second uses antisymmetry!) □

Example 2.4. Let $X = S^2$ be the unit sphere in R^3. For any $p \in S^2$, we have

$$T_p S^2 = \{q \in \mathbb{R}^3 \mid p \cdot q = 0\}.$$

Consider the 2-form ω defined by the equation $\omega_p(q, r) = p \cdot (q \times r)$. This is clearly smooth, antisymmetric, and non-degenerate. Since $\dim X = 2$, all 2-forms are closed, so ω is symplectic. Let f be the height function: $f(p) := p_z$. This is the restriction of a linear function on $\mathbb{R}^3$, so its derivative is the restriction of the same linear function. That is, for any $r \in T_p X$, $df_p(r) = r_z$. Then $v_f(p)$ is the unique vector such that $p \cdot (v_f(p) \times r) = r_z$. I claim that the following vector does the trick: $v_f(p) = (-p_y \ p_x \ 0)$.

Draw the picture. Compare to the usual gradient, which is $\nabla_f(p) = (-p_z p_x \ \ -p_z p_y \ \ 1 - p_z^2)$.

Let's give a few necessary conditions for a manifold to admit a symplectic form.

Proposition 2.5. *If X is a non-empty symplectic n-manifold, then n is even.*

Proof. By elementary linear algebra, any finite-dimensional vector space with a non-degenerate antisymmetric bilinear form admits a basis $e_1, f_1, \ldots, e_k, f_k$ such that $\langle e_i, e_j\rangle = 0 = \langle f_i, f_j\rangle$ and $\langle e_i, f_j\rangle = \delta_{ij}$. In particular, it must be even-dimensional. □

Proposition 2.6. *Suppose that X is a non-empty $2n$-manifold. A closed 2-form $\omega \in \Omega^2(X)$ is symplectic if and only if $\wedge^k\omega$ is a volume form. In particular, every symplectic manifold is orientable.*

Proof. This is a purely local statement: we are saying that an antisymmetric bilinear form ω_p on a $2k$-dimensional vector space V is non-degenerate if and only if its n^{th} wedge power is non-zero. Indeed, if we choose a basis for V and write the bilinear form as a matrix, then its n^{th} wedge power will equal the determinant of the matrix times the generator of $\wedge^{2n}(V)$. □

We can do even better than this if X is compact.

Proposition 2.7. *Suppose that X is a compact, nonempty, $2n$-manifold and $\omega \in \Omega^2(X)$ is symplectic. Let $[\omega] \in H^2(X;\mathbb{R})$ be the de Rham class of ω. Then $[\omega]^n \neq 0$. In particular, if $n > 0$, then $H^2(X;\mathbb{R}) \neq 0$.*

Proof. Suppose $[\omega]^n = 0$; this means that $\omega^n = d\theta$ for some $\theta \in \Omega^{2n-1}(X)$. Then $\int_X \omega^n = \int_X d\theta = 0$ by Stokes' Theorem. This contradicts the fact that ω^n is a volume form. □

Example 2.8. S^4 does not admit a symplectic form, despite being even-dimensional and orientable.

2.2 Examples

In the previous section we saw one example of a symplectic manifold (the 2-sphere) and lots of non-examples (anything odd-dimensional, non-orientable, or compact with trivial second Betti number). Now let's write down some more examples.

Example 2.9. Let $X = \mathbb{R}^{2n}$, and let $\omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$. For every $p \in X$, $T_pX = \mathbb{R}^{2n}$, and ω_p is the bilinear form discussed in the proof of Proposition 2.5. Note that $H^2(\mathbb{R}^{2n}; \mathbb{R}) = 0$, but that's okay because $\mathbb{R}^{2n}$ is non-compact. Sometimes it will be convenient to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, in which case ω becomes $\frac{i}{2}\big(dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n\big)$.

Example 2.10. By Proposition 2.6, any volume form on an orientable surface is symplectic.

Example 2.11. The $2n$-torus T^{2n} can be made symplectic, either by choosing a volume form on T^2 and writing $T^{2n} = (T^2)^n$, or by noting that $T^{2n} \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, and the action of $\mathbb{Z}^{2n}$ on $\mathbb{R}^{2n}$ preserves the symplectic form from Example 2.9.

Example 2.12. Let Y be any manifold, and let $X := T^*Y$ be its cotangent bundle, and let $\pi : X \to Y$ be the projection. We will denote an element of X by a pair (q, p), where $q \in Y$ and $p \in T_qY$. With this notation, $\pi(q, p) = q$. The **canonical 1-form** $\alpha \in \Omega^1(X)$ is defined by the formula

$$\alpha_{(q,p)}(v) := p\Big(d\pi_{(q,p)}(v)\Big).$$

Proposition 2.13. *The 2-form $d\alpha$ is symplectic.*[3]

Proof. Being symplectic is a local condition, so we may assume that $Y = \mathbb{R}^n$, in which case we can identify X with $\mathbb{R}^{2n}$. Let $y_1, \ldots, y_n$ be the coordinate functions on Y, and $x_1, \ldots, x_n$ the coordinate functions on the fibers. Then $\alpha = x_1 dy_1 + \ldots x_n dy_n$, and $d\alpha = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$, which we know is symplectic. □

There is something very confusing about Proposition 2.13. First of all, you are probably more used to thinking about tangent bundles than about cotangent bundles. What does the cotangent bundle of Y look like? Well, you can always choose a Riemannian metric on Y (even if its dimension is odd!), and you can use that metric to identify T_q^*Y with T_qY for every q. In particular, this means that T^*Y is diffeomorphic to TY.

Why then, you might ask, didn't we state the proposition about tangent bundles? The answer is that the symplectic structure on T^*Y is canonical, while the identification of T^*Y with TY is

[3] Note that $d\alpha$ is exact, so its cohomology class is trivial. This doesn't contradict Proposition 2.7 because cotangent bundles are non-compact.

not (it involves the choice of a Riemannian metric). Here is an important practical implication: suppose that a Lie group G acts on Y. There is an induced action of G on both TY and T^*Y. The induced action on T^*Y will preserve the symplectic form; that is, if $g \in G$ and ρ_g is the corresponding automorphism of Y, then $\rho_g^* d\alpha = d\alpha$. The corresponding statement is false for TY.

Exercise 2.14. *Prove that the action of G preserves the symplectic form on T^*Y. Hint: Show that it preserves the canonical 1-form.*

Exercise 2.15. *Consider the example of $\mathbb{R}^\times$ acting on $\mathbb{R}$ by scalar multiplication. Show explicitly that the induced cotangent action on $\mathbb{R}^2$ preserves the symplectic form $dx \wedge dy$, while the induced tangent action does not.*

Another confusing point is that the proof of Proposition 2.13 was entirely local. If we immediately reduced to the case where $Y = \mathbb{R}^n$, it seems as if the distinction between the tangent bundle and the cotangent bundle is completely irrelevant, as both can be canonically identified with $\mathbb{R}^{2n}$. The answer to this objection is that there is no way to write down a 1-form on TY such that, given any local diffeomorphism from $\mathbb{R}^n$ to Y, the induced 1-form on $\mathbb{R}^{2n}$ is $x_1 dy_1 + \ldots x_n dy_n$. You can only do this on a cotangent bundle.

Finally, a fact that's important in my own work is that TY and T^*Y are not necessarily isomorphic if you are working in a category other than the category of smooth manifolds. For example, everything that we have written above would make sense in the category of smooth algebraic varieties over an arbitrary algebraically closed field. In this category, T^*Y is always symplectic, while TY is not.

Example 2.16. Our goal in this example will be to define a symplectic form on an arbitrary coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$.[4] For any $\xi \in \mathfrak{g}^*$, consider the skew-symmetric bilinear form on $\mathfrak{g}$ given by the formula

$$\omega_\xi(a, b) := \langle \xi, [a, b] \rangle.$$

We define the **kernel** of ω_ξ to be the set of $a \in \mathfrak{g}$ such that $\omega_\xi(a, b) = 0$ for all $b \in \mathfrak{g}$.

Lemma 2.17. *The kernel of ω_ξ is equal to the Lie algebra stabilizer $\mathfrak{g}_\xi$ for the coadjoint action.*

Proof. Recall that $\mathfrak{g}_\xi$ is defined as the kernel of the map $\mathfrak{g} \to T_\xi \mathfrak{g}^* \cong \mathfrak{g}^*$ taking a to $\hat{a}_\xi$. By Exercise 1.35, $\hat{a}_\xi = a \cdot \xi = \mathrm{ad}_a^*(\xi)$. That means that, for all $b \in \mathfrak{g}$,

$$\langle \hat{a}_\xi, b \rangle = \langle \mathrm{ad}_a^*(\xi), b \rangle = -\langle \xi, \mathrm{ad}_a(b) \rangle = -\langle \xi, [a, b] \rangle = -\omega_\xi(a, b).$$

The lemma follows. $\square$

Let $\mathcal{O} = G \cdot \xi \subset \mathfrak{g}^*$ be the coadjoint orbit containing ξ. By Proposition 1.46, the map $a \mapsto \hat{a}_\xi$ induces an isomorphism

$$T_\xi \mathcal{O} \cong \mathfrak{g}/\mathfrak{g}_\xi = \mathfrak{g}/\ker \omega_\xi.$$

[4] If we choose a G-invariant inner product on $\mathfrak{g}$, we can G-equivariantly identify $\mathfrak{g}$ with $\mathfrak{g}^*$. However, our construction will naturally make sense only for the coadjoint action. If you find this confusing, you should reread the discussion of tangent and cotangent bundles.

It follows that ω_ξ descends to a non-degenerate skew-symmetric bilinear form on $T_\xi\mathcal{O}$. Considering all ξ at once, we obtain a non-degenerate 2-form ω.

Proposition 2.18. *The 2-form ω on $\mathcal{O}$ is closed, thus $(\mathcal{O}, \omega)$ is symplectic.*

Proof. Since the map $a \mapsto \hat{a}_\xi$ is a surjection from $\mathfrak{g}$ to $T_\xi\mathcal{O}$, it is enough to show that

$$d\omega_\xi(\hat{a}_\xi, \hat{b}_\xi, \hat{c}_\xi) = 0$$

for all $a, b, c \in \mathfrak{g}$. I didn't actually write down the definition of the exterior derivative of a 2-form, but if you look it up (for example, on the Wikipedia page), you'll find that if u, v, w are vector fields, then

$$\begin{aligned} d\omega(u,v,w) &= d\big(\omega(v,w)\big)(u) - d\big(\omega(u,w)\big)(v) + d\big(\omega(u,v)\big)(w) \\ &\quad -\omega([u,v],w) + \omega([u,w],v) - \omega([v,w],u). \end{aligned}$$

Here the brackets are Lie brackets of vector fields, but remember that the map $a \mapsto \hat{a}$ is an antihomomorphism of Lie algebras (Remark 1.36) so $[\hat{a}, \hat{b}] = -\widehat{[a,b]}$. Also, note that $\omega_\xi(\hat{b}_\xi, \hat{c}_\xi) = \langle \xi, [b,c] \rangle$ is linear in ξ, so

$$d\omega(\hat{b}, \hat{c})_\xi(\hat{a}_\xi) = \langle \hat{a}_\xi, [b,c] \rangle = \langle \mathrm{ad}_a^*(\xi), [b,c] \rangle = -\langle \xi, \mathrm{ad}_a([b,c]) \rangle = -\langle \xi, [a,[b,c]] \rangle.$$

Putting it all together, we have

$$\begin{aligned} d\omega_\xi(\hat{a}_\xi, \hat{b}_\xi, \hat{c}_\xi) &= -\langle \xi, [a,[b,c]] \rangle + \langle \xi, [b,[a,c]] \rangle - \langle \xi, [c,[a,b]] \rangle \\ &\quad +\langle \xi, [[a,b],c] \rangle - \langle \xi, [[a,c],b] \rangle + \langle \xi, [[b,c],a] \rangle. \end{aligned}$$

Both lines vanish by the Jacobi identity, thus $d\omega = 0$. □

Remark 2.19. For a long time, I thought that the map from $\mathfrak{g}$ to $\mathrm{VF}(X)$ was a homomorphism rather than an antihomomorphism of Lie algebras, which would have the effect of changing the signs in the second line of our expression for $d\omega_\xi(\hat{a}_\xi, \hat{b}_\xi, \hat{c}_\xi)$. In this case the two lines would cancel each other out, and the Jacobi identity would not have been necessary! I thank Justin Hilburn for returning the Jacobi identity to its rightful place at the center of this argument.

Example 2.20. Consider the group $U(n)$ of $n \times n$ unitary matrices. By Proposition 1.26, we know that coadjoint orbits for $U(n)$ are the same as conjugacy classes of Hermitian matrices. Furthermore, you know from your extensive linear algebra training that every Hermitian matrix is diagonalizable, and that the diagonalization of a Hermitian matrix is unique up to permuting the diagonal entries. That is, we have a bijection

$$\begin{array}{rcl} \{(\lambda_1,\ldots,\lambda_n) \mid \lambda_1 \geq \ldots \geq \lambda_n \in \mathbb{R}\} & \longleftrightarrow & \Big\{U(n)\text{-orbits in } \mathfrak{u}(n)^*\Big\} \\ \lambda & \longmapsto & \mathcal{H}_\lambda := U(n)\cdot \operatorname{diag}(\lambda) \\ \operatorname{Spectrum}(\Sigma) & \longleftarrow & U(n)\cdot\Sigma. \end{array}$$

By Proposition 2.18, $\mathcal{H}_\lambda$ is a symplectic manifold for every λ. So let's think about what this manifold $\mathcal{H}_\lambda$ is.

Proposition 2.21. *Suppose that $\lambda_1 > \ldots > \lambda_n$ are distinct. Then $\mathcal{H}_\lambda$ is diffeomorphic to the manifold of orthogonal frames in $\mathbb{C}^n$, or equivalently the manifold of flags in $\mathbb{C}^n$.*

Proof. A Hermitian matrix with spectrum λ is completely determined by its eigenspaces, which are pairwise orthogonal. □

More generally, by letting some of the eigenvalues coincide, we can get any partial flag manifold. An important special case is when $n-1$ of the eigenvalues coincide.

Proposition 2.22. *Suppose that $\lambda_1 > \lambda_2 = \ldots = \lambda_n$. Then $\mathcal{H}_\lambda$ is diffeomorphic to $\mathbb{CP}^{n-1}$.*

Proof. A Hermitian matrix with spectrum λ is completely determined by its λ_1-eigenspace. □

Note that the symplectic form on $\mathcal{H}_\lambda$ depends non-trivially on λ, even when the multiplicities do not change. In particular, we've just defined an infinite family of symplectic forms on $\mathbb{CP}^n$, one for every λ satisfying the condition of Proposition 2.22.

The next proposition will provide us with a huge wealth of new examples. Recall that an **almost complex** structure J on a manifold X is an automorphism J_p of T_pX for every $p \in X$ (smoothly varying, of course) such that $J_p^2 = -1$. If an almost complex structure can be induced by local diffeomorphisms with $\mathbb{C}^n$, we call it a **complex structure**. (Not every almost complex structure is complex.)

Suppose that J is an almost complex structure and ω is a symplectic form. We say that ω and J are **compatible** if, for all $p \in X$, the following two conditions hold:

- for all $u, v \in T_pX$, we have $\omega_p(J_pu, J_pv) = \omega_p(u, v)$
- for all $v \in T_pX$, $\omega_p(u, Ju) > 0$.

Note that, if ω and J are compatible, then $g(-,-) := \omega(-, J-)$ is a Riemannian metric on X.

Proposition 2.23. *Suppose that (X, ω) is symplectic, that X is equipped with an almost complex structure J, and that J and ω are compatible in the sense that $\omega_p(J_pu, J_pv) = \omega_p(u, v)$ for any $p \in X$ and $u, v \in T_pX$. Suppose that $Z \subset X$ is an almost complex submanifold. Then $(Z, \omega|_Z)$ is symplectic.*

Proof. When we restrict to Z, the Riemannian metric g remains nondegenerate (this is true for any positive-definite bilinear form), thus so does ω. □

Remark 2.24. A compatible triple (X, ω, J) is called an **almost Kähler manifold**. If J is a complex structure, then it is called a **Kähler manifold**. An important fact that we will need in the section on Morse theory is that every symplectic manifold admits a compatible almost complex structure (though not necessarily a compatible complex structure) [Au, II.2.5]. Furthermore, if a compact Lie group G is acting symplectically on X, then we can take this almost complex structure to be compatible with the G-action.

Example 2.25. If we take X to be $\mathbb{C}^n$, it is not hard to show that the symplectic form of Example 2.9 is compatible with the complex structure. In particular, this means that every complex submanifold of $\mathbb{C}^n$ has a canonical symplectic form. Similarly, the symplectic form on $\mathcal{H}_\lambda \cong \mathbb{CP}^{n-1}$ from Proposition 2.22 is compatible with the complex structure, so every complex projective variety (closed complex submanifold of $\mathbb{CP}^{n-1}$) is symplectic. More generally, any complex submanifold of $\mathbb{C}^n \times \mathbb{CP}^m$ is symplectic.

Exercise 2.26. *Prove that the symplectic forms on $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$ are compatible with the complex structures.*

Proposition 2.23 and Remark 2.24 have the following important corollary.

Corollary 2.27. *Suppose that G is a compact group acting symplectically on (X, ω). Then $(X^G, \omega|_{X^G})$ is symplectic.*

Proof. Choose a G-equivariant compatible almost complex structure. The submanifold $X^G \subset X$ will be almost complex, and therefore symplectic by Proposition 2.23. □

2.3 Darboux's theorem

A manifold has no local invariants other than dimension: every n-manifold is locally diffeomorphic to $\mathbb{R}^n$. (This is not a theorem, it is the definition of a manifold!) Riemannian manifiolds behave very differently: they have curvature. In particular, it is *not* true that every Riemannian n-manifold is locally isometric to $\mathbb{R}^n$ with the Euclidean metric. (Such a Reimannian manifold is called **flat**, and flat Riemannian manifolds are very unusual!) Darboux's theorem says that symplectic manifolds are like manifolds, or like flat Riemannian manifolds.

Theorem 2.28. *Let (X, ω) be a symplectic manifold of dimension $2n$. For every $p \in X$, we can find a neighborhood U of p in X, a neighborhood V of 0 in $\mathbb{R}^{2n}$, and a diffeomorphism $\phi : V \to U$ such that $\phi^*\omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$.*

Remark 2.29. Theorem 2.28 would fail if we dropped the condition that $d\omega = 0$ in the definition of a symplectic form. (This is clear, because closedness is local, and the form on V is closed.) Thus you can think of the condition $d\omega = 0$ as a symplectic version of "flatness".

To prove Darboux's theorem, we begin with the following analytical lemma, known as "the Moser trick"; for the proof, I am completely following [Au, II.1.9], with just a little bit of added commentary.

Lemma 2.30. *Suppose that ω_0 and ω_1 are two symplectic forms on X that agree at the point p. Then we can find a neighborhood U of p and a map $\psi : U \to X$ fixing p such that $\psi^*\omega_1 = \omega_0$.*

Proof. For all $t \in [0,1]$, let $\omega_y = \omega_0 + t(\omega_1 - \omega_0)$. This form is nondegenerate at p, and therefore in a neighborhood of p. By compactness of $[0,1]$, we can find a neighborhood U of p such that ω_t is nondegenerate on U for all t (see Exercise 2.31). We may assume that U is contractible. Since $\omega_0 - \omega_1$ is closed and U is contractible, there exists a 1-form $\eta \in \Omega^1(U)$ such that $d\eta = \omega_0 - \omega_1$. We may also choose η such that $\eta_p = 0$. (If $\eta_p \neq 0$, pick a closed 1-form η' with $\eta'_p = \eta_p$, and subtract η' from η.) Since ω_t is symplectic, there is a unique vector field $v_t \in \mathrm{VF}(U)$ such that $i_{v_t}\omega_t = \eta$; note that we must have $v_t(p) = 0$.

We regard v_t as a time-dependent vector field on U, and let ϕ_t denote the map from U to X obtained by flowing along this time dependent vector field from time 0 to time t. Okay, it may not actually be defined on all of U, because some points may rush off to infinity in finite time. But since $v_t(p) = 0$ for all t, everything is moving very slowly near t, which means that we can shrink U so that ϕ_t is defined on U for all $t \in [0,1]$ (again, we are using compactness of $[0,1]$). Then

$$
\begin{aligned}
\frac{d}{dt}\left[\phi_t^*\omega_t\right] &= \lim_{h\to 0} \frac{\phi_{t+h}^*\omega_{t+h} - \phi_t^*\omega_t}{h} \\
&= \lim_{h\to 0} \frac{\phi_{t+h}^*\omega_t - \phi_t^*\omega_t}{h} + \lim_{h\to 0} \frac{\phi_{t+h}^*\omega_{t+h} - \phi_{t+h}^*\omega_t}{h} \\
&= \phi_t^* L_{v_t}\omega_t + \phi_t^*\frac{d\omega_t}{dt} \\
&= \phi_t^*\left[L_{v_t}\omega_t + \frac{d\omega_t}{dt}\right].
\end{aligned}
$$

We have $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$ and $L_{v_t}\omega_t = di_{v_t}\omega_t + i_{v_t}d\omega_t = d\eta = \omega_0 - \omega_1$, so two terms cancel and we have $\frac{d}{dt}\left[\phi_t^*\omega_t\right] = 0$. This means that $\phi_t^*\omega_t$ does not depend on t. In particular, we have $\phi_1^*\omega_1 = \phi_0^*\omega_0 = \omega_0$, and we are done. □

Exercise 2.31. *Fill in the details justifying the existence of the neighborhood U in the proof of Lemma 2.30. Hint: For all $t \in [0,1]$, show that there is a neighborhood $U_t \times W_t$ be a neighborhood of (p,t) in $X \times [0,1]$ such that ω_s is nondegenerate on U_t for all $s \in W_t$. Now use compactness.*

Proof of Theorem 2.28: By picking a Riemannian metric, we can find a local diffeomorphism from T_pX to X taking 0 to p. Transporting the constant form ω_p on T_pX along this local diffeomorphism, we obtain a symplectic form ω' in a neighborhood W of p such that $\omega'_p = \omega_p$ and (W, ω') is locally diffeomorphic to (T_pX, ω_p). As we saw in the proof of Proposition 2.5, any vector space with a constant symplectic form is symplectomorphic to $(\mathbb{R}^{2n}, dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n)$. Now apply Lemma 2.30 to the symplectic forms ω and ω' on the manifold W. □

Remark 2.32. Suppose that G is a compact group acting symplectically on X and $p \in X^G$. We have just shown that (X, ω) is locally symplectomorphic to (T_pX, ω_p). In fact, this local symplectomorphism can be made G-equivariant; we will need this later.

3 Hamiltonian actions

In this section we define moment maps, which will be crucial to the section on Morse theory. We also use moment maps (and the related theory of symplectic reduction) to introduce lots of new examples of symplectic manifolds, including toric varieties, polygon spaces, and quiver varieties.

3.1 Hamiltonian vector fields

Let (X, ω) be a symplectic manifold. We say that a vector field v on X is **symplectic** if $L_v\omega = 0$; that is, if ω is preserved by flowing along v. We have already seen a nice set of examples of symplectic vector fields. Recall that, for $f \in C^\infty(X)$, the **symplectic gradient** of f is defined to be the unique vector field v_f such that $i_{v_f}\omega = df$ (Definition 2.2). If $v = v_f$ for some f, we say that v is **Hamiltonian** and that f is a **Hamiltonian function** for v. Proposition 2.3 says that all Hamiltonian vector fields are symplectic.

It is then natural to wonder whether or not all symplectic vector fields are Hamiltonian. By Cartan's magic formula, a vector field v is symplectic if and only if the 1-form $i_v\omega$ is closed. On the other hand, v is Hamiltonian if and only if $i_v\omega$ is exact. Thus a symplectic vector field has a cohomological obstruction to being Hamiltonian, namely the class $[i_v\omega] \in H^1(X; \mathbb{R})$.

Example 3.1. Let $X = T^2 \cong S^1 \times S^1$, and let ω be a translation invariant volume form. Let v be the vector field corresponding to rotation in the first S^1; then v is clearly symplectic. However, $[i_v\omega]$ evaluates to ± 1 on the homology class $[\{1\} \times S^1] \in H_1(X; \mathbb{R})$, so v is not Hamiltonian.

Remark 3.2. If you feel a little bit shaky with this dose of de Rham cohomology, don't worry about it. The only thing that you need to retain from this section is the definition of what it means for f to be a Hamiltonian function for v, and the fact that a given vector field, even if it is symplectic, may or may not admit a Hamiltonian function.

3.2 Moment maps

Let (X, ω) be a symplectic manifold, and let G be a Lie group acting on X and preserving ω. (That is, if ϕ_g is the automorphism of X corresponding to $g \in G$, then $\phi_g^*\omega = \omega$.) For every $a \in \mathfrak{g}$, we get a symplectic vector field $\hat{a}$ on X.

Definition 3.3. A **moment map** for this action is a smooth function $\mu : X \to \mathfrak{g}^*$ such that

- μ is G-equivariant (for all $p \in X$ and $g \in G$, $\mu(g \cdot p) = g \cdot \mu(p)$, where G acts on $\mathfrak{g}^*$ coadjointly)
- for all $a \in \mathfrak{g}$, μ_a is a Hamiltonian function for $\hat{a}$, where $\mu_a(p) := \langle \mu(p), a \rangle$.

If a moment map exists, we say that the action is **Hamiltonian**.

Let's go over the most important examples.

Example 3.4. Let $X = \mathbb{C}$, let $\omega = \frac{i}{2} dz \wedge d\bar{z}$ be the standard symplectic form on $\mathbb{C}$, and let $G = U(1)$ act by complex scalar multiplication. We have

$$\mathfrak{g} = \{\text{skew-Hermitian } 1 \times 1 \text{ matrices}\} = i\mathbb{R}$$

and

$$\mathfrak{g}^* = \{\text{Hermitian } 1 \times 1 \text{ matrices}\} = \mathbb{R},$$

with the pairing between them being $\langle \xi, a \rangle = -i\xi a$ for $\xi \in \mathbb{R}$ and $a \in i\mathbb{R}$. This may seem a little bit stilted, but it's actually very convenient to be able to think of $\mathfrak{g}$ and $\mathfrak{g}^*$ as different vector spaces, rather than calling them both $\mathbb{R}$. Also, the statement that $\mathfrak{g} = i\mathbb{R}$ should conform to your geometric intuition about the tangent space at 1 to the unit complex numbers.

Proposition 3.5. *This action is Hamiltonian with moment map* $\mu(z) = -\frac{1}{2}|z|^2$.

Proof. Equivariance (or invariance, since the coadjoint action is trivial) is clear. To check the second condition, we start by computing $\hat{a}_z \in T_z\mathbb{C} = \mathbb{C}$ for $a \in i\mathbb{R}$ and $z \in \mathbb{C}$. By definition, $\hat{a}_z$ is the image of a under the derivative at 1 of the map $U(1) \to \mathbb{C}$ taking t to tz, thus $\hat{a}_z = az$.

Next, for any $z \in \mathbb{C}$ and $u, v \in T_z\mathbb{C} = \mathbb{C}$, we have $\omega_z(u, v) = \frac{i}{2}(u\bar{v} - \bar{u}v)$. Thus for any $v \in T_z\mathbb{C} = \mathbb{C}$, we have

$$(i_{\hat{a}}\omega)_z(v) = \omega_z(az, v) = \frac{i}{2}\big(az\bar{v} - \bar{a}\bar{z}v\big) = \frac{ia}{2}\big(z\bar{v} + \bar{z}v\big).$$

Here we have used the fact that $a \in i\mathbb{R}$, so $\bar{a} = -a$.

On the other hand, we have $\mu_a(z) = -i\mu(z)a = \frac{ia}{2}|z|^2 = \frac{ia}{2}z\bar{z}$, so $d\mu_a = \frac{ia}{2}(\bar{z}dz + zd\bar{z})$. Evaluating at v, we have

$$(d\mu_a)_z(v) = \frac{ia}{2}\big(\bar{z}v + z\bar{v}\big).$$

This completes the proof. □

Remark 3.6. Note that the moment map in Proposition 3.5 is not unique. The condition that characterizes a moment map depends only on equivariance and the derivative, hence $-\frac{1}{2}|z|^2 + c$ is also a moment map for any $c \in \mathbb{R}$.

Example 3.7. Let $U(1)$ act on S^2 by rotation, where S^2 has the symplectic form that we defined in Example 2.4. That is, $\omega_p(q, r) := p \cdot (q \times r)$. If we write

$$S^2 = \{(z, r) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + r^2 = 1\},$$

then the action takes the form $t \cdot (z, r) := (tz, r)$.

Proposition 3.8. *This action is Hamiltonian with moment map* $\mu(z, r) = r$.

Proof. As in the previous example, equivariance (invariance) is clear. For any point $(z, r) \in S^2$ and any element $a \in i\mathbb{R}$, we have

$$\hat{a}_{(z,r)} = (az, 0) \in T_{(z,r)}S^2 \subset \mathbb{C} \oplus \mathbb{R}.$$

Taking $a = i$ and switching back to real coordinates, we have

$$\hat{i}_p = (-p_y \; p_x \; 0) \, .$$

Also in real coordinates, we have $\mu_i(p) = -i\mu(p)i = \mu(p) = p_z$. We showed in Example 2.4 that the symplectic gradient of the height function was $(-p_y \; p_x \; 0)$ so the moment map equation checks with $a = i$. All other values of a follow by $\mathbb{R}$-linearity. □

Remark 3.9. The proof of Proposition 3.8 illustrates an important point. By definition, a moment map is a choice of a $U(1)$-invariant Hamiltonian function for $\hat{a}$ for every $a \in \mathfrak{u}(1)$, with the extra condition that these functions depend linearly on a. This means that it really just involves choosing a $U(1)$-invariant Hamiltonian function for a single value of a, and then extending by linearity.

More generally, a moment map for a G-action can be specified by choosing a basis for $\mathfrak{g}$ and picking a Hamiltonian function for the vector field induced by each basis element. (However, when G is nonabelian, the equivariance condition becomes much harder to state in terms of a basis.)

Remark 3.10. Archimedes observed that the lateral projection from the cylinder $S^1 \times (-1, 1)$ to S^2 is area-preserving (think of what this means for Greenland). More precisely, the pullback of ω along this projection is equal to $d\theta \wedge dr$, where $d\theta$ is the (inexact, and therefore poorly named) rotation-invariant 1-form that takes the value 1 on the tangent vector $(0\;1\;0) \in T_{(1,0,r)}\big(S^1 \times (-1, 1)\big)$. This makes it clear why r is a moment map for the rotation action.

Exercise 3.11. *Prove Archimedes' theorem.*

Example 3.12. Consider the action of $U(1)$ on T^2 by rotating the first factor. This action is symplectic (if we choose a rotation invariant volume form on T^2), but we saw in Example 3.1 that the vector field $\hat{i}$ is not Hamiltonian, so there is no moment map.

Example 3.13. Consider the canonical action of $U(n)$ on $\mathbb{C}^n$, where $\mathbb{C}^n$ has its standard symplectic form $\omega = \frac{i}{2}\big(dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n\big)$. The following Proposition generalizes Proposition 3.5.

Proposition 3.14. *This action is Hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)^*$ taking the column vector z to the $n \times n$ Hermitian matrix $-\frac{1}{2}zz^*$.*

Proof. First, we should convince ourselves that zz^* is Hermitian; indeed, $(zz^*)^* = (z^*)^*z^* = zz^*$. Next, let's check equivariance: for all $M \in U(n)$,

$$\mu(Mz) = -\frac{1}{2}(Mz)(Mz)^* = -\frac{1}{2}Mzz^*M^* = -\frac{1}{2}Mzz^*M^{-1} = M \cdot \mu(z).$$

Now we move on to the moment map condition. First, for all $A \in \mathfrak{u}(n)$ and $z \in \mathbb{C}^n$, we have $\mu_A(z) = \frac{1}{2}\operatorname{tr}(i\mu(z)A) = \frac{1}{2}\operatorname{tr}(izz^*A)$, so

$$(d\mu_A)_z(v) = \frac{1}{2}\operatorname{tr}(ivz^*A) + \frac{1}{2}\operatorname{tr}(izv^*A)$$

by the product rule. For all $u, v \in T_z\mathbb{C}^n = \mathbb{C}^n$, $\omega_z(u,v) = \frac{i}{2}(v^*u - u^*v)$. For any $A \in \mathfrak{u}(n)$, it is easy to check that $\hat{A}_z = A \cdot z = Az$ (the first of the two equalities is Exercise 1.35), thus we have

$$\left(i_{\hat{A}}\omega\right)_z(v) = \omega_z(Az, v) = \frac{i}{2}\left(v^*Az - z^*A^*v\right).$$

We now use the facts that $A^* = -A$, that a 1×1 matrix is equal to its own trace, and that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. We get

$$\frac{i}{2}\left(v^*Az - z^*A^*v\right) = \frac{i}{2}\left(v^*Az + z^*Av\right) = \frac{i}{2}\operatorname{tr}\left(v^*Az\right) + \frac{i}{2}\operatorname{tr}\left(z^*Av\right) = \frac{i}{2}\operatorname{tr}\left(zv^*A\right) + \frac{i}{2}\operatorname{tr}\left(vz^*A\right).$$

This is equal to $(d\mu_A)_z(v)$, so we are done. □

Here's an easy one.

Proposition 3.15. *Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. Then the inclusion of $\mathcal{O}$ into $\mathfrak{g}^*$ is a moment map for the action of G.*

Proof. Equivariance is tautological; we only need to check the moment map equation. For any $a, b \in \mathfrak{g}$, and $\xi \in \mathcal{O}$,

$$(d\mu_a)_\xi(\hat{b}_\xi) = \langle \hat{b}_\xi, a\rangle = \langle \operatorname{ad}_b^*(\xi), a\rangle = -\langle \xi, \operatorname{ad}_b(a)\rangle = -\langle \xi, [b,a]\rangle = \langle \xi, [a,b]\rangle = \omega_\xi(\hat{a}_\xi, \hat{b}_\xi).$$

Since every element of $T_\xi\mathcal{O}$ has the form $\hat{b}_\xi$ for some $b \in \mathfrak{g}$, this proves that $d\mu_a = i_{\hat{a}}\omega$. □

Example 3.16. Let Y be a smooth manifold with an action of G. Recall that the cotangent bundle T^*Y is equipped with a canonical 1-form $\alpha \in \Omega^1(T^*Y)$ (Example 2.12) and that $\omega := d\alpha$ is symplectic (Proposition 2.13). You already showed that the induced action of G on T^*Y preserves the symplectic form (Exercise 2.14). The following proposition says that it is in fact Hamiltonian.

Proposition 3.17. *Consider the map $\mu : T^*Y \to \mathfrak{g}^*$ given by the formula $\mu_p(a) := -\alpha_p(\hat{a}_p)$ for all $p \in T^*Y$ and $a \in \mathfrak{g}$. This is a moment map for the induced action of G on T^*Y.*

Proof. We leave equivariance as an exercise. For the moment map condition, we fix an element $a \in \mathfrak{g}$, and we have

$$d\mu_a = -di_{\hat{a}}\alpha = -L_{\hat{a}}\alpha + i_{\hat{a}}d\alpha = i_{\hat{a}}d\omega,$$

where the vanishing of $L_{\hat{a}}\alpha$ follows from the fact that the action of G on T^*Y preserves α. □

Exercise 3.18. *Show that the map defined in Proposition 3.17 is equivariant.*

Let's now develop a few simple techniques to generate new Hamiltonian actions out of old ones. The first two are completely trivial (but useful!), so I won't dignify them with proofs.

Proposition 3.19. *Let G act on X and let $Z \subset X$ be a G-equivariant symplectic submanifold. Then the restriction of a moment map for the action on X is a moment map for the action on Z.*

Proposition 3.20. *Let G_X act on X with moment map μ_X and G_Y act on Y with moment map μ_Y. Then $G_X \times G_Y$ acts on $X \times Y$ with moment map $\mu_X \times \mu_Y$.*

Proposition 3.21. *Suppose that G acts on X with moment map $\mu_G : X \to \mathfrak{g}^*$ and that $j : H \to G$ is a homomorphism of Lie groups. The action of H on X is Hamiltonian with moment map $\mu_H := j^* \circ \mu_G : X \to \mathfrak{h}^*$.*

Proof. For any $h \in H$ and $p \in X$,

$$\mu_H(h \cdot p) = j^* \circ \mu_G\big(j(h) \cdot p\big) = j^*\big(\operatorname{Ad}^*_{j(h)} \mu_G(p)\big) = \operatorname{Ad}^*_h \circ j^* \circ \mu_G(p) = \operatorname{Ad}^*_h \circ \mu_H(p).$$

For any $a \in \mathfrak{h}$, we have

$$(\mu_H)_a = \langle j^* \circ \mu_G, a \rangle = \langle \mu_G, j(a) \rangle = (\mu_G)_{j(a)},$$

so $(\mu_H)_a$ is a Hamiltonian function for the vector field $\widehat{j(a)}$. But $\widehat{j(a)} = \hat{a}$, so we are done. □

Corollary 3.22. *Suppose that G acts on X with moment map μ_X and on Y with moment map μ_Y. Then the diagonal action of G on $X \times Y$ is Hamiltonian with moment map $\mu(p,q) = \mu_X(p) + \mu_Y(q)$.*

Proof. This follows from Proposition 3.20 and Proposition 3.21 applied to the diagonal embedding $G \hookrightarrow G \times G$. □

Exercise 3.23. *We can find a moment map for the action of T^n on $\mathbb{C}^n$ in two different ways: via Propositions 3.5 and 3.20, or via Example 3.13 and Proposition 3.21. Check that these two procedures give the same moment map.*

Example 3.24. Let $U(1)$ act on $\mathbb{C}^n$ by the formula $t \cdot (z_1, \ldots, z_n) = (t^{e_1} z_1, \ldots, t^{e_n} z_n)$. Then Exercise 3.23 and Proposition 3.21 tell us that this action is Hamiltonian with moment map

$$\mu(z) = -\frac{1}{2}\left(e_1|z_1|^2 + \ldots + e_n|z_n|^2\right).$$

We'll conclude this section with a theorem that will be important to keep in mind when we introduce toric varieties. We have seen that we have a Hamiltonian action of T^n on $\mathbb{C}^n$, and of course the dimension of T^n is half of the dimension of $\mathbb{C}^n$. We've also seen that we have a Hamiltonian action of $U(1)$ on S^2, and again, the dimension of $U(1)$ is half of the dimension of S^2. The theorem says that this is the best we can do.

Theorem 3.25. *Let (X, ω) be a nonempty, connected, symplectic manifold of dimension $2n$, and suppose that T^k acts effectively*[5] *and Hamiltonianly on X. Then $n \geq k$.*

Proof. The condition that T^k acts effectively can be rephrased as saying that the intersection of all of the stabilizer subgroups of T^k is trivial:

$$\bigcap_{p \in T^k} T^k_p = \{e\}.$$

We will need to use a stronger statement, namely that there exists a point $p \in X$ such that $T^k_p = \{e\}$. For arbitrary Lie groups, this would not be true (for example, think about the action of SO(3) on S^2). However, it is true for a torus; the set of such p is open and dense by [Au, I.2.5].[6]

Fix a point p with $T^k_p = \{e\}$. Let $L := \{\hat{a}_p \mid a \in \mathfrak{t}^k\} \subset T_pX$. Since $T^k_p = \{e\}$,

$$\ker(\mathfrak{t}^k \twoheadrightarrow L) = \mathfrak{t}^k_p = 0,$$

so $\mathfrak{t}^k \cong L$. For any $a, b \in \mathfrak{t}^k$, we have

$$\omega_p(\hat{a}_p, \hat{b}_p) = (i_{\hat{a}}\omega)_p\,(\hat{b}_p) = (d\mu_a)_p\,(\hat{b}_p) = 0$$

by T^k-invariance of μ. Thus ω_p restricts to zero on L. Since ω_p is nondegenerate, we have

$$2n = \dim T_pX = \dim L + \dim L^{\perp}.$$

Since $L \subset L^{\perp}$, $\dim L + \dim L^{\perp} \geq 2k$, so $n \geq k$. □

Remark 3.26. A **toric variety** is a connected symplectic manifold with an effective Hamiltonian torus action that attains this bound, along with the extra requirements that the moment map should be proper and $H^*(X)$ should be finite dimensional.

3.3 Symplectic reduction

Let G act on (X, ω) with moment map $\mu : X \to \mathfrak{g}^*$. Since μ is equivariant and $0 \in \mathfrak{g}^*$ is fixed by the coadjoint action, the action of G on X preserves the level set $\mu^{-1}(0)$. The **symplectic quotient** of X by G is

$$X /\!\!/ G := \mu^{-1}(0)/G.$$

If 0 is a regular value of μ, then $\mu^{-1}(0)$ is a manifold (Proposition 1.37). If moreover G acts freely on $\mu^{-1}(0)$, then the quotient is a manifold. The following results say that these two contingencies

[5] This means that there is no element of T^k that acts trivially on X. The theorem would obviously fail without this assumption.

[6] The point is that X can be nicely partitioned by the conjugacy class of the stabilizer subgroup. In particular, there is always a subgroup $H \subset G$ such that the stabilizer subgroup of every point contains a conjugate of H, and on a dense open set it is equal to a conjugate of H. If G is abelian, that means that H is contained in the stabilizer of every point. For the action of SO(3) on S^2, all points have stabilizer conjugate to SO(2) $\subset$ SO(3), however the actual stabilizer varies enough so that the intersection of all of the stabilizers is trivial.

are related.

Lemma 3.27. *For all $p \in X$, $\mathfrak{g}_p$ (a subspace of $\mathfrak{g}$) is naturally dual to* $\operatorname{coker} d\mu_p$ *(a quotient of $\mathfrak{g}^*$).*

Proof. We need to show that the image of $d\mu_p$ is exactly the perp space to $\mathfrak{g}_p$. For all $a \in \mathfrak{g}$, we have

$$\begin{aligned} a \perp \operatorname{im}(d\mu_p) &\Leftrightarrow \langle d\mu_p(v), a\rangle = 0 \text{ for all } v \in T_pX \\ &\Leftrightarrow (d\mu_a)_p(v) = 0 \text{ for all } v \in T_pX \\ &\Leftrightarrow \omega_p(\hat{a}_p, v) = 0 \text{ for all } v \in T_pX \\ &\Leftrightarrow \hat{a}_p = 0 \\ &\Leftrightarrow a \in \mathfrak{g}_p, \end{aligned}$$

where the penultimate equivalence comes from nondegeneracy of ω_p. □

Corollary 3.28. *The point $0 \in \mathfrak{g}^*$ is a regular value for μ if and only if the action of G on $\mu^{-1}(0)$ is locally free. In particular, if the action on $\mu^{-1}(0)$ is free, then $X/\!\!/G$ is a manifold of dimension* $\dim X - 2\dim G$.

Proof. By Lemma 3.27, $d\mu_p$ is surjective for all $p \in \mu^{-1}(0)$ if and only if $\mathfrak{g}_p = 0$ for all $p \in \mu^{-1}(0)$. □

Example 3.29. Consider the diagonal action of $U(1)$ on $\mathbb{C}^n$ with moment map

$$\mu(z) = -\frac{1}{2}\left(|z_1|^2 + \ldots + |z_n|^2\right) + c.$$

Then $\mu^{-1}(0)$ is the sphere of radius $\sqrt{2c}$. If $c > 0$, then $U(1)$ acts freely, and the quotient is $\mathbb{CP}^{n-1}$, which is a manifold of dimension $2n-2$. If $c < 0$, then $\mu^{-1}(0)$ is empty; in particular, $U(1)$ acts freely, and the quotient is a manifold of dimension $2n-2$ (the empty set). If $c = 0$, then everything goes wrong.

Example 3.30. Recall from Exercise 1.28 that the coadjoint action of $SO(3)$ is isomorphic to the standard representation $\mathbb{R}^3$; in particular, the coadjoint orbits are the spheres of various radii. (Even if you didn't do this exercise, it's easy to convince yourself that the statement is true. How many ways can you think of to fill up a 3-dimensional vector space with surfaces that admit transitive $SO(3)$-actions?) Let S^2_r be the sphere of radius r. Fix positive numbers $r_1, \ldots, r_n$, and consider the diagonal action of $SO(3)$ on $S^2_{r_1} \times \ldots \times S^2_{r_n}$. By Proposition 3.15 and Corollary 3.22, this action is Hamiltonian with moment map $\mu(v_1, \ldots, v_n) = v_1 + \ldots + v_n \in \mathbb{R}^3 \cong \mathfrak{so}(3)^*$. Let

$$\operatorname{Pol}(r_1, \ldots, r_n) := S^2_{r_1} \times \ldots \times S^2_{r_n}/\!\!/SO(3) = \big\{(v_1, \ldots, v_n) \mid |v_i| = r_i \text{ and } v_1 + \ldots + v_n = 0\big\}.$$

We call this a **polygon space** because it parameterizes polygons in $\mathbb{R}^3$ with edge lengths $r_1, \ldots, r_n$, up to translation and rotation.

Proposition 3.31. *Suppose that there does not exist any subset $S \subset \{1, \ldots, n\}$ with the property that $\sum_{i \in S} r_i = \sum_{i \notin S} r_i$. Then $\mathrm{Pol}(r_1, \ldots, r_n)$ is a compact manifold of dimension $2(n-3)$.*

Proof. The only way in which $SO(3)$ can fail to act freely on $\mu^{-1}(0)$ is if there exists a colinear n-tuple $(v_1, \ldots, v_n)$ with $|v_i| = r_i$ for all i and $v_1 + \ldots + v_n = 0$. □

Our next goal is to show that, assuming that G acts freely on $\mu^{-1}(0)$, the manifold $X/\!\!/G$ inherits a natural symplectic form from X. Before formally stating the theorem, let's think about how this is going to work. Let $\pi : \mu^{-1}(0) \to X/\!\!/G$ be the quotient map. Then for all $p \in \mu^{-1}(0)$, $T_{\pi(p)}(X/\!\!/G)$ is naturally a subquotient of T_pX. If we can show that the sub is exactly the perp space (via ω_p) to the stuff that we are dividing by, then ω_p will descend to a nondegenerate bilinear form on the subquotient.

Consider the diagram

$$X \overset{i}{\hookleftarrow} \mu^{-1}(0) \overset{\pi}{\twoheadrightarrow} X/\!\!/G.$$

Theorem 3.32. *Suppose that G acts freely on $\mu^{-1}(0)$, so that $X/\!\!/G$ is smooth. There exists a unique 2-form $\omega_{\mathrm{red}} \in \Omega^2(X/\!\!/G)$ such that $\pi^*\omega_{\mathrm{red}} = i^*\omega$. Furthermore, ω_{red} is symplectic.*

Proof. First note that π is a submersion (Proposition 1.37), which means that the map on tangent spaces at every point is surjective. This means that the pullback map on forms (of any degree) is injective. In particular, if ω_{red} exists, then it is unique. Furthermore, we would have

$$\pi^* d\omega_{\mathrm{red}} = d\pi^*\omega_{\mathrm{red}} = di^*\omega = i^*d\omega = i^*0 = 0,$$

which would imply that $d\omega_{\mathrm{red}} = 0$. Thus we only need to show that ω_{red} exists and is nondegenerate.

For all $p \in \mu^{-1}(0)$, we have

$$T_{\pi(p)}(X/\!\!/G) \cong T_p\mu^{-1}(0)/\mathfrak{g},$$

where the inclusion of $\mathfrak{g} \hookrightarrow T_p\mu^{-1}(0) \subset T_pX$ is given by the map $a \mapsto \hat{a}_p$ (Proposition 1.37). We need to show the subspace $\mathfrak{g} \subset T_pX$ is the perp space to $T_p\mu^{-1}(0)$. We know that

$$\dim \mathfrak{g} + \dim T_p\mu^{-1}(0) = \dim \mathfrak{g}^* + \dim \ker d\mu_p = \dim T_pX,$$

so it is enough to show that every element of $\mathfrak{g}$ is perpendicular to every element of $\ker d\mu_p$. Choose any $a \in \mathfrak{g}$ and $v \in \ker d\mu_p$. Then

$$\omega_p(\hat{a}_p, v) = (i_{\hat{a}}\omega)_p(v) = (d\mu_a)_p(v) = \langle d\mu_p(v), a\rangle = \langle 0, a\rangle = 0.$$

This proves existence and nondegeneracy of ω_{red}. □

Remark 3.33. This is probably the simplest way to think about a symplectic form on $\mathbb{CP}^{n-1}$. Note that we get one for every $c > 0$.

We end this section with a proposition that will be very important for the construction of toric varieties in the next section. Suppose that G acts on X with moment map $\mu : X \to \mathfrak{g}^*$. Let $H \subset G$

be a $i : H \hookrightarrow G$ be a normal subgroup, so that $i^* \circ \mu : X \to \mathfrak{h}^*$ is a moment map for H. Consider the reduction $X /\!\!/ H = \mu^{-1}(\ker i^*)/H$; assume that H acts freely on $\ker i^*$, so that $X /\!\!/ H$ is smooth and symplectic. The action of G on X descends to an action of G/H on $X /\!\!/ H$, and the moment map μ descends to a map

$$\bar{\mu} : X /\!\!/ H \to \ker i^* \cong (\mathfrak{g}/\mathfrak{h})^*.$$

Proposition 3.34. *The map $\bar{\mu}$ is a moment map for the action of G/H on $X /\!\!/ H$.*

Exercise 3.35. *Prove Proposition 3.34.*

3.4 Toric varieties

Toric varieties can be introduced in two different ways: abstractly or constructively. Let's start with some definitions.

Definition 3.36. A **rational polyhedron** is a subset of $\mathbb{R}^d$ defined by finitely many linear inequalities of the form

$$a \cdot \xi \leq r,$$

where $a \in \mathbb{Z}^d$ and $r \in \mathbb{R}$. (The fact that I am using the letter ξ should give you the hint that I will eventually want $\mathbb{R}^n$ to be the dual of a Lie algebra.) A rational polyhedron $P \subset \mathbb{R}^d$ is called **Delzant** if $\dim P = d$ and P "looks locally like $\mathbb{R}^d_{\geq 0}$". That is, for all $\xi \in P$, there should exist a point $x \in \mathbb{R}^d$ and a matrix $M \in GL(n; \mathbb{Z})$ that takes a neighborhood of 0 in $P - \xi$ to a neighborhood of 0 in $\mathbb{R}^d_{\geq 0} - x$. It is called a **polytope** if it is bounded.

Example 3.37. All rational polyhedra in $\mathbb{R}$ have the form $[a, b]$, where a is allowed to be $-\infty$ and b is allowed to be ∞. They are all Delzant unless $a = b$. Draw some pictures to indicate what can go wrong in $\mathbb{R}^2$ or $\mathbb{R}^3$.

- **Toric varieties, abstract approach:** A toric variety of dimension $2d$ is defined to be a connected symplectic $2d$-manifold with an effective Hamiltonian action of T^d such that the moment map is proper and $H^*(X)$ is finite dimensional. (Note that we already saw this definition in Remark 3.26.)

- **Toric varieties, constructive approach:** To any Delzant polyhedron $P \subset \mathbb{R}^d$ we will associate a symplectic $2d$-manifold $X(P)$. This manifold will come equipped with an effective Hamiltonian action of T^d. The moment map $X(P) \to (\mathfrak{t}^d)^* \cong \mathbb{R}^d$ will be proper, and the image will be P.

It is a theorem that these two approaches are the same. More precisely, we have the following statement.

Theorem 3.38. *Let X be a connected symplectic $2d$-manifold with an effective Hamiltonian action of T^d such that the moment map is proper and $H^*(X)$ is finite dimensional. Let $P \subset \mathbb{R}^d$ be the image of the moment map. Then P is a Delzant polytope, and X is T^d-equivariantly symplectomorphic over P to $X(P)$.*

This thoerem is proven for compact X by Delzant [De, 2.1]. I have no idea where to find a proof for arbitrary X, nor have I even seen it stated before for arbitrary X, but I know in my heart that it is true. Furthermore, I think it would not be so hard to generalize Delzant's proof, but it would be a little bit messy because it would involve generalizing the Atiyah-Guillemin-Sternberg theorem to the non-compact setting. We will (mostly) prove Theorem 3.38 for compact X in the next section.[7]

Remark 3.39. By assuming that X is compact, we (and Delzant) don't have to worry about properness of the moment map or finite dimensionality of $H^*(X)$. However, let me explain why those two properties have to be part of the statement of the theorem. Without properness, the image of the moment map can be the shape of a (life sized, three dimensional) elephant. Without finite dimensionality of $H^*(X)$, it can be the convex hull of the integer points on a parabola.

This section will be devoted to the construction and basic properties of $X(P)$. We will assume for now that P has at least one vertex; for example, we won't allow P to be the upper half-plane in $\mathbb{R}^2$. At the end we will return to the general case and do away with this assumption. (Note that, if you only care about compact toric varieties, then you need P to be a polytope, in which case this assumption is superfluous.)

Let $F_1, \ldots, F_n$ be the facets (codimension 1 faces) of a rational polyhedron P (we won't assume that P is Delzant until Proposition 3.43), and let $a_1, \ldots, a_n \in \mathbb{Z}^d$ be the primitive outward normal vectors. This means that a_i is not an integer multiple of any other element of $\mathbb{Z}^d$, and there exist real numbers $r_1, \ldots, r_n$ such that

$$P = \{\xi \in \mathbb{R}^d \mid a_i \cdot \xi \leq r_i \text{ for all } i\}.$$

Furthermore, this is the minimal number of inequalities needed to define P.

Note that each vector a_i is a d-tuple of integers; these form the columns of an $n \times d$ integer matrix A. This matrix A defines a Lie group homomorphism $\pi : T^n \to T^d$ in the only way you can possibly imagine; rather than writing the general formula, let's just do an example.

Example 3.40. If P is a triangle in $\mathbb{R}^2$, we get a homomorphism $T^3 \to T^2$.

Let $K := \ker(\pi) \subset T^n$. We now have a lot of groups floating around, and it will be helpful to write down some short exact sequences that record how they relate to each other. We have

$$1 \longrightarrow K \stackrel{i}{\longrightarrow} T^n \stackrel{A}{\longrightarrow} T^d \longrightarrow 1.$$

Differentiating at the identity, we get

$$0 \longrightarrow \mathfrak{k} \stackrel{i}{\longrightarrow} \mathfrak{t}^n \stackrel{A}{\longrightarrow} \mathfrak{t}^d \longrightarrow 0,$$

[7] We will show that X is equivariantly homeomorphic over P to $X(P)$, and that the homeomorphism is a symplectomorphism over the interior of P. All that we will be missing is the statement that the homeomorphism is smooth over the boundary of P.

where the second map is again given by the matrix A. Dualizing, we have

$$0 \longleftarrow \mathfrak{k}^* \overset{i^*}{\longleftarrow} (\mathfrak{t}^n)^* \overset{A^t}{\longleftarrow} (\mathfrak{t}^d)^* \longleftarrow 0.$$

If we identify $(\mathfrak{t}^n)^*$ with $\mathbb{R}^n$ and $(\mathfrak{t}^d)^*$ with $\mathbb{R}^d$ via the dot product, then the second map is just given by the transpose of A.

Our plan is to take a symplectic quotient of $\mathbb{C}^n$ by K, where the action of K on $\mathbb{C}^n$ is induced by the standard action of T^n. However, we need to specify a moment map for this action, which requires a choice of n real numbers. This is good, because we have n real numbers that we haven't used yet! Consider the smooth function $\tilde{\mu} : \mathbb{C}^n \to \mathbb{R}^n$ given by the formula

$$\tilde{\mu}(z_1, \ldots, z_n) = \left(-\frac{1}{2}|z_1| + r_1, \ldots, -\frac{1}{2}|z_n| + r_n\right) \in \mathbb{R}^n \cong (\mathfrak{t}^n)^*.$$

By Remark 3.6 and Corollary 3.20, $\tilde{\mu}$ is a moment map for the action of T^n. By Proposition 3.21, $i^* \circ \tilde{\mu}$ is a moment map for the action of K.

Definition 3.41. $X(P) := \mathbb{C}^n /\!\!/ K$.

Example 3.42. $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{C}$, weighted $\mathbb{CP}^2$.

Note that $X(P)$ inherits an action of $T^n/K \cong T^d$ and a map

$$\mu : X(P) = \tilde{\mu}^{-1}(\ker i^*)/K \to \ker i^*.$$

Furthermore, we may identify $\ker i^*$ with $\mathbb{R}^d$ via the isomorphism $A^t : \mathbb{R}^d \to \ker i^*$.

Theorem 3.43. *If P is Delzant, then $X(P)$ is smooth, and therefore symplectic. The map $\mu : X(P) \to \mathbb{R}^d$ is a proper moment map with image P.*

Proof. Properness of μ follows immediately from properness of $\tilde{\mu}$ (whether or not $X(P)$ is smooth). Furthermore, we have

$$\begin{aligned} \operatorname{im}\mu &= \operatorname{im}\tilde{\mu} \,\cap\, \ker i^* \\ &= \{\xi \in \mathbb{R}^d \mid A^t\xi \in \operatorname{im}\tilde{\mu}\} \\ &= \{\xi \in \mathbb{R}^d \mid (A^t\xi)_i \leq r_i \;\forall\, i\} \\ &= \{\xi \in \mathbb{R}^d \mid a_i \cdot \xi \leq r_i \;\forall\, i\} \\ &= P. \end{aligned}$$

It's easy to see from this argument that, for $(z_1, \ldots, z_n) \in \tilde{\mu}^{-1}(\ker i^*)$, we have $\tilde{\mu}(z_1, \ldots, z_n) \in F_i$ if and only if $z_i = 0$.

To prove smoothness, we need to show that K acts freely on $\mu^{-1}(0)$. Let $(t_1, \ldots, t_n) \in K \subset T^n$ be an element of K that fixes an element $(z_1, \ldots, z_n)$ of $\mu^{-1}(0)$; we will show that we must have

$t_i = 1$ for all i. What is clear is that we must have either $t_i = 1$ or $z_i = 0$ for all i. Let

$$S := \{i \mid z_i = 0\} \subset \{1, \ldots, n\};$$

then we must have $t_i = 1$ for all $i \notin S$. We know that $\tilde{\mu}(z_1, \ldots, z_n) \in F_i$ for all $i \in S$; in particular, $\bigcap_{i \in S} F_i$ cannot be empty. Since P is Delzant, $\{a_i \mid i \in S\}$ is contained in a basis for $\mathbb{Z}^d$. This means that the restriction of the map $A : T^n \to T^d$ to the coordinate subtorus $T^S \subset T^n$ is injective. Since $(t_1, \ldots, t_n) \in T^S \cap \ker A$, we must have $t_i = 1$ for all i.

The fact that μ is a moment map follows immediately from Proposition 3.34. □

Let's expand some of the ideas that we used in the proof of Theorem 3.43 in order to understand better the fibers of the map from $X(P)$ to P. For each i, let $T_i \subset T^d$ be the image of the i^{th} coordinate circle in T^n. That is, it is the subtorus defined by the vector a_i.

Definition 3.44. For any $\xi \in P$, let $T_\xi \subset T^d$ be the subtorus generated by T_i for all i such that $\xi \in F_i$.

Example 3.45. Write down the five subtori for $\mathbb{CP}^2$.

Proposition 3.46. *For any $\xi \in P$, T^d acts transitively on $\mu^{-1}(\xi)$. For any $p \in \mu^{-1}(\xi)$, $T^d_p = T_\xi$.*

Proof. The statement that T^d acts transitively on the fibers of μ follows from the statement that T^n acts transitively on the fibers of $\tilde{\mu}$, which is clear. If $p \in X(P)$ is the image of a point $(z_1 \ldots, z_n) \in \tilde{\mu}^{-1}(\ker i^*)$, then the stabilizer in T^d of p is the image of the stabilizer in T^n of $(z_1 \ldots, z_n)$. This is the coordinate torus T^S, where $S = \{i \mid z_i = 0\} = \{i \mid \mu(p) \in F_i\}$, and its image in T^d is $T_{\mu(p)}$. □

Corollary 3.47. *For any rational polyhedron P (Delzant or not), $X(P)$ is homeomorphic to $Y(P) := \big(P \times T^d\big)/\sim$, where $(\xi, t) \sim (\xi', t')$ if and only if $\xi = \xi'$ and t and t' differ by an element of T_ξ.*

Proof. First we will construct a continuous bijection from $f : Y(P) \to X(P)$. To do this, it is sufficient to find a section $\sigma : P \to X(P)$ of μ, and then let $f[\xi, t] = t \cdot \sigma(\xi)$. That section can be obtained by asking each z_i to be a positive real number. More explicitly, we have $A^t\xi = (x_1, \ldots, x_n) \in \mathbb{R}^n$, with $x_i \leq r_i$ for all i. That means that the point

$$\left(\sqrt{2(r_1 - x_1)}, \ldots, \sqrt{2(r_n - x_n)}\right) \in \mathbb{C}^n$$

is sent to $A^t\xi$ by $\tilde{\mu}$. Let $\sigma(\xi)$ be the image of this point in $X(P)$.

It is clear that f is a continuous bijection. If P is a polytope, then $Y(P)$ is compact, so f must be a homeomorphism. More generally, we need to show that the inverse is continuous. This is easy and unenlightening; see [Pr, §2.1]. □

Example 3.48. $\mathbb{CP}^1$, $\mathbb{C}$, $\mathbb{C}^2$, $\widetilde{\mathbb{C}}^2$. More generally, chopping a corner induces a symplectic blow up.

Remark 3.49. Corollary 3.47 implies that $X(P)$ is connected, that the action of T^d on $X(P)$ is effective, and that the moment map $\mu : X(P) \to \mathbb{R}^d$ is proper. Combining this with Theorem 3.43, we see that it satisfies the definition of an abstract toric variety.

Remark 3.50. In the statement of Corollary 3.47, I wrote "homeomorphic" rather than "diffeomorphic" or "symplectomorphic" because it is not so easy to directly define a smooth structure on $Y(P)$. However, $Y(P)$ has a dense open set

$$\mathring{Y}(P) := \mathring{P} \times T^d \subset (\mathfrak{t}^d)^* \times T^d \cong T^*T^d,$$

which has a natural symplectic form (Proposition 2.13). It is not too hard to show that the restriction to $\mathring{Y}(P)$ of our homeomorphism is a symplectomorphism onto its image (Exercise below).

Exercise 3.51. *Show that the restriction to $\mathring{Y}(P)$ of the homeomorphism $f : Y(P) \to X(P)$ is a symplectomorphism onto its image. Hint: Show that the symplectic form that we defined on $\mathring{Y}(P)$ is the unique symplectic form for which the projection to $\mathring{P}$ is a moment map for the T^d-action.*

Finally, we need to explain the construction of $X(P)$ when P has no vertices. Let's start with the most basic example of such a polyhedron, namely $\mathbb{R} \subset \mathbb{R}$. We want Corollary 3.47 to hold, so $X(\mathbb{R})$ had better be homeomorphic $\mathbb{R} \times T^1$. Furthermore, we want the projection to $\mathbb{R}$ to be a moment map for the rotation action by T^1; this means that we should endow $X(\mathbb{R})$ with the symplectic form $d\theta \wedge d\xi$, where θ is the coordinate on T^1 and ξ is the coordinate on $\mathbb{R}$.

More generally, let $P \subset \mathbb{R}^d$ be a rational polyhedron. Changing coordinates, we may write

$$P = Q \times \mathbb{R}^m \subset \mathbb{R}^{d-m} \times \mathbb{R}^m = \mathbb{R}^d,$$

where $Q \subset \mathbb{R}^{d-m}$ is a rational polyhedron with at least one vertex. Then we define

$$X(P) := X(Q) \times \mathbb{R}^m \times T^m.$$

Note that if P is Delzant, then so is Q, and therefore $X(P)$ has a natural symplectic form. The projection to P is a moment map for the action of $T^d = T^{d-m} \times T^m$, and Proposition 3.46, Corollary 3.47, and Remark 3.50 all hold for $X(P)$.

The following proposition will be very useful in the next section. Let $\xi \in P$ be a vertex. By Proposition 3.47, $\mu^{-1}(\xi)$ is a single point in $X(P)$, and that point is fixed by T^d. Call that point p, and consider the action of T^d on $T_pX(P)$. Let $\eta_1, \ldots, \eta_d \in \mathbb{Z}^d \cong \mathrm{Hom}(T^d, U(1))$ be the additive inverses of the primitive integer vectors along the d edges emanating from ξ.

Proposition 3.52. *The space $T_pX(P)$ is T^d-equivariantly isomorphic to $\mathbb{C}_{\eta_1} \oplus \ldots \oplus \mathbb{C}_{\eta_d}$, where T^d acts on $\mathbb{C}_{\eta_i}$ via the associated homomorphism to $U(1)$.*

Proof. Let $\rho_1, \ldots, \rho_d \in \mathbb{Z}^d$ be the weights for the T^d-action on $T_pX(P)$, so that

$$T_pX(P) \cong \mathbb{C}_{\rho_1} \oplus \ldots \oplus \mathbb{C}_{\rho_d}.$$

We want to show that $\{\rho_1, \ldots, \rho_d\} = \{\eta_1, \ldots, \eta_d\}$.

By Theorem 2.28 and Remark 2.32, $X(P)$ is locally symplectomorphic near p to $T_pX(P)$. By Proposition 3.46, the image under μ of a neighborhood of p in $X(P)$ is a neighborhood of ξ in P (the point is that the fibers of μ are connected). Thus, if we compute the moment map for the action of T^d on $T_pX(P)$ and apply it to a neighborhood of 0 in $T_pX(P)$, we should get a neighborhood of ξ in P.

By Proposition 3.5, $U(1)$ acts on $\mathbb{C}$ with moment map $z \mapsto -\frac{1}{2}|z|^2 + c$. By Proposition 3.21, T^d acts on $\mathbb{C}_{\rho_i}$ with moment map $z \mapsto -\frac{1}{2}|z|^2\rho_i + \xi_i$, where $\xi_i \in \mathbb{R}^d$. By Corollary 3.22, T^d acts on T_pX with moment map

$$z \mapsto -\frac{1}{2}|z_1|^2\rho_1 + \xi_1 - \ldots - \frac{1}{2}|z_d|^2\rho_d + \xi_d.$$

If we want the moment map to take 0 to $\xi = \mu(p)$, then we must have $\xi_1 + \ldots + \xi_d = \xi$. The image of this moment map is equal to

$$\xi - \mathbb{R}_{\geq 0}\rho_1 - \ldots - \mathbb{R}_{\geq 0}\rho_d.$$

The only way that this can look like P near ξ is if ρ_i is a positive multiple of η_i for all i (up to permutation). Thus it remains only to show that each ρ_i is primitive.

If this were not the case, then $\rho_1, \ldots, \rho_d$ would not span $\mathbb{Z}^d$. This in turn would imply that the exists an element of T^d which acts trivially on $T_pX(P)$, and therefore on a neighborhood of p in $X(P)$. But this cannot happen, since we know that T^d acts freely over $\mathring{P}$. □

Remark 3.53. Since Proposition 3.52 is a purely local statement, one can also prove it by assuming that $P = \mathbb{R}^d_{\geq 0}$. This gets a little bit confusing because the negative edge vectors coincide with the perp vectors to the facets; to clarify this, one just needs to think about the fact that these vectors naturally live in dual vector spaces, and the negative edge vectors are the ones that transform correctly. There is also some subtlety with signs, but it all works out, and it's helpful to think about this confusingly trivial example.

It will also be nice to have a generalization of Proposition 3.52 in which a vertex is replaced by a face of arbitrary dimension. Let F be a face of P, and let $X(P)_F = \mu^{-1}(F) \subset X(P)$. If F is a vertex, then $X(P)_F$ is the single point that appears in Proposition 3.52.

Proposition 3.54. *For any Delzant polyhedron P and any face F, $X(P)_F$ is a T-submanifold of $X(P)$, and it is T-equivariantly isomorphic to $X(F)$.*

Before proving Proposition 3.54, let's try to make sense of the statement. Choose an element $\xi \in \mathring{F}$, and consider the torus $T_\xi \subset T^d$. This subtorus does not depend on the choice of ξ, so we might as well call it T_F. We have $\dim T_F = \operatorname{codim} F$. Furthermore, it's easy to convince yourself that $\mathrm{Lie}(T^d/T_F)^*$, which is a *subspace* of $(\mathfrak{t}^d)^*$ whose dimension is equal to that of F, is equal to the linear span of $F - \xi$. Since F is a Delzant polyhedron inside of this linear span, $X(F)$ is naturally a toric variety for the torus T^d/T_F. In particular, T^d acts on $X(F)$ with kernel T_F, just as T^d acts on $X(P)_F$ with kernel T_F. So the statement of the proposition makes sense!

Exercise 3.55. *Prove Proposition 3.54.*

Example 3.56. Think about all of the toric subvarieties of $\mathbb{CP}^2$, and of $\mathbb{C}^2$ blown up at a point.

Now Proposition 3.52 admits a relatively straightforward generalization. Let $\xi \in F \subset P$ be a vertex contained in a face of P. (Note that ξ is now a vertex of F, not an element of the relative interior of F.) On the level of spaces, this gives us a T-fixed point $p \in X(F) \subset X(P)$. Consider the normal space

$$N_p(X(P), X(F)) := T_pX(P)/T_pX(F).$$

Let $\eta_1, \ldots, \eta_k \in \mathbb{Z}^d \cong \mathrm{Hom}(T^d, U(1))$ be the additive inverses of the primitive integer vectors along the d edges emanating from ξ that *do not* lie in F (here k is the codimension of F in P).

Proposition 3.57. *The space $N_p(X(P), X(F))$ is T^d-equivariantly isomorphic to $\mathbb{C}_{\eta_1} \oplus \ldots \oplus \mathbb{C}_{\eta_k}$.*

Proof. This follows from two applications of Proposition 3.52, one for $X(P)$ and the other for $X(F)$. □

4 Morse theory

In this section we apply Morse theory (or its generalization, Morse-Bott theory) to $U(1)$ moment maps and use it to compute Poincaré polynomials of lots of symplectic manifolds, including arbitrary toric varieties and flag manifolds. We also use these techniques to prove the Atiyah-Guillemin-Sternberg theorem (Theorem 4.32), which characterizes the image of a moment map for an arbitrary torus action, and Delzant's theorem (Theorem 3.38), which asserts that all abstract toric varieties arise via the construction studied in the previous section.

4.1 Morse theory oversimplified

Let X be an n-manifold, and let $f \in C^\infty(X)$ be a smooth function. Recall that a point $p \in X$ is called a **critical point** if $df_p = 0$. The function f is called a **Morse function** if, near any critical point p, we can choose coordinates $x_1, \ldots, x_n$ such that

$$f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2 + f(p)$$

for some k. The number k is called the **Morse index** of p.

Remark 4.1. Now for a few apologies.

- The "official" definition of a Morse function is that, near every critical point, there exists a coordinate chart such that the Hessian (the matrix of second derivatives) is nonsingular. It is an important theorem, called the **Morse Lemma**, that we can always find coordinates for which f has this simple form.

- It is also a theorem that the index k is independent of the choice of chart, but this much easier than the Morse Lemma. Without choosing coordinates, the Hessian may be regarded as a symmetric bilinear form on T_pX, and k is the dimension of the largest subspace of T_pX on which this form is negative definite.

- After the Morse Lemma, the next big theorem in Morse theory is that Morse functions are generic; that is, the set of Morse functions is dense and open in $C^\infty(X)$ (with respect to some natural topology). We won't need this fact.

Example 4.2. Draw a few 2-dimensional examples, including T^2 (everybody's favorite), S^2 with a weird function on it (so that there are more than two critical points), and at least one non-compact surface.

Our main example of Morse functions will be moment maps. Suppose that $U(1)$ acts on a symplectic manifold X with moment map $\mu : X \to \mathbb{R}$, and that the fixed point set $X^{U(1)}$ is discrete. Choose a $U(1)$-equivariant almost complex structure on X that is compatible with the symplectic form. For any fixed point $p \in X^{U(1)}$, T_pX is a complex representation of $U(1)$, thus there exist integers $e_1, \ldots, e_n$ such that

$$T_pX \cong \mathbb{C}_{e_1} \oplus \ldots \oplus \mathbb{C}_{e_n},$$

where $U(1)$ acts on $\mathbb{C}_{e_i}$ by the formula $t \cdot z = t^{e_i} z$. Since the fixed point set is discrete and X looks locally like T_pX near p, we must have $e_i \neq 0$ for all i. We define the **equivariant index** of p to be the number of i such that $e_i > 0$. Our overuse of the word "index" is justified by the following proposition.

Proposition 4.3. *With the above hypotheses, the moment map μ is a Morse function, the critical points are equal to the fixed points, and the Morse index of a fixed point is equal to 2 times the equivariant index.*

Proof. By Example 3.24, $U(1)$ acts on the symplectic vector space T_pX with moment map

$$z \mapsto -\frac{1}{2}\left(e_1|z_1|^2 + \ldots + e_n|z_n|^2\right) + c.$$

By the equivariant Darboux theorem (Theorem 2.28 and Remark 2.32), X is $U(1)$-equivariantly symplectomorphic near p to T_pX, so μ may be written in the above form near p. This means that we have real coordinates $x_1, y_1, \ldots, x_n, y_n$, and

$$\mu(x,y) = -\frac{1}{2}\Big(e_1(x_1^2 + y_1^2) + \ldots + e_n(x_n^2 + y_n^2)\Big) + \mu(p).$$

Scaling x_i and y_i by $|e_i|^{-\frac{1}{2}}$, we obtain the desired result. □

Remark 4.4. Note that we never needed the Morse lemma, because the equivariant Darboux theorem gave us a nice chart.

Remark 4.5. It's interesting to think about where periodicity was used. That is, every smooth function on a compact symplectic manifold is a moment map for an action of $\mathbb{R}$, namely the flow along the symplectic gradient. But we've already seen that it's not the case that every Morse function on a compact symplectic manifold has even indices. What makes $U(1)$ actions different from $\mathbb{R}$ actions? The answer is that we can always choose our almost complex structures to be $U(1)$ equivariant because $U(1)$ is compact, but the same is not true for $\mathbb{R}$.

Given a function $f \in C^\infty(X)$ and a real number $a \in \mathbb{R}$, let $X_a := f^{-1}(-\infty, a]$. The following theorem, which might be called the fundamental theorem of Morse theory, describes the change in topology of X_a as a increases. We will not give a proof; see Remark 4.7 for references.

Theorem 4.6. *Suppose that $f : X \to \mathbb{R}$ is proper and Morse.*

- *If $a < b$ and f has no critical values in $[a, b]$, then X_a is a deformation retract of X_b.*
- *If f has no critical values in $[a, \infty)$, then X_a is a deformation retract of X.*
- *Suppose that $a < b < c$, that $p_1, \ldots, p_r$ are critical points of f with $f(p_i) = b$ for all i, and that f has no other critical values in the interval $[a, c]$. Let m_i be the Morse index of p_i. Then there exist disjoint embeddings $g_i : S^{m_i - 1} \hookrightarrow \partial X_a$ such that X_c is homotopy equivalent to*

$$X_a \bigcup_{g_1, \ldots, g_r} \left(D^{m_1} \sqcup \ldots \sqcup D^{m_r}\right).$$

Remark 4.7. If we added the assumption that there is only one critical point for each critical value (which I do not want to do), then a proof of this theorem could be found in any reference on Morse theory; the standard one is Milnor's book [Mi, 3.1 & 3.2]. The only reference that I found allowing multiple critical points at a single level is a paper of Palais [Pa], where he proves a much stronger theorem. First of all, he gets diffeomorphisms of handlebodies rather than homotopy equivalences. Second, he works in infinite dimensions! Since we will not need either of these improvements and since it would take some effort to state them precisely, I am sticking with the weaker statement.

Remark 4.8. I did not give the definition of g_i as part of the statement of the theorem because I wanted to keep it succinct, but actually it is fairly easy to do. If $\epsilon = b - a$ is sufficiently small, then g_i is the inclusion of

$$\left\{(x_1, \ldots, x_n) \mid x_1^2 + \ldots + x_{m_i}^2 = \epsilon \text{ and } x_j = 0 \text{ for all } j > m_i\right\} \cong S^{m_i - 1}$$

into the coordinate neighborhood of p.

Example 4.9. Revisit the examples from Example 4.2.

When all of the Morse indices are even, as is the case for $U(1)$ moment maps, Theorem 4.6 allows us to give a very simple description of the Poincaré polynomial of X.

Corollary 4.10. *Suppose that $f : X \to \mathbb{R}$ is proper, bounded below, and Morse. Suppose further that there are only finitely many critical points and that all Morse indices $m(p)$ are even. Then*

$$\mathrm{Poin}_X(t) = \sum_{p \in \mathrm{Crit}(f)} t^{m(p)}.$$

Proof. Since f is bounded below, X_a is empty for $a \ll 0$. By the second item in Theorem 4.6, X is homotopy equivalent to X_a for $a \gg 0$. By the first item in Theorem 4.6, the topology of X_a only changes when a crosses over a critical value. Thus we only need to study such changes, which are covered by the third item in Theorem 4.6.

In the notation of Theorem 4.6, we will consider the long exact sequence in cohomology associated to the pair (X_c, X_a):

$$H^{2k-1}(X_a) \to H^{2k}(X_c, X_a) \to H^{2k}(X_c) \to H^{2k}(X_a) \to H^{2k+1}(X_c, X_a) \to H^{2k+1}(X_c) \to H^{2k+1}(X_a).$$

We may assume inductively that $H^{\mathrm{odd}}(X_a) = 0$. By Theorem 4.6, we have $H^*(X_c, X_a) \cong \mathbb{Z}^r$, with degrees given by the Morse indices $m_1, \ldots, m_r$. Since these indices are all even, we have $H^{\mathrm{odd}}(X_c, X_a) = 0$. This implies that $H^{\mathrm{odd}}(X_c) = 0$, and that $H^*(X_c) \cong H^*(X_a) \oplus \mathbb{Z}^r$, which is what we want. □

Theorem 4.6 has another important corollary when all Morse indices are even.

Corollary 4.11. *Suppose that X is connected, $f : X \to \mathbb{R}$ is proper and Morse, and there are no critical points of index 1 or $n-1$.*[8] *Then*

- *all level sets of f are connected*
- *every local minimum/maximum of f is a global minimum/maximum, and there is at most one global minimum/maximum.*

The proof of Corollary 4.11 that I want to give is kind of intuitive and hand-wavy. For a more careful cohomological proof (with the extra assumption that X is compact), see [Au, IV.3.1]. (Audin actually proves the generalized version of this statement that we will encounter two sections later in Corollary 4.28.)

Proof. To prove connectedness of the level sets of f, it is sufficient to consider only regular values. Suppose that $f^{-1}(a)$ is disconnected. The only way to connect it by "going up" is by encountering a critical point of index 1. The only way to connect it by "going down" is by encountering a critical point for $-f$ of index 1, which is the same as a critical point for f of index $n-1$.

Suppose that p is a local minimum; this is the same as a critical point of index 0. Let $b = f(b)$, and choose $a < b < c$ such that b is the only critical value in $[a, b]$. By Theorem 4.6, X_c is homotopic to the disjoint union of X_a and a point (and possibly some more stuff attached to X_a if there are other critical points at level b). The only way for this to happen without introducing disconnected

[8] In particular, this holds when n is even and all indices are even.

level sets is if $X_a = \emptyset$, which means that b is a global minimum of f. The existence of more than one critical point of index 0 at level b would also contradict connectedness of X_c. For maxima, we can apply the same argument to $-f$. □

4.2 Poincaré polynomials of symplectic manifolds

Proposition 4.3 and Corollary 4.10 give us a recipe for computing Poincaré polynomials of symplectic manifolds whenever we have a $U(1)$ action with finitely many fixed points and a moment map that is proper and bounded below. To summarize, we have the following theorem.

Theorem 4.12. *Suppose that $U(1)$ acts on X with finitely many fixed points, and that we have a moment map $\mu : X \to \mathbb{R}$ that is proper and bounded below.*[9] *Then*

$$\mathrm{Poin}_X(t) = \sum_{p \in X^{U(1)}} t^{2e_p},$$

where $e(p)$ is the equivariant index of p.

Example 4.13. Let $P \subset \mathbb{R}^2$ be the standard simplex and let $X = X(P) \cong \mathbb{CP}^2$. Choose an integer vector $a \in \mathbb{Z}^2 \cong \mathrm{Hom}(U(1), T)$ that is not perpendicular to any of the three edges. By Proposition 3.46, this implies that the induced action of $U(1)$ on X does not fix any points that lie over the edges, thus $X^{U(1)} = X^T$, which consists only of three isolated points. By Proposition 3.21, μ_a is a moment map for the action of $U(1)$ on X.

By Proposition 3.52, for all $p \in X^T$, we have $T_pX \cong \mathbb{C}_{\eta_1} \oplus \mathbb{C}_{\eta_2}$, where η_1 and η_2 are the negative edge vectors at p. This gives T_pX as a representation of T. As a representation of $U(1)$, we have $T_pX \cong \mathbb{C}_{e_1} \oplus \mathbb{C}_{e_2}$, where $e_i = a \cdot \eta_i$. Apply Theorem 4.12, and we see that

$$P_X(t) = 1 + t^2 + t^4.$$

The technique of Example 4.13 generalizes immediately to arbitrary toric varieties. Let $P \subset \mathbb{R}^d$ be a Delzant polyhedron, and let $a \in \mathbb{Z}^d$ be a vector that is not perpendicular to any edge, with the additional property that the dot product with a is bounded below on P.

Proposition 4.14. *The action of $U(1)$ on $X(P)$ induced by a has isolated fixed points indexed by the vertices of P. The moment map μ_a is proper and bounded below, and the equivariant index of the fixed point corresponding to a vertex ξ is equal to the number e_ξ of edge vectors at ξ whose dot product with a is negative. Thus we have*

$$\mathrm{Poin}_{X(P)}(t) = \sum_{\xi} t^{2e_\xi},$$

where ξ runs over the vertices of P.

[9]Note that if X is compact, μ is automatically proper and bounded below.

Example 4.15. Compute some more Poincaré polynomials.

Proposition 4.14 allows us to prove the following awesome theorem, which gives a general formula for the Poincaré polynomial of a toric variety. Let $f_k(P)$ be the number of k-dimensional faces of P (so $f_0(P)$ is the number of vertices, and $f_d(P) = 1$).

Theorem 4.16. *Let P be a Delzant polyhedron with at least one vertex. Then*

$$\operatorname{Poin}_{X(P)}(t) = \sum_{k=0}^{d} f_k(P)(1-t^2)^k t^{2(d-k)}.$$

Remark 4.17. If you want to compute the Poincaré polynomial of a particular toric variety, it will be much faster to use Proposition 4.14 directly. The cool thing about Theorem 4.16 is that we don't have to break symmetry by choosing a vector $a \in \mathfrak{t}$. Also see Remark 4.18 for a nice application

Proof. Let $a \in \mathbb{Z}^d$ be a vector that is not perpendicular to any edge, with the additional property that the dot product with a is bounded below on P. By definition of a, every face of P has a unique "lowest" vertex (where we measure height by dot product with a). At the vertex ξ, the edges that point up from ξ span a simplicial cone of dimension $d - e_\xi$. That means that the number of k-faces that obtain their minimum at ξ is equal to $\binom{d-e_\xi}{k}$. We therefore have

$$f_k(P) = \sum_{\xi} \binom{d - e_\xi}{k}.$$

It is surprisingly tricky to complete the proof from here. We will do it by messing around with the expression $t^{2d} \operatorname{Poin}_{X(P)}(t^{-1})$.[10] We have

$$\begin{aligned}
t^{2d} \operatorname{Poin}_{X(P)}(t^{-1}) &= \sum_{\xi} t^{2(d-e_\xi)} \\
&= \sum_{\xi} \big((t^2-1)+1\big)^{2(d-e_\xi)} \\
&= \sum_{\xi} \sum_{k=1}^{d-e_\xi} \binom{d-e_\xi}{k} (t^2-1)^k \\
&= \sum_{k=1}^{d} \sum_{\xi} \binom{d-e_\xi}{k} (t^2-1)^k \\
&= \sum_{k=1}^{d} f_k(P)(t^2-1)^k.
\end{aligned}$$

[10] By Poincaré duality, this is the Poincaré polynomial for compactly supported cohomology. If you are philosophically inclined, you might take this argument as evidence that compactly supported cohomology is sometimes more natural than ordinary cohomology.

Multiplying through by t^{-2d}, we get

$$\mathrm{Poin}_{X(P)}(t^{-1}) = \sum_{k=1}^{d} f_k(P)(t^2-1)^k t^{-2d} = \sum_{k=1}^{d} f_k(P)(t^2-1)^k t^{-2d}.$$

Finally, inverting t, we get

$$\mathrm{Poin}_{X(P)}(t) = \sum_{k=1}^{d} f_k(P)(t^{-2}-1)^k t^{2d} = \sum_{k=1}^{d} f_k(P)(1-t^2)^k t^{2(d-k)},$$

and we are done. □

Remark 4.18. For a long time, it was an unsolved problem to determine which vectors $(f_0, \ldots, f_d)$ could arise as face vectors of Delzant polytopes.[11] McMullen had conjectured a certain set of necessary and sufficient conditions, but he had not proved either direction. Then Stanley [St] (building off of the work of Danilov [Da]) proved Theorem 4.16, and observed that this puts certain restrictions on the numbers $f_0(P), \ldots, f_d(P)$. Specifically:

- The Poincaré polynomial of $X(P)$ has non-negative coefficients.
- By Poincaré duality, the Poincaré polynomial of $X(P)$ is symmetric.
- By the Lefschetz theorem, the Betti numbers satisfy $b_0 \leq b_2 \leq \ldots \leq b_{\lfloor \frac{d}{2} \rfloor}$.[12]
- The cohomology ring of $X(P)$ is generated in degree 2 (we will prove this in the next section), therefore the Betti numbers can't grow too quickly.

These conditions turn out to translate precisely into McMullen's linear inequalities, thus Stanley had proven the necessity half of McMullen's conjecture. Sufficiency was proven by Billera and Lee [BL] around the same time.

We have now used Theorem 4.12 to compute Poincaré polynomials of toric varieties in general. Next, we will use it to compute Poincaré polynomials of flag manifolds. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a decreasing sequence real numbers, and let $\mathcal{H}_\lambda$ be the space of $n \times n$ Hermitian matrices with spectrum λ. By Example 2.20, $\mathcal{H}_\lambda$ is a coadjoint orbit for $U(n)$; this action is Hamiltonian by Proposition 3.15. By Proposition 2.21, $\mathcal{H}_\lambda$ is diffeomorphic to the manifold of orthogonal frames in $\mathbb{C}^n$, or (equivalently) to the manifold of flags in $\mathbb{C}^n$. Consider the subgroup $T^n \subset U(n)$ of diagonal matrices, which also acts on $\mathcal{H}_\lambda$.

Exercise 4.19. *Show that the fixed point set $\mathcal{H}_\lambda^T$ consists of diagonal matrices whose entries are given by a permutation of λ. We will call these matrices $\{A_\sigma \mid \sigma \in S_n\}$.*

[11] Actually, the problem was phrased in terms of simple rational polytopes, which means that we require P to look like a simplicial cone near each vertex, but not necessarily like an orthant. This means working with symplectic toric orbifolds, rather than manifolds.

[12] This uses the fact that $X(P)$ admits the structure of a complex projective variety, which we haven't proven. One can prove it using the Kirwan-Ness theorem, which roughly says that symplectic quotients are the same as GIT quotients.

Consider the inclusion of $U(1)$ into T^n taking t to the diagonal matrix with entries $t, t^2, \ldots, t^n$; its fixed point set is the same as the T-fixed point set. To compute the Poincaré polynomial of $\mathcal{H}_\lambda$, we need to compute the equivariant index of each A_σ.

Exercise 4.20. *For each $i < j$, we have a natural subgroup $U(2)_{ij} \subset U(n)$. Consider the projective line*

$$\mathbb{CP}^1_{ij,\sigma} := U(2)_{ij} \cdot A_\sigma.$$

Show that the action of T^n on $\mathcal{H}_\lambda$ restricts to an action on $\mathbb{CP}^1_{ij,\sigma}$, and that T^n acts on the tangent space $T_{A_\sigma}\mathbb{CP}^1_{ij,\sigma} \cong \mathbb{C}$ via the formula $t \cdot z = t_{\sigma_i} t_{\sigma_j}^{-1} z$. Use this to draw the following conclusions:

- *For all $\sigma \in S_n$, $T_{A_\sigma}\mathcal{H}_\lambda = \bigoplus_{i<j} T_{A_\sigma}\mathbb{CP}^1_{ij,\sigma}$.*
- *$U(1)$ acts on $T_{A_\sigma}\mathbb{CP}^1_{ij,\sigma}$ with weight $\sigma_i - \sigma_j$.*
- *The equivariant index of A_σ is $d_\sigma := |\{(i,j) \mid \sigma_i > \sigma_j\}|$ (the "length" of σ).*
- $\mathrm{Poin}_{\mathcal{H}_\lambda}(t) = \displaystyle\sum_{\sigma \in S_n} t^{2d_\sigma}$.

Example 4.21. Taking $n = 3$, this tells us that $\mathrm{Poin}_{\mathrm{Flag}(\mathbb{C}^3)} = 1 + 2t^2 + 2t^4 + t^6$.

Remark 4.22. The computation in Exercise 4.20 generalizes to generic coadjoint orbits of arbitrary compact Lie groups, with T^n replaced by a maximal torus and S_n replaced by the Weyl group. That is, the cohomology of G/T (a generic coadjoint orbit of G) has total dimension $|W|$, with each element $w \in W$ contributing one dimension's worth of cohomology in degree $2\ell(w)$.

4.3 Morse-Bott theory

You may have noticed that we have not yet used Corollary 4.11. In fact, we will never use it in its current form; we will need a stronger version. The issue is that we would like to be able to apply this corollary to arbitrary $U(1)$ moment maps, even when the fixed point set has positive dimension. We can indeed do this, but to make it work we need to introduce a generalization of Morse theory, called Morse-Bott theory.

Let X be an n-manifold and $f \in C^\infty(X)$ a smooth function. We say that f is **Morse-Bott** if, near any critical point p, we can choose coordinates $x_1, \ldots, x_n$ such that

$$f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{k+\ell}^2 + f(p)$$

for some k, ℓ. The number k is called the **Morse-Bott index** of p. Note that this implies that the set of critical points of f is a submanifold of X, and that the dimension of the component containing p is $n - k - \ell$. Furthermore, it implies that the Morse-Bott index is a locally constant function on $\mathrm{Crit}(f)$. Thus, we may talk about the Morse-Bott index of a critical component.

Example 4.23. Any Morse function is Morse-Bott. The constant function is Morse-Bott. The height function on a doughnut resting normally is Morse-Bott.

Once again, our main examples will come from $U(1)$ moment maps. Suppose that $U(1)$ acts on a symplectic manifold X with moment map $\mu : X \to \mathbb{R}$. Choose a $U(1)$-equivariant almost complex structure on X that is compatible with the symplectic form. For any fixed point $p \in X^{U(1)}$, T_pX is a complex representation of $U(1)$, thus there exist integers $e_1, \ldots, e_n$ such that

$$T_pX \cong \mathbb{C}_{e_1} \oplus \ldots \oplus \mathbb{C}_{e_n},$$

where $U(1)$ acts on $\mathbb{C}_{e_i}$ by the formula $t \cdot z = t^{e_i} z$. We again define the **equivariant index** of p to be the number of i such that $e_i > 0$. This number is locally constant on the fixed point set, so we may talk about the equivariant index of a fixed component.

Proposition 4.24. *The moment map μ is a Morse-Bott function, the critical points are equal to the fixed points, and the Morse-Bott index of a fixed point (or a fixed component) is equal to 2 times the equivariant index.*

Proof. The proof is identical to the proof of Proposition 4.3. By Example 3.24, $U(1)$ acts on the symplectic vector space T_pX with moment map

$$z \mapsto -\frac{1}{2}\left(e_1|z_1|^2 + \ldots + e_n|z_n|^2\right) + c.$$

By the equivariant Darboux theorem, X is $U(1)$-equivariantly symplectomorphic near p to T_pX, so μ may be written in the above form near p. This means that we have real coordinates $x_1, y_1, \ldots, x_n, y_n$, and

$$\mu(x, y) = -\frac{1}{2}\Big(e_1(x_1^2 + y_1^2) + \ldots + e_n(x_n^2 + y_n^2)\Big) + \mu(p).$$

Scaling x_i and y_i by $|e_i|^{-\frac{1}{2}}$ whenever $e_i \neq 0$, we obtain the desired result. □

Example 4.25. Look at an action of $U(1)$ on $\mathbb{CP}^2$ whose fixed point set is the union of a point and a line.

We now explain how the fundamental theorem (Theorem 4.6) generalizes to Morse-Bott theory. The first two statements are identical, and the third is only a slight modification. A sketch of the proof can be found in [Au, IV.2.5].

Theorem 4.26. *Suppose that $f : X \to \mathbb{R}$ is proper and Morse-Bott.*

- *If $a < b$ and f has no critical values in $[a, b]$, then X_a is a deformation retract of X_b.*
- *If f has no critical values in $[a, \infty)$, then X_a is a deformation retract of X.*
- *Suppose that $a < b < c$, that $Y_1, \ldots, Y_r$ are critical components of f with $f(Y_i) = b$ for all i, and that f has no other critical values in the interval $[a, c]$. Let m_i be the Morse-Bott index of Y_i. Then there exist disjoint closed submanifolds $S_i \subset \partial X_a$ such that S_i is diffeomorphic to the boundary of a D^{m_i}-bundle over Y_i, and X_c is homotopy equivalent to the space obtained by gluing in the full disk bundles.*

Remark 4.27. The actual definition of the disk bundle and the inclusion of its boundary into X_a is a straightforward generalization of Remark 4.8.

The proof of Corollary 4.28 is identical to the proof of Corollary 4.11.

Corollary 4.28. *Suppose that X is connected, $f : X \to \mathbb{R}$ is proper and Morse-Bott, and there are no critical points of index 1 or $n-1$. Then*

- *all level sets of f are connected*
- *every local minimum/maximum of f is a global minimum/maximum, and the set of global minima/maxima is connected.*

Note that, by Proposition 4.24, Corollary 4.28 applies to any proper $U(1)$ moment map on a connected symplectic manifold.

Exercise 4.29. *Prove the following Morse-Bott analogue of Corollary 4.10. Suppose that $U(1)$ acts on X with a moment map $\mu : X \to \mathbb{R}$ that is proper and bounded below. Then*

$$\mathrm{Poin}_X(t) \;=\; \sum_{Y \in X^{U(1)}} t^{2e_Y}\,\mathrm{Poin}_Y(t),$$

where Y ranges over the critical components and e_Y is the equivariant index of Y.

4.4 The Atiyah-Guillemin-Sternberg theorem

The following important result, sometimes called the **Atiyah lemma**, is the main step in the proof of the Atiyah-Guillemin-Sternberg theorem.

Lemma 4.30. *Let T be a torus acting on a connected symplectic manifold X with proper moment map $\mu : X \to \mathfrak{t}^*$. The fibers of μ are connected.*

Sketch of proof: This is a messy proof whose guts I don't want to go through, either in class or in these notes. I will just sketch the argument; full details can be found on pages 115-6 of [Au, §IV.4]. It's enough to show that the zero fiber is connected, since the fiber over ξ of μ is the same as the fiber over 0 of $\mu - \xi$, which is also a moment map.

We proceed by induction on the dimension of T. When $\dim T = 1$, this follows from Corollary 4.28. For arbitrary T, let $H \subset T$ be a codimension 1 subtorus, and let $\mu_H : X \to \mathfrak{h}^*$ be the H-moment map obtained by composing μ with the projection $\mathfrak{t}^* \to \mathfrak{h}^*$. Let $Y := X/\!\!/H = \mu_H^{-1}(0)/H$. Our inductive hypothesis implies that Y is connected. As Audin explains, we may reduce to the case where 0 is a regular value of μ_H, which is equivalent to saying that H acts locally freely on $\mu_H^{-1}(0)$. To simplify the argument, I will assume further that H acts freely on $\mu_H^{-1}(0)$, so that Y is a symplectic manifold (see Remark 4.31).

Since μ is T-invariant, it is also H-invariant, thus the restriction of μ to $\mu_H^{-1}(0)$ descends to a map

$$\bar{\mu} : Y \to \ker(\mathfrak{t}^* \to \mathfrak{h}^*) \cong \mathrm{Lie}(T/H)^*.$$

By Proposition 3.34, $\bar{\mu}$ is a moment map for the action of $T/H \cong U(1)$ on Y, thus its fibers are connected. But we have $\bar{\mu}^{-1}(0) \cong \mu^{-1}(0)/H$, so $\mu^{-1}(0)$ must also be connected. □

Remark 4.31. If we don't assume that H acts freely on $\mu_H^{-1}(0)$, then we need to do Morse-Bott theory on the smooth manifold $\mu_H^{-1}(0)$ or on the symplectic orbifold $X /\!\!/ H$. Since we have already developed a lot of machinery for Morse-Bott theory on symplectic manifolds, it's very helpful to assume that this action is free.

We can now use the Atiyah lemma to prove the Atiyah-Guillemin-Sternberg theorem.

Theorem 4.32. *Let T be a torus acting on a compact connected symplectic manifold X with moment map $\mu : X \to \mathfrak{t}^*$. Then μ is a locally constant function on X^T, and the image of μ is equal to the convex hull of $\mu(X^T)$. This image is called the* **moment polytope** *of X.*

Before proving Theorem 4.32, let's look at a few examples.

Example 4.33. If $X = X(P)$ is the toric variety associated to a polytope P, then $\mu(X) = P$, and the theorem follows from Proposition 3.46. In general, X^T need not map to vertices of the moment polytope. For example, if $T = U(1)$, then the moment polytope has only two vertices, but there might be lots of critical points of the moment map. We can see this explicitly for $X = \mathbb{CP}^2$.

Example 4.34. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a weakly decreasing n-tuple of real numbers, and let $\mathcal{H}_\lambda$ be the space of Hermitian matrices with spectrum λ. By Proposition 3.15, the inclusion of $\mathcal{H}_\lambda$ into the space $\mathfrak{u}(n)^*$ of $n \times n$ Hermitian matrices is a moment map for the $U(n)$-action. We are interested in torus actions, so let $T^n \subset U(n)$ be the diagonal subtorus, and consider the composition

$$\mathcal{H}_\lambda \subset \mathfrak{u}(n)^* \twoheadrightarrow \mathfrak{t}^* \cong \mathbb{R}^n,$$

which is a moment map for the action of T by Proposition 3.21.

Exercise 4.35. *Show that the projection $\mathfrak{u}(n)^* \twoheadrightarrow (\mathfrak{t}^n)^* \cong \mathbb{R}^n$ takes a Hermitian matrix to its diagonal.*

By Exercises 4.19 and 4.35, the image of $\mathcal{H}_\lambda^T$ is equal to the set of permutations of λ. Applying Theorem 4.32, we obtain the following classical result, known as the **Schur-Horn theorem**.

Theorem 4.36. *A vector $\nu \in \mathbb{R}^n$ appears as the diagonal of a Hermitian matrix with spectrum λ if and only if ν lies in the convex hull of the set of permutations of λ.*

Proof of Theorem 4.32: For any $p \in X^T$, we know that the cokernel of $d\mu_p$ is dual to the stabilizer Lie algebra $\mathfrak{t}_p = \mathfrak{t}$, thus $d\mu_p = 0$. Since this holds at every fixed point, μ is locally constant on X^T.

Next we will show that the image of μ is convex. If not, then there exists an affine line $L \subset \mathfrak{t}^*$ such that $\mu(X) \cap L$ is disconnected. We may assume that L is rational, so that there exists a codimension 1 subtorus $H \subset T$ such that $\mathfrak{h}$ is perp to L. The moment map $\mu_H : X \to \mathfrak{h}^*$ is given by composing μ with the projection along L. Thus μ_H has a disconnected fiber, which contradicts the Atiyah lemma.

Since the image of μ is convex, it contains the convex hull of $\mu(X^T)$. To prove equality, we need to show that every isolated maximum of a linear function on $\mu(X)$ is equal to $\mu(p)$ for some $p \in X^T$.

Let $0 \neq a \in \mathfrak{t}$ be given, and suppose that $\xi \in \mu(X)$ is the unique maximum of the function given by dot product with a. That means that $\mu^{-1}(\xi) \subset X$ is the maximum of the function $\mu_a : X \to \mathbb{R}$. By Proposition 3.21, this function is a moment map for the action of $\mathbb{R}$ on T given by the homomorphism $\mathbb{R} \to T$ whose derivative takes $1 \in \mathbb{R} = \mathrm{Lie}(\mathbb{R})$ to $a \in \mathfrak{t}$. Thus $\mu^{-1}(\xi)$ is fixed by this action of $\mathbb{R}$. Perturbing a slightly, we may assume a is as irrational as possible, which means that the image of this homomorphism is dense in T. It follows that $\mu^{-1}(\xi)$ is fixed by all of T. □

4.5 Proof of Delzant's theorem

We are now ready to prove the compact case of Delzant's theorem (Theorem 3.38), which says that every compact abstract toric variety is the symplectic manifold associated to some Delzant polytope. Out strategy for proving the theorem breaks into two steps:

- Use Lemma 4.30 and Theorem 4.32 to show that P is a Delzant polytope.
- Show that, for all $\xi \in P$, T^d acts transitively on $\mu^{-1}(\xi)$ with stabilizer T_ξ (Definition 3.44).

Once we establish these two items, we can choose any section $\sigma : P \to X$, and define a map $f : Y(P) \to X$ by putting $f[\xi, t] = t \cdot \sigma(\xi)$ (just like the map $Y(P) \to X(P)$ from the proof of Corollary 3.47). This is clearly a continuous bijection, and therefore a homeomorphism. By the same argument outlined in the hint for Exercise 3.51, it is a symplectomorphism over the interior of P. That means that we have homeomorphisms $X(P) \leftarrow Y(P) \to X$, both of which are symplectomorphisms on a dense open set. The only thing that will remain to show, which we will not prove, is that the composition $X(P) \to X$ is smooth over the boundary of P.

We will first show that the image P of the moment map, which is a polytope by Theorem 4.32, is in fact Delzant. Fix an element $p \in X^T$. The tangent space T_pX is a linear representation of T^d, and is therefore isomorphic to

$$\mathbb{C}_{\eta_1} \oplus \ldots \oplus \mathbb{C}_{\eta_d}$$

for some weights $\eta_1, \ldots, \eta_d \in \mathbb{Z}^d \cong \mathrm{Hom}(T^d, U(1))$.[13] By Darboux's theorem (Theorem 2.28), there is a neighborhood of p in X that is symplectomorphic to a neighborhood of 0 in T_pX, where T_pX is equipped with the constant symplectic form ω_p. By Remark 2.32, we can choose this symplectomorphism to be T^d-equivariant.

Lemma 4.37. *The elements $\eta_1, \ldots, \eta_d \in \mathbb{Z}^d$ form a basis for $\mathbb{Z}^d$.*

[13] At this point we should stop and go over what we already do or don't know. If P is Delzant and $X = X(P)$, then we know by Proposition 3.52 that $\eta_1, \ldots, \eta_d$ are the negative edge vectors at the vertex $\mu(p) \in P$. However, we do not yet know that $X = X(P)$, or that P is Delzant. We don't even know yet that $\mu(p)$ is a vertex of P.

Proof. First we will show that they form a basis for $\mathbb{R}^d$. If not, then there exists a primitive nonzero vector $a \in \mathbb{Z}^d \subset \mathbb{R}^d \cong \mathfrak{t}^d$ that is perpenducular to every η_i. Consider the subgroup $U(1) \subset T^d$ defined by the element $a \in \mathbb{Z}^d \cong \mathrm{Hom}(U(1), T^d)$. This subgroup acts trivially on $\mathbb{C}_{\eta_1} \oplus \ldots \oplus \mathbb{C}_{\eta_d}$, and therefore it acts trivially in a neighborhood of p. But this contradicts the fact that the set of points with trivial stabilizer is dense (see the proof of Theorem 3.25).

Next we will show the stronger statement that they form a basis for $\mathbb{Z}^d$. If not, then there exists a nonzero primitive vector $a \in \mathbb{Z}^d$ and an integer $k > 1$ such that $a \cdot \eta_i$ is a multiple of k for all i. That means that the k^{th} roots of unity in $U(1) \subset T^d$ act trivially in a neighborhood of p, and we obtain the same contradiction. □

Let us now describe the moment map μ near p. By the above discussion, it is sufficient to understand the moment map for the action of T^d on $\mathbb{C}_{\eta_1} \oplus \ldots \oplus \mathbb{C}_{\eta_d}$. As we saw in the proof of Proposition 3.52, T^d acts on T_pX with moment map

$$z \mapsto -\frac{1}{2}|z_1|^2\eta_1 + \xi_1 - \ldots - \frac{1}{2}|z_d|^2\eta_d + \xi_d.$$

By the equivariant Darboux theorem, this means that we can choose equivariant coordinates $z_1, \ldots, z_d$ for X near p such that, in these coordinates, we have

$$\mu(z_1, \ldots, z_d) = -\frac{1}{2}|z_1|^2\eta_1 - \ldots - \frac{1}{2}|z_d|^2\eta_d + \mu(p).$$

Lemma 4.38. *For any $p \in X^T$, $\mu(p)$ is a vertex of P.*

Proof. Choose an element $a \in \mathfrak{t}^d \cong \mathbb{R}^d$ such that $a \cdot \eta_i > 0$ for all i. In a neighborhood of p, we have

$$\mu_a(z_1, \ldots, z_d) = -\frac{1}{2}|z_1|^2 a \cdot \eta_1 - \ldots - \frac{1}{2}|z_d|^2 a \cdot \eta_d + a \cdot \mu(p),$$

so μ_a has an isolated local maximum at p. By Corollary 4.28, this is a global maximum. That means that $\mu(p)$ is an isolated global maximum of the linear function on P given by dot product with a, thus $\mu(p)$ is a vertex of P. □

For all $p \in X^T$, let

$$C_p := -\mathbb{R}_{\geq 0}\eta_1 - \ldots - \mathbb{R}_{\geq 0}\eta_d + \mu(p) \subset \mathbb{R}^d.$$

This is an orthant with vertex $\mu(p)$, and it is Delzant by Lemma 4.37.

Lemma 4.39. *For any $p \in X^T$, P looks locally like C_p near $\mu(p)$.*

Proof. Let U be an equivariant Darboux neighborhood of p in X. For any $\xi \in \mu(U)$, I claim that $\mu^{-1}(\xi) \subset U$. This is because $\mu^{-1}(\xi) \cap U$ is a closed subset of X, therefore a clopen subset of $\mu^{-1}(\xi)$. By Lemma 4.30, it must be all of $\mu^{-1}(\xi)$.

This tells us that $\mu(U)$ is a neighborhood of $\mu(p)$ in P. But our formula for μ in the coordinates $z_1, \ldots, z_n$ demonstrates that $\mu(U)$ is a neighborhood of $\mu(p)$ in C_p, so we are done. □

Corollary 4.40. *The polytope P is Delzant.*

Proof. It is enough to check the Delzant condition at each vertex, and Lemma 4.39 does that. □

Next we will prove that, for all $\xi \in P$, T^d acts transitively on $\mu^{-1}(\xi)$ with stabilizer T_ξ.

Lemma 4.41. *For all $\xi \in P$ and $p \in \mu^{-1}(\xi)$, T_ξ is equal to the identity component of T^d_p.*

Proof. Let $K = T^d_p$. First we will show that $T_\xi \subset K$. Since T_ξ is connected, it is enough to show that $\mathfrak{t}_\xi \subset \mathfrak{t}^d_p = \mathfrak{k}$. By Lemma 3.27, this is equivalent to showing that the image of $d\mu_p$ is perp to $\mathfrak{t}_\xi$. This is clear from the definition of $\mathfrak{t}_\xi$: any direction in the image of $d\mu_p$ must stay in P, and therefore in every facet containing ξ.

To finish the proof, we will show that the dimension of K is less than or equal to the dimension of T_ξ, which is the codimension of the smallest face containing ξ. Let $Z \subset X$ be the component of X^K containing p. By Corollary 2.27, Z is a symplectic submanifold of X. By Proposition 3.19, the action of T^d on Z is Hamiltonian. This descends to an effective Hamiltonian action of T^d/K on Z. By Theorem 3.25, $\dim Z \geq 2(d - \dim K)$. Consider the vector space

$$(T_pZ)^\perp := \{v \in T_pX \mid \omega_p(v, w) = 0 \text{ for all } w \in T_pZ\}.$$

This is a linear representation of K, so we may write

$$(T_pZ)^\perp \cong \mathbb{C}_{\eta_1} \oplus \ldots \oplus \mathbb{C}_{\eta_r},$$

where $\eta_1, \ldots, \eta_r \in \mathrm{Hom}(K, U(1)) \cong \mathbb{Z}^{\dim K}$. The inequality $\dim Z \geq 2(d - \dim K)$ translates to the inequality $r \leq \dim K$.

I claim that $\eta_1, \ldots, \eta_r$ form a basis for $\mathbb{Z}^{\dim K}$. If this were not true, then by our inequality, they could not span $\mathbb{Z}^{\dim K}$. This means that there exists a nontrivial element $k \in K$ that acts trivially on $(T_pZ)^\perp$, and therefore on all of T_pX. Since T_pX is locally K-equivariantly diffeomorphic to X, this means that k acts trivially in a neighborhood of p. But this contradicts the fact that the set of points in X with trivial stabilizer is dense.

Let $a_1, \ldots, a_r \in \mathbb{R}^{\dim K} \cong \mathfrak{k} \subset \mathfrak{t}^d$ be the dual basis. We know that μ_{a_i} has a critical point at p, and the fact that $a_i \cdot \eta_j \geq 0$ for all j implies that it is a local maximum, and therefore a global maximum by Corollary 4.28. This in turn implies that ξ lies on the face of P determined by the linear function a_i. Since $a_1, \ldots, a_r$ are linearly independent, the codimension of the smallest face containing ξ is at least r. Thus $\dim T_\xi \geq r = \dim K$. This proves the lemma. □

Corollary 4.42. *An element $\xi \in P$ is a regular value for μ if and only if $\xi \in \mathring{P}$.*

Proof. This is Lemmas 3.27 and 4.41, along with the fact that $\mathfrak{t}_\xi = 0$ if and only if $\xi \in \mathring{P}$. □

Now we are ready to prove the desired statement for $\xi \in \mathring{P}$.

Proposition 4.43. *If $\xi \in \mathring{P}$, then T^d acts freely and transitively on $\mu^{-1}(\xi)$.*

Proof. We first show that the action is free. Since the map $\mu^{-1}(\mathring{P}) \to \mathring{P}$ is proper and all values are regular (by Corollary 4.42), it is a T^d-equivariant fiber bundle. Since $\mathring{P}$ is contractible, this fiber bundle is can be trivialized, therefore every point has the same stabilizer. Since we know that some point has a trivial stabilizer, all points must have a trivial stabilizer.

Now we show that the action of T^d on $\mu^{-1}(\xi)$ is transitive. Since ξ is a regular value of μ, $\mu^{-1}(\xi)$ has dimension d, and is therefore a union of finitely many T^d-orbits. By connectedness, it is a single T^d-orbit. □

Let $F \subset P$ be a face. The following lemma and proposition are the analogues in the abstract setting of Proposition 3.54, which we already proved (or at least stated) for concrete toric varieties.

Lemma 4.44. *The vector subspace* $-\xi + \operatorname{Span} F \subset \mathbb{R}^d \cong (\mathfrak{t}^d)^*$ *is canonically isomorphic to* $\operatorname{Lie}(T^d/T_\xi)^*$*.*

Proof. We have $\operatorname{Lie}(T^d/T_\xi) \cong \mathfrak{t}^d/\mathfrak{t}_\xi$, and its dual is isomorphic to the subspace of $(\mathfrak{t}^d)^*$ consisting of vectors that are perpendicular to $\mathfrak{t}_\xi$. By definition, this is $-\xi + \operatorname{Span}(F)$. □

Proposition 4.45. *Let F be any face of P, and let $X_F := \mu^{-1}(F) \subset X$. Let $\xi \in \mathring{F}$.*

1. *T_ξ acts trivially on X_F.*
2. *X_F is a symplectic submanifold on X.*
3. *The action of T^d/T_ξ on X_F is effective.*
4. *The map*
$$-\xi + \mu : X_F \to -\xi + F \subset -\xi + \operatorname{Span} F \cong \operatorname{Lie}(T^d/T_\xi)^*$$
is a moment map for the action of T^d/T_ξ on X_F.

These statements imply that X_F is a toric variety for the torus T^d/T_ξ, and F is its moment polytope.

Proof. Lemma 4.41 implies that every element of X_F is fixed by T_ξ. Furthermore, it implies that elements of X near X_F are not fixed by T_ξ, therefore X_F is a connected component of X^{T_ξ}. This proves parts (1) and (2).

By choosing Darboux coordinates for X near a T-fixed point, we can show that stabilizers of elements of X near fixed points are connected. In conjunction with Lemma 4.41, this means that there is some point of X_F with T-stabilizer exactly equal to T_ξ. Thus its T^d/T_ξ-stabilizer is trivial. This proves part (3). Proposition 3.19 tells us that μ_{X_F} is a moment map for the action of T; part (4) is an easy extension of this statement. □

Exercise 4.46. *Fill in the details of the proofs of parts (2)–(4) above.*

By Propositions 4.43 and 4.45, the action of T^d on $\mu^{-1}(\xi)$ is transitive with stabilizer T_ξ for any $\xi \in P$. Thus the compact case of Theorem 3.38 (modulo smoothness of the symplectomorphism over the boundary of P) is proved.

Exercise 4.47. *Show that every compact toric variety of dimension $2d$ has at least $d+1$ fixed points. Show that this assertion fails without compactness.*

5 Equivariant cohomology

In this section we study equivariant cohomology rings of symplectic manifolds with Hamiltonian group actions.

5.1 The Borel space

Let G be a Lie group. Let EG be a contractible space on which G acts freely, and let $BG := EG/G$. Such a space always exists, but since we will mostly be interested in torus actions, I will only give a construction when G is a torus.

First, let's consider the case where $G = U(1)$. For every positive integer n, $U(1)$ acts freely on $S^{2n+1} \subset \mathbb{C}^{n+1}$. The sphere S^{2n+1} is not contractible, but it gets closer and closer to being contractible as n increases. In the limit, we have a free action of $U(1)$ on $S^{2\infty+1} \subset \mathbb{C}^{\infty+1}$, and $S^{2\infty+1}$ is contractible. More precisely, $S^{2\infty+1}$ may be defined as the direct limit of S^{2n+1}, thus $\pi_k(S^{2\infty+1})$ is the direct limit of $\pi_k(S^{2n+1})$. Since $\pi_k(S^{2n+1}) = \{0\}$ for $n \geq k/2$, this direct limit is trivial. Since all of the homotopy groups of $S^{2\infty+1}$ vanish, it is contractible. We then have $BU(1) \cong \mathbb{CP}^\infty$, which you may think of as the direct limit of $S^{2n+1}/U(1) \cong \mathbb{CP}^n$. If G is a torus of dimension d, then $G \cong U(1)^d$, so we may take $EG = \big(S^{2\infty+1}\big)^d$ and $BG \cong \big(\mathbb{CP}^\infty\big)^d$.

Now let X be a space with an action of G, and let

$$X_G := \big(X \times EG\big)/G.$$

This is called the **Borel space** for the action of G on X. Observe that X_G is a fiber bundle over BG with fiber X. If the action is trivial, then it's the trivial bundle. One should think of the interestingness of the bundle as precisely reflecting the interestingness of the action.

Remark 5.1. If G acts freely on X, then X_G is also a fiber bundle over X/G with fiber EG, and is therefore homotopy equivalent to X/G. One way to think about the Borel space is that we just want to take a quotient, but that's only a good idea when the action is free. Thus we first "correct" X by replacing it with a homotopy equivalent space on which the action is free, and then we take a quotient.

Remark 5.2. At this point you should be a little bit nervous about how much the construction of X_G depends on the choice of EG. The answer is that it doesn't, at least up to homotopy. That is, let E and E' be two different contractible spaces on which G acts freely. Then we have

$$X_G \;:=\; \big(X \times E\big)/G \;\longleftarrow\; \big(X \times E \times E'\big)/G \;\longrightarrow\; \big(X \times E'\big)/G \;=:\; X'_G.$$

The left arrow is a fiber bundle with fiber E' and the right arrow is a fiber bundle with fiber E, so they are both homotopy equivalences.

Definition 5.3. For any G-space X, $H^*_G(X) := H^*(X_G)$.

Note that, since X_G is a fiber bundle over BG with fiber X, we have maps

$$H^*(BG) \to H^*_G(X) \to H^*(X).$$

The following theorem is straightforward.

Theorem 5.4. *Equivariant cohomology is a contravariant functor from the category of G-spaces (with G-equivariant maps) to the category of algebras over $H^*(BG)$.*

Proof. A G-equivariant map $X \to Y$ (over a point) induces a map $X_G \to Y_G$ (over BG). Now take cohomology. □

On a few occasions, we will need to make use of the Leray-Serre spectral sequence associated to the fiber bundle $X_G \to BG$, so we might as well give a quick review of this gadget now. The following theorem can be found in [McC, §5].

Theorem 5.5. *Let $F \hookrightarrow E \twoheadrightarrow B$ be a fiber bundle, and assume that the base B is simply-connected. Then there exists a spectral sequence with*

$$E_2^{p,q} = H^p(B;\mathbb{Q}) \otimes H^q(F;\mathbb{Q}) \qquad \text{and} \qquad \bigoplus_{p+q=n} E_\infty^{p,q} \cong H^n(E;\mathbb{Q}).$$

The last isomorphism is non-canonical, but it is compatible with the maps from $H^n(B;\mathbb{Q})$ and to $H^n(F;\mathbb{Q})$.

If B is not simply-connected, then we instead have $E_2^{p,q} = H^p(B;\{H^q(F;\mathbb{Q})\})$, where $\{H^q(F;\mathbb{Q})\}$ is the local system on B whose fibers over a given point is the q^{th} cohomology of the preimage of that point.

Example 5.6. Illustrate Theorem 5.5 for $U(1) \hookrightarrow S^{2n+1} \twoheadrightarrow \mathbb{CP}^n$ and also for $S^1 \hookrightarrow K \twoheadrightarrow S^1$.

5.2 Thinking about (equivariant) cohomology classes

I want to digress for a few minutes about how to think about cohomology classes, both ordinary and equivariant. This material ought to be covered in a first graduate topology class, but (unfortunately) is often isn't.

Let's start with homology. Let X be any manifold, and let Y be a compact oriented submanifold of dimension k. Then we get a class $[Y] \in H_k(X)$. One way to think of this is that we choose a triangulation of Y, and this gives us a closed singular k-chain on X. Another (completely equivalent) perspective is that the orientation of Y determines an element of $H_k(Y)$, and the class $[Y]$ is the pushforward of this element along the inclusion map from Y to X. If Y is the boundary of a compact oriented submanifold-with-boundary, then $[Y] = 0$. It is not necessarily the case that all homology classes on X are spanned by classes of this form, but it's true for most of the spaces that I encounter in my daily life, and it's close enough to being true that this is how I think of a homology class.

How about cohomology classes? If X happens to be compact and oriented, then we have Poincaré duality, which tells us that $H^k(X) \cong H_{\dim X-k}(X)$, thus we can think of cohomology classes as being represented by closed oriented submanifolds of codimension k. Intuitively, a cohomology class of degree k is something that pairs with a homology class of degree k and produces a number. The pairing between closed oriented submanifolds of dimension k and closed oriented submanifolds of codimension k is given by wiggling them until they are transverse and then counting intersection points with signs determined by the orientations.

What happens if X is still oriented but not compact? In this case, a cohomology class of degree k should still be something that we can combine with a homology class of degree k to get a number. A compact oriented submanifold of codimension k provides such a thing, but so does a closed oriented submanifold of codimension k (since closed intersected with compact is compact). Thus, for any closed oriented submanifold Z of codimension k, we get a class $[Z] \in H^k(X)$. It is zero if Z is the boundary of a closed oriented submanifold.

The pairing between cohomology and homology (cap product) is given by transverse intersection, and the multiplication on cohomology (cup product) is also given by transverse intersection. For a more rigorous treatment of this material, read about "Borel-Moore homology". A good place to start is [CG, §2.6].

Example 5.7. Consider the case of $X = \mathbb{C} \smallsetminus \{0\}$. We have

$$H_0(X) = \mathbb{Z} \cdot [\{1\}], \qquad H_1(X) = \mathbb{Z} \cdot [S^1], \qquad \text{and} \qquad H_2(X) = 0.$$

We also have

$$H^0(X) = \mathbb{Z} \cdot 1 = \mathbb{Z} \cdot [X], \qquad H_1(X) = \mathbb{Z} \cdot [R_+], \qquad \text{and} \qquad H^2(X) = 0.$$

Note that we have a class $[S^1] \in H^1(X)$, but it is zero because S^1 is the boundary of a closed submanifold. (Alternatively, it is zero because it intersects trivially with all 1-dimensional compact submanifolds.) Similarly, we have a class $[\{1\}] \in H^2(X)$, because it is zero because $\{1\}$ is the boundary of a closed submanifold. (Or because it intersects trivially with all 2-dimensional compact submanifolds, of which there are none.)

More generally, we always have a map from $H_{\dim X-k}(X)$ to $H^k(X)$ taking $[Y]$ to $[Y]$, but if X is not compact, this map need not be either surjective or injective. Surjectivity fails because not all closed submanifolds of dimension $\dim X - k$ are compact, and injectivity fails because not all closed submanifolds-with-boundary of dimension $\dim X - k + 1$ are compact.

Remark 5.8. I didn't say anything about what happens when X is not oriented. This is not so relevant for these notes, since symplectic manifolds are always oriented. However, the answer is that a cohomology class on X is represented by a closed cooriented submanifold $Z \subset X$, which means that, for every $p \in Z$, we have an oriention of the normal space $N_pZ = T_pX/T_pZ$. If X is already oriented, then a coorientation of Y induces an orientation of Y, and vice versa. In general a coorientation is exactly what you need to be able to intersect transversely with a compact oriented submanifold of complementary dimension and get a well-defined number.

Example 5.9. A good illustration of this perspective is the definition of the Euler class of a vector bundle. Let X be a manifold (not necessarily oriented), and let $V \to X$ be an oriented vector bundle (meaning that all of the fibers are oriented) of rank k. The orientation of the bundle is exactly the data of a coorientation of the zero section $X \subset V$. By definition, the **Euler class** of the bundle is the element $[X] \in H^k(V) \cong H^k(X)$. The isomorphism from $H^k(V)$ to $H^k(X)$ is given by taking a submanifold of V, wiggling it so that it's transverse to the zero section, and then intersecting it with the zero section. If you wiggle the submanifold $X \subset V$, you get a generic section of the bundle. Then when you intersect with the zero section, you get the vanishing set of the generic section. Thus the Euler class is also equal to the class of the zero set of a generic section, interpreted as a cooriented codimension k submanifold of X.

Remark 5.10. This perspective provides a good framework to think about the pushforward in cohomology. If $f : Z \to X$ is a proper map and $k = \dim X - \dim Z$, then we get a pushforward map $f_* : H^*(Z) \to H^{*+k}(X)$. If f is a closed inclusion, then we have $f_*[Y]_Z = [Y]_X$ for any $Y \subset Z \subset X$. (In particular $[Z]_X = f_*[Z]_Z = f_*1_Z$.) More generally, we have $f_*[Y]_Z = [f(Y)]_X$. The set $f(Y)$ may not be smooth, but it will be "smooth enough" to define a cohomology class. When f is a fiber bundle, this pushforward is also known as "integration along the fibers".

Example 5.11. Suppose that $f : Z \to X$ is a closed inclusion of codimension k, and consider the class $f^*f_*1_Z = f^*[Z]_X \in H^k(Z)$. Intuitively, this is the class on Z obtained by intersecting Z with itself (after some wiggling for transversity) inside of X. Since all of this happens "near Z", we may as well replace X with the normal bundle N to Z in X. Thus we have $f^*f_*1_Z = e(N)$.

Everything described above generalizes to equivariant cohomology, provided that we only use equivariant maps. That is, if $Z \subset X$ is a closed, cooriented G-submanifold of codimension k, then we get a class $[Z]_G \in H^k_G(X)$. (One way to see this is to note that it induces a codimension k inclusion of Z_G into X_G.) A proper G-equivariant map $Z \to X$ of oriented G-manifolds[14] induces a pushforward on equivariant cohomology. If the map is a closed inclusion, then $[Z]_G$ is the pushforward of $1 \in H^0_G(Z)$. If $V \to X$ is a G-equivariant vector bundle of rank k, then the class $[X]_G \in H^k_G(V) \cong H^k_G(X)$ is called the **equivariant Euler class**.

The last important observation is that, if $Z \subset X$ is a closed, cooriented G-submanifold of codimension k, then the map $H^k_G(X) \to H^k(X)$ takes $[Z]_G$ to $[Z]$. This can be seen by noting that this map is given by restricting along the inclusion $X \subset X_G$, and the submanifolds X and Z_G intersect transversely at Z. For example, this implies that the image of the equivariant Euler class of an equivariant vector bundle is equal to the ordinary Euler class.

[14] I don't feel like trying to define a cooriented map, so I'll just assume that X and Z are oriented.

5.3 Torus equivariant cohomology of a point

We have already observed that $BU(1) \cong \mathbb{CP}^\infty$, therefore

$$H^*_{U(1)}(*) \cong H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[u],$$

where x is an element of degree 2. Similarly, we have

$$H^*_{T^n}(*) \cong \mathbb{Z}[u_1, \ldots, u_n].$$

However, I want us to understand these isomorphisms more geometrically and more functorially. We'll start with $U(1)$.

For every integer e, let $\mathbb{C}_e$ be the 1-dimensional complex representation of $U(1)$ given by the formula $t \cdot z = t^e z$. The Borel space of $\mathbb{C}_e$ is a bundle over $BU(1)$ with fiber $\mathbb{C}_e$, and is therefore homotopy equivalent to $BU(1)$. Thus, we have $H^*_{U(1)}(\mathbb{C}_e) = H^*_{U(1)}(*)$. Consider the class

$$\alpha_e := [0]_{U(1)} \in H^2_{U(1)}(\mathbb{C}_e) \cong H^2_{U(1)}(*).$$

Proposition 5.12. *For any $e \in \mathbb{Z}$, $\alpha_e = e \cdot u$, where $u \in H^2(\mathbb{CP}^\infty)$ is the canonical generator. That is, u is the element that restricts to the class of a point in $H^2(\mathbb{CP}^1) \cong H^2(\mathbb{CP}^\infty)$.*

Proof. As noted in the statement of the proposition, we have $H^2(\mathbb{CP}^\infty) \cong H^2(\mathbb{CP}^1)$. Thus, while we are interested in the class of the codimension 2 submanifold

$$\{0\}_{U(1)} = \big(\{0\} \times S^{2\infty+1}\big)/U(1) \subset \big(\mathbb{C}_e \times S^{2\infty+1}\big)/U(1) = (\mathbb{C}_e)_{U(1)},$$

we might as well consider the submanifold

$$\big(\{0\} \times S^3\big)/U(1) \subset \big(\mathbb{C}_e \times S^3\big)/U(1).$$

This is, by definition, the Euler class of the line bundle over $\mathbb{CP}^1$ with fiber $\mathbb{C}_e$, which is usually denoted $\mathcal{O}(e)$. Thus the proposition is saying that the Euler class of $\mathcal{O}(e)$ is equal to e times the class of a point in $H^2(\mathbb{CP}^1)$. This "well-known" argument is relegated to the following exercise. $\square$

Exercise 5.13. *If you haven't seen the bundle $\mathcal{O}(e)$ before, show that the Euler class of $\mathcal{O}(1)$ is equal to the class of a point. You can do this by finding an algebraic section that meets the zero section transversely in exactly one point. It's not so hard to generalize the argument to all positive e. To deal with negative e, you should convince yourself that $\mathcal{O}(-e)$ is dual to $\mathcal{O}(e)$, and that dual line bundles have opposite Euler classes.*

This gives a very satisfying description of $H^2_{U(1)}(*)$, in the sense that we have figured out how to represent every class as a submanifold (not a submanifold of a point, but a submanifold of a complex line with a certain $U(1)$-action). Next we would like to do something similar for a higher dimensional torus.

Let $\eta \in \mathbb{Z}^n \cong \mathrm{Hom}(T^n, U(1))$ be given, and consider the 1-dimensional representation $\mathbb{C}_\eta$ of T^n. Let

$$\alpha_\eta := [0]_{T^n} \in H^2_{T^n}(\mathbb{C}_\eta) \cong H^2_{T^n}(*).$$

The following proposition is a direct generalization of Proposition 5.12.

Proposition 5.14. *We have* $\alpha_\eta = \eta_1 u_1 + \ldots \eta_n u_n \in H^2(\mathbb{CP}^\infty \times \ldots \times \mathbb{CP}^\infty)$.

Remark 5.15. The coordinate-free statement is that, for any torus T, there is a canonical isomorphism from the character lattice $\mathrm{Hom}(T, U(1))$ to $H^2_T(*)$ taking η to $[0]_T \in H^2_T(\mathbb{C}_\eta) \cong H^2_T(*)$.

Finally, given a arbitrary linear representations $W \subset V$ of T, we would like to be able to compute the class $[W]_T \in H^*_T(V) \cong H^*_T(*)$. We know that any linear representation of a torus is isomorphic to a sum of one-dimensional representations, so we may write $V/W \cong \mathbb{C}_{\eta_1} \oplus \ldots \oplus \mathbb{C}_{\eta_k}$ for some collection of chartacters $\eta_1, \ldots, \eta_k \in \mathrm{Hom}(T, U(1))$.[15]

Proposition 5.16. *We have* $[W]_T = \eta_1 \cdot \ldots \cdot \eta_k \in H^{2k}_T(*)$.

Proof. Consider the projection $\pi_i : V \to V/W \to \mathbb{C}_{\eta_i}$. Since π is transverse to $0 \in \mathbb{C}_{\eta_i}$, we have $[\pi_i^{-1}(0)]_T = \pi^*[0]_T \in H^2_T(V) \cong H^2_T(*)$, so $[\pi_i^{-1}(0)] = \eta_i$. The statement then follows from the fact that $W \subset V$ is the transverse intersection of $\pi_i^{-1}(0)$ for all i. □

5.4 Other groups

Throughout these notes, the two types of groups on which we've focused have been tori and $U(n)$. The purpose of this section will be to understand $U(n)$-equivariant cohomology in terms of T^n-equivariant cohomology. First, it turns out, we need to address the case of finite groups.

Lemma 5.17. *Suppose that W is a finite group acting on a space X. Then* $H^*_W(X) \cong H^*(X)^W$.

Proof. Consider the Leray-Serre spectral sequence associated to the fiber bundle $X_W \to BW$ (Theorem 5.5). The group $E_2^{p,q}$ is equal to the p^{th} cohomology of BW with coefficients in the local system whose fibers are $H^q(X; \mathbb{Q})$.

Since W is discrete, the category of local systems on BW is equivalent to the category of representations of W, and this equivalence takes the global sections functor to the functor which takes the W-invariant space of a representation. Since W is finite, the invariants functor is exact. Thus we may conclude that $E_2^{p,q} = 0$ if $p > 0$, and $E_2^{0,q} = H^q(Y)^W$. The spectral sequence degenerates at the E_2 page, and we are done. □

Example 5.18. A good non-example is $\mathbb{Z}$ acting on $\mathbb{R}$ by translation ($\mathbb{Z}$ is discrete but not finite). In this case, $H^*(\mathbb{R}; \mathbb{Q})^{\mathbb{Z}}$ is trivial, but $H^*_{\mathbb{Z}}(\mathbb{R}; \mathbb{Q}) = H^*(\mathbb{R}/\mathbb{Z}; \mathbb{Q})$ is not. The reason is that the invariants functor is not exact, and the first derived invariants of the trivial representation of $\mathbb{Z}$ are nontrivial.

[15] Unlike in the statement of Proposition 5.14, each η_i is now being used to denote a character of T, rather than one coordinate of a single character η. Sorry for the inconsistent notation!

Now let G be a compact Lie group acting on X. Let $T \subset G$ be a maximal torus, let $N \subset G$ be the normalizer of T, and let $W = N/T$ be the Weyl group. For example, we can take $G = U(n)$ and $T = T^n$, in which case N is the group spanned by T^n and the permutation matrices, and $W \cong S_n$.

Theorem 5.19. *The Weyl group W acts on $H_T^*(X; \mathbb{Q})$, and there is a canonical isomorphism $H_G^*(X; \mathbb{Q}) \cong H_T^*(X; \mathbb{Q})^W$.*

Example 5.20. We have $H_{U(n)}^*(*; \mathbb{Q}) \cong H_{T^n}^*(*; \mathbb{Q})^{S_n}$, which is the ring of symmetric polynomials in n variables.

Before proving Theorem 5.19, we will need an important lemma.

Lemma 5.21. *We have $H^*(G/N; \mathbb{Q}) = \mathbb{Q}$.*

Proof. First, observe that $G/N \cong (G/T)/W$, and that the action of W on G/T is free. For example, if $T = U(n)$, then G/T is isomorphic to the space of orthogonal frames in $\mathbb{C}^n$, and S_n permutes the lines in a frame. We have shown that the cohomology of G/T all lies in even degree, with total dimension $|W|$ (Remark 4.22). Since W acts freely on G/T, we have $\chi(G/N) = \frac{1}{|W|}\chi(G/T) = 1$. By Lemma 5.17,

$$H^*(G/N; \mathbb{Q}) \cong H^*((G/T)/W; \mathbb{Q}) \cong H_W^*(G/T; \mathbb{Q}) \cong H^*(G/T; \mathbb{Q})^W,$$

so $H^*(G/N; \mathbb{Q})$ is also concentrated in even degree. This means that $\dim H^*(G/N; \mathbb{Q}) = 1$, and we are done. □

Proof of Theorem 5.19: Let E be a contractible space on which G acts freely. Since N and T are subgroups of G, they both act freely on E, as well. Thus we may take

$$X_G = (X \times E)/G, \qquad X_N = (X \times E)/N, \qquad \text{and} \qquad X_T = (X \times E)/T.$$

This means that X_N is a fiber bundle over X_G with fiber G/N, and X_T is a fiber bundle over X_N with fiber $N/T = W$.

Applying Lemma 5.21 to the Leray-Serre spectral sequence of the first fiber bundle, we see that $H^*(X_G; \mathbb{Q}) \cong H^*(X_N; \mathbb{Q})$. Applying Lemma 5.17 to the second fiber bundle, we see that

$$H^*(X_N; \mathbb{Q}) \cong H^*(X_T/W; \mathbb{Q}) \cong H_W^*(X_T; \mathbb{Q}) \cong H^*(X_T; \mathbb{Q})^W.$$

Putting these isomorphisms together, we see that $H_G^*(X; \mathbb{Q}) \cong H_T^*(X; \mathbb{Q})^W$. □

5.5 The equivariant cohomology of the 2-sphere

In this section we will attempt to understand $H_{U(1)}^*(S^2)$ as an algebra over $H_{U(1)}^*(*) \cong \mathbb{Z}[u]$.

Proposition 5.22. *We have an isomorphism of graded rings $H_{U(1)}^*(S^2) \cong \mathbb{Z}[x, y]/\langle xy \rangle$, where x and y have degree 2. This is an algebra over $\mathbb{Z}[u]$ via the map $u \mapsto x - y$.*

Proof. Let $n, s \in S^2$ be the north and south poles, and consider the map

$$f : \mathbb{Z}[x, y]/\langle xy\rangle \to H^*_{U(1)}(S^2)$$

taking x to $[n]_{U(1)}$ and y to $[s]_{U(1)}$. This is well-defined because n and s are disjoint, thus the product $[n]_{U(1)} \cdot [s]_{U(1)} = 0$. We need to show that f is both injective and surjective. It is obviously injective and surjective in degree zero, so we can restrict our attention to positive degrees.

Let $U := S^2 \setminus \{s\}$ and $V := S^2 \setminus \{s\}$. We will apply the Mayer-Vietoris sequence in equivariant cohomology to the open cover $S^2 = U \cup V$. (Note that this is simply the Mayer-Vietoris sequence in ordinary cohomology applied to the open cover $S^2_{U(1)} = U_{U(1)} \cup V_{U(1)}$.) We get a long exact sequence of the form

$$\ldots \to H^{k-1}_{U(1)}(U \cap V) \to H^k_{U(1)}(S^2) \to H^k_{U(1)}(U) \oplus H^k_{U(1)}(V) \to H^k_{U(1)}(U \cap V) \to \ldots .$$

The action of $U(1)$ on $U \cap V$ is free, so we have $H^*_{U(1)}(U \cap V) = H^*(U \cap V/U(1)) = H^*(*)$. In particular, this means that

$$H^k_{U(1)}(S^2) \xrightarrow{\simeq} H^k_{U(1)}(U) \oplus H^k_{U(1)}(V)$$

when $k > 1$. It's easy to see that it's also an isomorphism when $k = 1$ (because the map to $H^0_{U(1)}(U \cap V)$ is surjective). Thus, we have reduced our problem to showing that the composition

$$\mathbb{Z}[x, y]/\langle xy\rangle \to H^*_{U(1)}(S^2) \to H^*_{U(1)}(U) \oplus H^*_{U(1)}(V)$$

is an isomorphism in positive degrees.

Since U and V are contractible, we have

$$H^*_{U(1)}(U) \oplus H^*_{U(1)}(V) \cong H^*_{U(1)}(*) \oplus H^*_{U(1)}(*) \cong \mathbb{Z}[u] \oplus \mathbb{Z}[u].$$

Where do x and y go under our composition? The element x maps first to $[n]_{U(1)}$. When we restrict to U, the element $n \in U$ looks like $0 \in \mathbb{C}_1$, so it's equivariant cohomology class is equal to u by Proposition 5.12. On the other hand, n is disjoint from V, so the restriction of $[n]_{U(1)}$ to V is zero. Thus $x \mapsto (u, 0)$. Similarly, $y \mapsto (0, -u)$. This gives us an isomorphism in positive degree.

Finally, we need to compute the image of $u \in H^2_{U(1)}(*)$ in

$$\mathbb{Z}\{x, y\} \cong H^2_{U(1)}(S^2) \cong H^2_{U(1)}(U) \oplus H^2_{U(1)}(V).$$

The image in the latter group is obviously (u, u), thus the image in $\mathbb{Z}\{x, y\}$ is the unique class that maps to (u, u), namely $x - y$. □

Here are two important points to take away from Proposition 5.22.

- The algebra $\mathbb{Z}[x, y]/\langle xy\rangle$ is a free module over $\mathbb{Z}[u]$ with basis 1 and x. Thus, as a module over $\mathbb{Z}[u]$, we have an isomorphism

$$\mathbb{Z}[u] \otimes H^*(S^2) \xrightarrow{\simeq} H^*_{U(1)}(S^2)$$

taking $[n]$ to $[n]_{U(1)}$. This isomorphism is not canonical! I also could have chosen the basis 1 and y, and I would have gotten a different isomorphism taking $[s]$ to $[s]_{U(1)}$. It is also not an isomorphism of rings. The best statement is that $H^*_{U(1)}(S^2)$ is a free module over $\mathbb{Z}[u]$ (with no natural choice of basis), and that the quotient ring $H^*_{U(1)}(S^2)/\langle u\rangle$ is canonically isomorphic to $H^*(S^2)$.

- The restriction map

$$H^*_{U(1)}(S^2) \to H^*_{U(1)}(U) \oplus H^*_{U(1)}(V) \cong H^*_{U(1)}(n) \oplus H^*_{U(1)}(s) \cong \mathbb{Z}[u] \oplus \mathbb{Z}[u]$$

is injective, and it is an isomorphism in high degree (in this case, in positive degree). In general, we will study the map given by restriction to the fixed point set.

5.6 Equivariant formality

Let X be a G-space, and consider the forgetful map $H^*_G(X) \to H^*(X)$ associated to the inclusion of X into X_G. We say that X is **G-equivariantly formal** if this map is surjective.[16] When this holds, we can say a lot about the structure of $H^*_G(X)$ using the Leray-Hirsch theorem [Ha, 4D.1].

Theorem 5.23. *Let $F \hookrightarrow E \twoheadrightarrow B$ be a fiber bundle such that the cohomology of F is finitely generated in each degree and the cohomology of E surjects onto the cohomology of F. Then there is a (noncanonical) isomorphism $H^*(E) \cong H^*(B) \otimes H^*(F)$ of graded modules over $H^*(B)$. More precisely, any set of classes in $H^*(E)$ that descend to a $\mathbb{Z}$-basis for $H^*(F)$ form a $H^*(B)$-basis for $H^*(E)$. We also have a canonical ring isomorphism $H^*(F) \cong H^*(E)/J$, where $J \subset H^*(E)$ is generated by the image of $H^{>0}(B)$.*

Remark 5.24. The last sentence is usually not included in the statement of the Leray-Hirsch theorem, but it follows easily from the previous statement, and we will want to have it is part of our corollary.

Corollary 5.25. *Suppose that X is G-equivariantly formal. Then there is a (noncanonical) isomorphism $H^*_G(X) \cong H^*_G(*) \otimes H^*(X)$ of graded modules over $H^*_G(*)$. Furthermore, we have a canonical graded ring isomorphism*

$$H^*(X) \cong H^*_G(X)/J,$$

where $J \subset H^*_G(X)$ is the ideal generated by the image of $H^{>0}_G(*)$.

[16]To be safe, we'll work with rational coefficients throughout this section.

The following theorem of Kirwan is proved by applying "Morse theory" to the norm squared of the moment map [Ki, §5]. (The quotes are there because $|\mu|^2$ is not Morse, nor even Morse-Bott; it is something weaker called *Morse-Bott-Kirwan*.)

Theorem 5.26. *Suppose that X is a compact symplectic manifold equipped with a Hamiltonian action of a compact Lie group G. Then X is G-equivariantly formal.*

We will not prove this theorem, but we will prove another theorem which will cover the case of toric varieties (even non-compact ones!) and flag manifolds, which serve as our main examples.

Theorem 5.27. *Suppose that G is a compact, connected Lie group, X is a G-space, and $H^k(X) = 0$ for all odd k. Then X is G-equivariantly formal.*

Example 5.28. If P is a Delzant polyhedron with at least one vertex, then $X(P)$ satisfies the hypotheses of Theorem 5.27. (Note that if P has no vertices, then the Corollary fails! Just think about $P = \mathbb{R}$.) The flag manifold $\mathrm{Flag}(\mathbb{C}^n)$ also satisfies the hypotheses, as does any coadjoint orbit of a compact Lie group G.

Proof of Theorem 5.27: Since $H^{\mathrm{odd}}(BG) = H^{\mathrm{odd}}(BT)^W = 0$ and $H^{\mathrm{odd}}(X) = 0$ vanishes, there can be no nontrivial differentials in the Leray-Serre spectral sequence. Thus $E_\infty = E_2$, and equivariant formality is clear. □

Using Theorem 5.27, we can derive a beautiful presentation for the (ordinary) cohomology ring of the flag manifold.

Theorem 5.29. *We have a natural isomorphism $H^*(\mathrm{Flag}(\mathbb{C}^n); \mathbb{Q}) \cong \mathbb{Q}[u_1, \ldots, u_n]/J$, where J is the ideal generated by symmetric polynomials of positive degree.*

Proof. We first compute the equivariant cohomology ring $H^*_{U(n)}(\mathrm{Flag}(\mathbb{C}^n); \mathbb{Q})$. This turns out to be very easy with a little trickery. We have

$$H^*_{U(n)}(\mathrm{Flag}(\mathbb{C}^n); \mathbb{Q}) = H^*_{U(n)}(U(n)/T^n; \mathbb{Q}) \cong H^*_{U(n)\times T^n}(U(n); \mathbb{Q}) \cong H^*_{T^n}(U(n)/U(n); \mathbb{Q}) \cong H^*_{T^n}(*; \mathbb{Q}),$$

which is isomorphic to $\mathbb{Q}[u_1, \ldots, u_n]$. By Corollary 5.25, we have

$$H^*(\mathrm{Flag}(\mathbb{C}^n); \mathbb{Q}) \cong H^*_{U(n)}(\mathrm{Flag}(\mathbb{C}^n); \mathbb{Q})/J,$$

where J is the ideal generated by positive degree elements of $H^*_{U(n)}(*; \mathbb{Q})$. Theorem 5.19 tells us that $H^*_{U(n)}(*; \mathbb{Q}) \subset H^*_{T^n}(*; \mathbb{Q})$ is isomorphic to the ring of symmetric polynomials; this completes the proof. □

5.7 (Equivariant) cohomology of toric varieties

Let $P \subset \mathbb{R}^d$ be a Delzant polyhedron with facets $F_1, \ldots, F_n$, and let $T = T^d$. Consider the ring homomorphism

$$\varphi : \mathbb{Z}[u_1, \ldots, u_n] \to H^*_T\big(X(P); \mathbb{Z}\big)$$

taking u_i to $[X(F_i)]_T \in H^2_T\big(X(P);\mathbb{Z}\big)$. Our goal will be to prove that φ is surjective and compute its kernel. We'll start by computing the kernel.

For any $S \subset \{1,\dots,n\}$, let

$$u_S := \prod_{i\in S} u_i \qquad \text{and} \qquad F_S := \bigcap_{i\in S} F_i.$$

Let

$$I := \langle u_S \mid F_S = \emptyset\rangle \subset \mathbb{Z}[u_1,\dots,u_n].$$

Proposition 5.30. $\ker\varphi = I$.

Proof. The fact that I is contained in the kernel of φ is clear: if $F_S = \emptyset$, then $\bigcap_{i\in S} X(F_i) = \emptyset$, so $\prod_{i\in S}\varphi(u_i) = 0$. Thus we must show that, for all $f \notin I$, $\varphi(f) \neq 0$.

Let $\xi \in P$ be a vertex, and let $F_{i_1},\dots,F_{i_d}$ be the facets that contain ξ. For any polynomial $f \in \mathbb{Z}[u_1,\dots,u_n]$, let $f_\xi \in \mathbb{Z}[u_{i_1},\dots,u_{i_d}]$ be the polynomial obtained from f by setting all of the other variables to zero. It is straightforward to see that

$$I = \{f \mid f_\xi = 0 \text{ for all vertices } \xi\}.$$

Suppose that $f \notin I$, and choose ξ such that $f_\xi \neq 0$. For simplicity, we will assume that $\xi \in F_1,\dots,F_d$.

Let $p \in X(P)$ be the fixed point corresponding to ξ, and consider the composition

$$\mathbb{Z}[u_1,\dots,u_n] \xrightarrow{\varphi} H^*_T\big(X(P);\mathbb{Z}\big) \to H^*_T(p;\mathbb{Z}).$$

What does this map do to u_i? If $i > d$, then $\xi \notin F_i$, so $p \notin X(F_i)$, so $u_i \mapsto 0$. In particular, this means that f and f_ξ map to the same place.

If $i < d$, then

$$u_i \mapsto [T_pX(F_i)]_T \in H^2_T\big(T_pX(P);\mathbb{Z}\big) \cong H^2_T(p;\mathbb{Z}).$$

By Proposition 5.16, this is equal to the weight of the 1-dimensional complex T-representation $T_pX(P)/T_pX(F_i)$, which is an element of $\operatorname{Hom}(T,U(1)) \cong H^2_T(*)$. By Proposition 3.57, this is equal to the inverse of the unique edge vector at ξ that is not contained in F. This tells us that $\mathbb{Z}[u_1,\dots,u_d]$ maps isomorphically to $H^*_T(p;\mathbb{Z})$. In particular, since $f_\xi \neq 0$, f_ξ does not map to zero, thus neither does f. So f cannot be in the kernel of φ. □

Before proving surjectivity, let us make explicit the interaction between the map φ and the $H^*_T(*)$-algebra structure on $H^*_T\big(X(P);\mathbb{Z}\big)$. We have

$$\mathbb{Z}\{u_1,\dots,u_n\} \cong \operatorname{Hom}(T^n,U(1)) \supset \operatorname{Hom}(T,U(1)) \cong H^2_T(*).$$

This defines a graded $H^*_T(*)$-algebra structure on $\mathbb{Z}[u_1,\dots,u_n]$.

Proposition 5.31. *The map φ is a homomorphism of $H^*_T(*)$-algebras.*

Proof. Let $\Phi : \mathbb{C}^n \to \mathfrak{k}^*$ be the moment map that we used to build $X(P) := \mathbb{C}^n /\!\!/ K$. The map φ can be interpreted geometrically as the restriction map

$$\mathbb{Z}[u_1, \ldots, u_n] \cong H^*_{T^n}(\mathbb{C}^n; \mathbb{Z}) \to H^*_{T^n}(\Phi^{-1}(0); \mathbb{Z}) \cong H^*_T(\Phi^{-1}(0)/K; \mathbb{Z}) = H^*_T\big(X(P); \mathbb{Z}\big).$$

This is because $u_i \in H^*_{T^n}(\mathbb{C}^n; \mathbb{Z})$ is represented by the submanifold $\{z_i = 0\}$, and this submanifold restricts and descends to the submanifold $X(F_i) \subset X(P)$. Thus φ is a map of $H^*_{T^n}(*)$-algebras, and in particular of $H^*_T(*)$-algebras. □

Proposition 5.32. *The map φ is surjective.*

Proof. By the exercise that follows, the classes $[X(F_i)] \in H^*\big(X(P); \mathbb{Z}\big)$ form a spanning set. By Theorem 5.23, this implies that the classes $[X(F_i)]_T \in H^*_T\big(X(P); \mathbb{Z}\big)$ span over $H^*_T(*)$.[17] Since φ is an $H^*_T(*)$-algebra homomorphism whose image contains $[X(F_i)]_T$ for all i, it must be surjective. □

Exercise 5.33. *Show that the classes $[X(F_i)] \in H^*\big(X(P); \mathbb{Z}\big)$ form a spanning set. Hint: Use Morse theory.*

Putting it all together, we obtain the following result.

Corollary 5.34. *We have*

$$H^*_T\big(X(P); \mathbb{Z}\big) \cong \mathbb{Z}[u_1, \ldots, u_n]/I \qquad \textit{and} \qquad H^*_T\big(X(P); \mathbb{Z}\big) \cong \mathbb{Z}[u_1, \ldots, u_n]/I + J,$$

where J is spanned by

$$H^2_T(*) \cong \operatorname{Hom}(T, U(1)) \subset \operatorname{Hom}(T^n, U(1)) \cong H^2_{T^n}(*) \cong \mathbb{Z}\{u_1, \ldots, u_n\}.$$

Example 5.35. Do a few examples, starting with S^2 and including a couple of toric surfaces.

5.8 The localization theorem

In the previous section, we got a lot of milage out of the restriction map

$$H^*_T(X) \to H^*_T(X^T) \cong H^*(X^T) \otimes H^*_T(*).$$

In this section, we will study this map in more detail. First, however, I need to review a little bit of commutative algebra. Most of the material here is drawn from [AB, §3]

Consider the ring

$$\mathbb{Q}[u_1, \ldots, u_n] \cong H^*_T(*; \mathbb{Q}) \cong \operatorname{Sym} \mathfrak{t}^* \cong \operatorname{Fun}(\mathfrak{t}).$$

For any element $f \in \mathbb{Q}[u_1, \ldots, u_n]$, we have a set

$$V(f) := \{a \in \mathfrak{t} \mid f(a) = 0\}.$$

[17] I'm cheating here, since I claimed only to state Theorem 5.23 with rational coefficients. I'm sure there is a fix, but I'm feeling lazy right now.

For any ideal $I \subset \mathbb{Q}[u_1, \ldots, u_n]$, we have

$$V(I) = \bigcap_{f \in I} V(f) = \{a \in \mathfrak{t} \mid f(a) = 0 \text{ for all } f \in I\}.$$

Let M be a module over $\mathbb{Q}[u_1, \ldots, u_n]$. We define the **annihilator**

$$\operatorname{Ann} M := \{f \in \mathbb{Q}[u_1, \ldots, u_n] \mid f \cdot m = 0 \text{ for all } m \in M\}$$

and the **support**

$$\operatorname{Supp} M := V(\operatorname{Ann} M) \subset \mathfrak{t}.$$

Example 5.36. If M is a free module, then $\operatorname{Ann} M = \{0\}$ and $\operatorname{Supp} M = \mathfrak{t}$. The converse is false. For example, if $n = 2$ and M consists of polynomials with no constant term, then M is not free, but its annihilator is still trivial.

Exercise 5.37. *If $M_1 \to M_2 \to M_3$ is an exact sequence of modules, show that*

$$\operatorname{Supp} M_2 \subset \operatorname{Supp} M_1 \cup \operatorname{Supp} M_3.$$

Lemma 5.38. *Suppose that M and M' are algebras (not just modules) over $\mathbb{Q}[u_1, \ldots, u_n]$, and $\varphi : M \to M'$ is a map of algebras.*[18] *Then* $\operatorname{Ann} M \subset \operatorname{Ann} M'$*, and therefore* $\operatorname{Supp} M' \subset \operatorname{Supp} M$*.*

Proof. If $f \in \operatorname{Ann} M$, then $f \cdot 1_M = 0$. This implies that $f \cdot 1_{M'} = f \cdot \varphi(1_M) = \varphi(f \cdot 1_M) = \varphi(0) = 0$. Then for any $m' \in M'$, $f \cdot m' = f \cdot (1_{M'} \cdot m') = (f \cdot 1_{M'}) \cdot m' = 0 \cdot m' = 0$. □

We define the **localization** of M to be the tensor product

$$\operatorname{Loc} M := M \otimes_{\mathbb{Q}[u_1, \ldots, u_n]} \mathbb{Q}(u_1, \ldots, u_n),$$

which is a vector space over the field $\mathbb{Q}(u_1, \ldots, u_n)$ of rational functions in n variables.

Example 5.39. If M is a free module, then $\operatorname{Loc} M$ is a vector space of dimension equal to the rank of M.

Example 5.40. If $\operatorname{Supp} M \neq \mathfrak{t}$, then $\operatorname{Loc} M = 0$. This is because there exists a nonzero polynomial f such that $f \cdot m = 0$ for all $m \in M$, thus

$$m \otimes 1 = (f \cdot m) \otimes (1/f) = 0 \otimes (1/f) = 0.$$

In this situation, we say that M is a **torsion** module.

Let's think about how these definitions apply to equivariant cohomology rings. That is, let X be a T-space, and let $M = H_T^*(X)$. Theorem 5.26 tells us that, if X is compact symplectic and the T-action is Hamiltonian, then M is free and $\operatorname{Supp} M = \mathfrak{t}$. The following results of Atiyah and Bott covers the opposite extreme [AB, 3.4].

[18] For example, if we have a map of T-spaces, we get a pullback map of equivariant cohomology rings.

Theorem 5.41. *If Y is a compact T-space with $Y^T = \emptyset$, then $H_T^*(Y)$ is torsion.*

Before proving Theorem 5.41, we will deduce the following corollary, which is the main point of the theorem [AB, 3.5].

Corollary 5.42. *Let X be a compact T-manifold. Then the kernel and cokernel of the restriction map $H_T^*(X) \to H_T^*(X^T)$ are both torsion. Equivalently, the localized restriction map $\operatorname{Loc} H_T^*(X) \to \operatorname{Loc} H_T^*(X^T)$ is an isomorphism.*[19]

Proof. The proof of Corollary 5.42 is closely related to the argument in the proof of Proposition 5.22. Let $U := X \smallsetminus X^T$ and let V be a small equivariant neighborhood of X^T, so that U and V cover X, and $H_T^*(V) \cong H_T^*(X^T)$. Then we have the Mayer-Vietoris sequence

$$\ldots \to H_T^{k-1}(U \cap V) \to H_T^k(X) \to H_T^k(U) \oplus H_T^k(V) \to H_T^k(U \cap V) \to \ldots.$$

Since $U \cap V \subset U = X \smallsetminus X^T$, Theorem 5.41 tells us that $H_T^*(U) \cong H_T^*(X \smallsetminus V)$ and $H_T^*(U \cap V) \cong H_T^*(\partial \bar{V})$ are torsion. Thus, after localizing, we get

$$\operatorname{Loc} H_T^*(X) \xrightarrow{\cong} \operatorname{Loc} H_T^*(V) \cong \operatorname{Loc} H_T^*(X^T),$$

which is what we needed. □

Using Corollary 5.42, we derive a result that we already saw "by hand" for toric varieties.

Corollary 5.43. *Suppose that X is a compact symplectic manifold equipped with a Hamiltonian T-action. Then the restriction map $H_T^*(X) \to H_T^*(X^T)$ is injective.*

Proof. Corollary 5.42 says that the kernel is torsion. But Theorem 5.26 says that $H_T^*(X)$ is free, and therefore has no nonzero torsion submodules. Thus the kernel is zero. □

Remark 5.44. We're going to ignore Corollary 5.43 for a while, and just focus on applications of localization for arbitrary (rather than symplectic) manifolds. However, we will come back to it in Section 5.10.

We now turn our attention to proving Theorem 5.41. We begin with the following lemmas [AB, 3.3].

Lemma 5.45. *Let $K \subset T$ be a closed subgroup. Then $\operatorname{Supp} H_T^*(T/K) = \mathfrak{k} \subset \mathfrak{t}$.*

Proof. We have $H_T^*(T/K) \cong H_{T\times K}^*(T) \cong H_K^*(T/T) = H_K^*(*) \cong \operatorname{Sym} \mathfrak{k}^* \cong \operatorname{Fun}(\mathfrak{k})$. The map from $H_T^*(*) \cong \operatorname{Fun}(\mathfrak{t})$ to $H_T^*(T/K) \cong \operatorname{Fun}(\mathfrak{k})$ is just the restriction map, so a function acts trivially if and only if it vanishes on $\mathfrak{k}$. □

Remark 5.46. I've finessed a technical point here, which is that $H_K^*(*) \cong \operatorname{Sym} \mathfrak{k}^*$ even if K is not connected. This follows from the fact that K is always isomorphic to a torus times a finite group, and the rational equivariant cohomology of a point for a finite group is trivial.

[19] This is equivalent because $\mathbb{Q}(u_1, \ldots, u_n)$ is a flat $\mathbb{Q}[u_1, \ldots, u_n]$-module, thus localization is an exact functor.

Lemma 5.45 extends trivially to neighborhoods of orbits as follows.

Corollary 5.47. *Let X be any T-space, and let $p \in X$ be a point with stabilizer $K \subset T$. Let U be a small T-equivariant neighborhood of the orbit $T \cdot p$ that retracts equivariantly onto the orbit. Then* $\operatorname{Supp} H_T^*(U) = \mathfrak{k}$.

Proof. We have $H_T^*(U) \cong H_T^*(T \cdot p) \cong H_T^*(T/K)$. $\square$

Proof of Theorem 5.41: For each $p \in Y$, let $U_p \subset Y$ be a small T-equivariant neighborhood of the orbit through p. By compactness, we may choose points $p_1, \ldots, p_r$ such that $Y = U_{p_1} \cup \ldots \cup U_{p_r}$. Let $Y_k = U_{p_1} \cup \ldots \cup U_{p_k}$, so that $Y_1 = U_{p_1}$ and $Y_r = U_{p_r}$. We will use induction to prove that $H_T^*(Y_k)$ is torsion for all k.

When $k = 1$, this follows from Corollary 5.47. Now assume that it holds for k, and consider the exact sequence

$$H^*(Y_k \cap U_{p_{k+1}}) \to H_T^*(Y_k) \to H_T^*(Y_k) \oplus H_T^*(U_{p_{k+1}}).$$

The right-hand term is torsion by our inductive hypothesis and Corollary 5.47, while the left-hand side is torsion by Lemma 5.38. Thus $H_T^*(Y_k)$ is torsion by Exercise 5.37. $\square$

5.9 The localization formula

Let X be a compact T-manifold. In Corollary 5.42, we learned that, modulo torsion, all of the data of $H_T^*(X)$ is "stored" at the fixed points. In this section we will put this result to work, allowing ourselves to compute intersection numbers in terms of local data.

Let $i : X^T \to X$ be the inclusion. Corollary 5.42 states that the localized restriction map

$$i^* : \operatorname{Loc} H_T^*(X) \to \operatorname{Loc} H_T^*(X^T) \cong H^*(X^T) \otimes_{\mathbb{Q}} \mathbb{Q}(u_1, \ldots, u_n)$$

is an isomorphism. It will be convenient to find a formula for its inverse. The first natural map to look at is the pushforward

$$i_* : \operatorname{Loc} H_T^*(X^T) \to \operatorname{Loc} H_T^*(X).$$

This is not quite the inverse to i^*, but Example 5.11 tells us exactly how it fails. That is, we have $i^* i_* 1 = e$, where e is the equivariant Euler class of the normal bundle to X^T in X.

Lemma 5.48. *More generally, for any $\alpha \in H_T^*(X^T)$, we have $i^* i_* \alpha = e \cdot \alpha$.*

Proof. We may replace X with the normal bundle to X^T in X, and thereby assume that $\alpha = i^*\beta$ for some $\beta \in H_T^*(X)$. Then the lemma follows from the adjunction formula, which says that, for any proper $f : Z \to X$, the pushforward $f_* : H^*(Z) \to H^{*+k}(X)$ is an $H^*(X)$-module homomorphism. That is, for all $\delta \in H^*(Z)$ and $\gamma \in H^*(X)$, we have $f_*(\delta \cdot f^*\gamma) = f_*\delta \cdot \gamma$.

In our case, this tells us that $i^* i_* \alpha = i^* i_* (1 \cdot i^* \beta) = i^* \left(i_* 1 \cdot \beta \right) = i^* i_* 1 \cdot i^* \beta = e \cdot \alpha$. $\square$

Proposition 5.49. *The image of e in* $\operatorname{Loc} H_T^*(X^T)$ *is invertible. Thus the inverse of i^* is given by first dividing by e and then applying i_*.*

Proof. For simplicity, we will focus on the case where X^T is finite, and leave the general case as an exercise. We have

$$\operatorname{Loc} H_T^*(X^T) \cong \bigoplus_{p \in X^T} \operatorname{Loc} H_T^{\dim X}(p) \cong \bigoplus_{p \in X^T} \mathbb{Q}(u_1, \ldots, u_n),$$

so we only need to show that each component is nonzero.

The normal bundle to p in X is simply the vector space T_pX, endowed with an action of T. Its equivariant Euler class is the element $[0]_T \in H_T^{\dim X}(T_pX) \cong H_T^{\dim X}(p)$, which is equal to the product of the weights for the action of T on T_pX by Proposition 5.16. Since each p is an isolated fixed point, these weights are all nonzero. $\square$

Exercise 5.50. *Repeat this proof in the more general case where X^T can have any dimension. (For example, the action of T on X could be trivial!)*

Corollary 5.51. *For any $\alpha \in H_T^*(X)$, we have*

$$\operatorname{Loc} \alpha = i_* \left(\frac{i^* \alpha}{e} \right) \in \operatorname{Loc} H_T^*(X).$$

Let $\pi : X \to *$ be the map to a point. Since X is compact, and therefore π is proper, we have a map

$$\pi_* : H_T^{*+\dim X}(X) \to H_T^*(*).$$

This map is often called **equivariant integration**, since the nonequivariant version

$$\pi_* : H^{\dim X}(X; \mathbb{R}) \to H^0(*; \mathbb{R}) \cong \mathbb{R}$$

is given by integration of top-degree forms over all of X, or equivalently by capping with the homology class $[X] \in H_{\dim X}(X)$. Thus we often write $\int_X \alpha := \pi_* \alpha$.

Lemma 5.52. *Equivariant integration kills all torsion. That is, if $\operatorname{Loc} \alpha = 0$, then $\int_X \alpha = 0$.*

Proof. If $\operatorname{Loc} \alpha = 0$, then there exists some nonzero $f \in \mathbb{Q}[u_1, \ldots, u_n]$ such that $f \cdot \alpha = 0$. Then $f \cdot \pi_* \alpha = \pi_*(f \cdot \alpha) = \pi_* 0 = 0$. But $H_T^*(*)$ has no torsion, so this implies that $\int_X \alpha = \pi_* \alpha = 0$. $\square$

Combining Corollary 5.51 and Lemma 5.52, we obtain the **localization formula**.

Theorem 5.53. *For any $\alpha \in H_T^*(X)$, we have*

$$\int_X \alpha = \int_{X^T} \frac{i^* \alpha}{e}.$$

Proof. Let ρ be the map from X^T to a point, so that $\int_{X^T} = \rho_*$. Then the theorem follows from applying π_* to both sides of Corollary 5.51, and noting that $\rho = \pi \circ i$, so $\rho_* = \pi_* \circ i_*$. Lemma 5.52 implies that there is no difference between integrating α and $\operatorname{Loc} \alpha$. $\square$

Remark 5.54. It can be confusing to think about disconnected manifolds, so Theorem 5.53 is often written as a sum over connected components of X^T:

$$\int_X \alpha = \sum_{F \subset X^T} \int_F \frac{i_F^* \alpha}{e_F},$$

where i_F is the inclusion of F into X, and e_F is the equivariant Euler class of its normal bundle.

Remark 5.55. The element $\frac{i_F^*\alpha}{e_F}$ lives in $\operatorname{Loc} H_T^*(F)$; if we do not localize, then we cannot invert e_F. That means that $\int_F \frac{i_F^*\alpha}{e_F}$ lives in $\mathbb{Q}(u_1, \ldots, u_n)$; indeed, it might not be a polynomial! The amazing thing is that, when we sum over all F, the denominators cancel and we get $\int_X \alpha \in \mathbb{Q}[u_1, \ldots, u_n]$. (We will see this in some explicit examples.)

Remark 5.56. Equivariant and nonequivariant integration are compatible in the following way: if $\alpha \in H^{\dim X}(X)$ and $\tilde{\alpha} \in H_T^{\dim X}(X)$ is a lift of α, we have

$$\int_X \tilde{\alpha} \in H_T^0(*) \cong \mathbb{Q} \qquad \text{and} \qquad \int_X \alpha \in H^0(*) \cong \mathbb{Q},$$

and they are equal. Thus, a good method for integrating a nonequivariant class is to lift it to an equivariant class and apply the localization formula. We will see this in the example below!

Example 5.57. For P a trapezoid, compute all intersection numbers of toric divisors on $X(P)$.

Example 5.58. Next, we'll use Theorem 5.53 to answer the first interesting question in enumerative geometry: Given four lines in $\mathbb{R}^3$ in generic position, how many lines intersect all four of them?

Let's start by reformulating the question. First of all, it's always cleaner to work over $\mathbb{C}$ rather than over $\mathbb{R}$. If our four complex lines are complexifications of real lines, then all of the lines that intersect all four of them will also be real. Second of all,it will be convenient to include "lines at infinity"; that is, we want to work in $\mathbb{CP}^3$ rather than $\mathbb{C}^3$. Equivalently, we can work in $\mathbb{C}^4$, and consider planes through the origin. Then we want to fix four planes in general position, and ask how many planes intersect the four fixed ones positive-dimensionally. If our original lines are generic enough, then this won't make a difference, since there won't be any lines at infinity meeting all four lines. Thus we will still be answering the question as stated.

Let $\mathrm{Gr}_2(\mathbb{C}^4)$ be the set of all planes (through the origin) in $\mathbb{C}^4$. We have already seen that this is a compact, oriented[20] manifold: it is a coadjoint orbit of $U(4)$. More precisely, it is the set of all matrices obtained by conjugating the matrix $\operatorname{diag}(1, 1, 0, 0)$ by an element of $U(4)$. We have $\dim \mathrm{Gr}_2(\mathbb{C}^4) = 8$, which is easiest to see back in the setting of real lines in $\mathbb{R}^3$ ($4 = 3 + 3 - 2$, $8 = 4 * 2$).

For a fixed plane P, let $X(P) \subset \mathrm{Gr}_2(\mathbb{C}^4)$ be the locus of planes whose intersection with P has positive dimension. We have $\dim X(P) = 6$, which is again easiest to see in $\mathbb{R}^3$ ($3 = 1+2$, $6 = 3*2$).

[20] Even better, it is symplectic, though this will not be relevant here.

We are then asking the following question: What is the cardinality of the set

$$X(P_1) \cap X(P_2) \cap X(P_3) \cap X(P_4)?$$

But this is of course a cohomological question, namely to compute the number

$$\int_{\mathrm{Gr}_2(\mathbb{C}^4)} [X(P_1)] \cdot [X(P_2)] \cdot [X(P_3)] \cdot [X(P_4)].$$

The first observation that I want to make is that $[X(P_i)] = [X(P_j)]$, so we might as well just integrate $[X(P)]^4$. We can take $P = P(1,2)$ to be the sum of the first two coordinate lines, which is preserved by the natural T^4-action on $\mathrm{Gr}_2(\mathbb{C}^4)$. This action has isolated fixed points (the six coordinate planes $P(i,j)$, so it should be pretty easy to apply the localization formula. However, there is a slight problem with this: $X(P)$ is not actually a submanifold! It is singular at the point P. It is a subvariety, which makes it good enough to define a cohomology class, but not good enough to use the localization formula. However, $X := X(P(1,2)) \cap X(P(3,4))$ **is** a submanifold representing the cohomology class $[X(P)]^2$, so we can use the localization formula to compute

$$\int_{\mathrm{Gr}_2(\mathbb{C}^4)} [X]^2.$$

As stated above, we have six fixed points, four of which lie in X: $P(1,3), P(1,4), P(2,3), P(2,4)$. For each of these four points, we need to understand the T^4-representation $T_{P(i,j)} \mathrm{Gr}_2(\mathbb{C}^4)$ along with the subrepresentation $T_{P(i,j)}X$. Then we may apply the localization formula.

Let $Y(i,j,k) \subset \mathrm{Gr}_2(\mathbb{C}^4)$ be the submanifold of planes formed by taking the direct sum of the i^{th} coordinate line and some line in $P(j,k)$. This is clearly a copy of $\mathbb{CP}^1$ connecting the points $P(i,j)$ and $P(i,k)$.

Exercise 5.59. *Verify the following three statements.*

- $T_{P(1,3)} \mathrm{Gr}_2(\mathbb{C}^4) = T_{P(1,3)}Y(1,3,4) \oplus T_{P(1,3)}Y(1,3,2) \oplus T_{P(1,3)}Y(3,1,2) \oplus T_{P(1,3)}Y(3,1,4)$.
- $T_{P(1,3)}X = T_{P(1,3)}Y(1,3,4) \oplus T_{P(1,3)}Y(3,1,2)$.
- $T_{P(1,3)}Y(1,3,4) \cong \mathbb{C}_{(0,0,1,-1)}$ *as a representation of* $\mathbb{C}^4$.

Note that all three statements obviously still hold when the numbers are permuted. Thus

$$T_{P(1,3)} \mathrm{Gr}_2(\mathbb{C}^4) \cong \mathbb{C}_{(0,0,1,-1)} \oplus \mathbb{C}_{(0,-1,1,0)} \oplus \mathbb{C}_{(1,-1,0,0)} \oplus \mathbb{C}_{(1,0,0,-1)}$$

and

$$T_{P(1,3)}X \cong \mathbb{C}_{(0,0,1,-1)} \oplus \mathbb{C}_{(1,-1,0,0)},$$

and similarly for $(1,4)$*,* $(2,3)$*, and* $(2,4)$*.*

Using Exercise 5.59, we can finish the computation! By the localization formula, we have

$$\begin{aligned}\int_{\mathrm{Gr}_2(\mathbb{C}^4)} [X]^2 &= \frac{i^*_{P(1,3)}[X]^2}{e_{P(1,3)}} + \frac{i^*_{P(1,4)}[X]^2}{e_{P(1,4)}} + \frac{i^*_{P(2,3)}[X]^2}{e_{P(2,3)}} + \frac{i^*_{P(2,4)}[X]^2}{e_{P(2,4)}} \\ &= \frac{(0,-1,1,0)\cdot(1,0,0,-1)}{(0,0,1,-1)\cdot(1,-1,0,0)} + \frac{(0,-1,0,1)\cdot(1,0,-1,0)}{(0,0,-1,1)\cdot(1,-1,0,0)} \\ &\quad + \frac{(-1,0,1,0)\cdot(0,1,0,-1)}{(0,0,1,-1)\cdot(-1,1,0,0)} + \frac{(-1,0,0,1)\cdot(0,1,-1,0)}{(0,0,-1,1)\cdot(-1,1,0,0)}.\end{aligned}$$

(Note that I got the first summand from Exercise 5.59, and the other three by permuting the numbers in the first.) This translates to

$$\frac{(-u_2+u_3)(u_1-u_4)}{(u_3-u_4)(u_1-u_2)} + \frac{(-u_2+u_4)(u_1-u_3)}{(-u_3+u_4)(u_1-u_2)} + \frac{(-u_1+u_3)(u_2-u_4)}{(u_3-u_4)(-u_1+u_2)} + \frac{(-u_1+u_4)(u_2-u_3)}{(-u_3+u_4)(-u_1+u_2)}.$$

Combining the fractions and simplifying, we obtain the number 2.

5.10 CS/GKM theory

We saw in Corollary 5.43 that the (unlocalized) restriction to the fixed points is injective in equivariant cohomology for a Hamiltonian group action on a compact symplectic manifold. In this section we will describe the image in the case where our group is a torus T. Let

$$X_k := \{p \in X \mid \dim \mathfrak{t}_p \geq \dim \mathfrak{t} - k\} \subset X.$$

Thus we have

$$X^T = X_0 \subset X_1 \subset \ldots \subset X_{\dim T} = X.$$

We will call X_k the **k-skeleton** of X.

Example 5.60. The k-skeleton of $X(P)$ is the preimage of the union of the k-faces of P.

Example 5.61. Let's describe the skeleta of the manifold $X := \mathrm{Flag}(\mathbb{C}^3)$ with its natural action of T^3. Remember that there are two other nice ways to describe X. First, we can describe it as the set of triples $\{(L_1, L_2, L_3)\}$ of pairwise orthogonal lines in $\mathbb{C}^3$. Equivalently, we can describe it as the conjugacy class of the Hermitian matrix $\mathrm{diag}(3,1,0)$. (Here I could have picked any three distinct real numbers; I just chose three particular ones for concreteness.)

The diagonal copy of $U(1)$ in T^3 acts trivially, thus every point of X has a stabilizer of dimension at least 1. In other words, we have $X_2 = X_3 = X$. By definition, X_0 is the set of points that are fixed by all of T^3, which we know to be the set of 6 coordinate flags. (Equivalently, it is the set of permutations of the standard orthogonal frame, or the set of diagonal matrices with entries 3, 1, and 0.) The interesting problem is therefore to compute X_1.

It is not hard to check that there are only three different 2-dimensional stabilizer groups that can arise, which we will call T_1, T_2, and T_3. The group T_1 is defined to be the subgroup of T^3

generated by $U(1) \times \{1\} \times \{1\}$ and the diagonal copy of $U(1)$; T_2 and T_3 are defined similarly. So we have $X_1 = X^{T_1} \cup X^{T_2} \cup X^{T_3}$.

To compute the set X^{T_1}, let's start by thinking of X in terms of triples of orthogonal frames. A triple (L_1, L_2, L_3) is fixed by $U(1) \times \{1\} \times \{1\}$ if and only if one of the three lines is equal to the first coordinate line. Thus X^{T_1} has three connected components, each of which is isomorphic to $\mathbb{CP}^1$. The same is true for X^{T_2} and X^{T_3}. Now let's think about it in terms of Hermitian matrices. In this picture X^{T_1} is the subset of X consisting of matrices for which the first coordinate line is one of the three eigenspaces. In other words, it is the set of block diagonal matrices of block type $(1, 2)$. Again, this set has three connected components indexed by the eigenvalue of the first coordinate line.

Finally, let's see what happens to the 1-skeleton under the moment map. We know that the image of the moment map is the convex hull in $\mathbb{R}^3$ of the set of permutations of the vector $(3, 1, 0)$. This is a 2-dimensional polytope, which we will project onto the first two coordinates (the third is then 4 minus the sum of the first two). We can see the images of the nine copies of $\mathbb{CP}^1$ in our picture.

Example 5.62. More generally, the codimension 1 stabilizer groups for T^n acting on $\mathrm{Flag}(\mathbb{C}^n)$ will be indexed by pairs $i < j$. The subgroup $T_{ij} \subset T^n$ will be the stuff generated by all of the coordinate copies of $U(1)$ except for the i^{th} one and the j^{th} one, plus the diagonal copy. (Thus, what I previously called T_1 is now being called T_{23}.) The set $\mathrm{Flag}(\mathbb{C}^n)^{T_{ij}}$ will be a disjoint union of lots of copies of $\mathbb{CP}^1$. If $p_\sigma \in \mathrm{Flag}(\mathbb{C}^n)^{T^n}$ is the fixed point corresponding to the element $\sigma \in T^n$, then there will be a copy of $\mathbb{CP}^1$ connecting p_σ with $p_{s_{ij}\sigma}$.

Given a class in $H_T^*(X^T)$, we would like to know whether or not it is the restriction of a class in $H_T^*(X)$. Of course, if it is the restriction of a class $\alpha \in H_T^*(X)$, then it is also the restriction of a class $\alpha|_{X_k} \in H_T^*(X_k)$ for every k. The following amazing theorem of Chang and Skjelbred [CS] says that it's enough to extend it just to the 1-skeleton!

Theorem 5.63. *If a class in $H_T^*(X^T)$ extends to $H_T^*(X_1)$, then it extends all the way to $H_T^*(X)$.*

Goresky, Kottwitz, and MacPherson use the Chang-Skjelbred theorem to give an explicit description of $H_T^*(X)$ when $\dim X_0 = 0$ and $\dim X_1 = 2$. (In particular, this includes toric varieties and $\mathrm{Flag}(\mathbb{C}^n)$, as we have seen above.) Such a space is called a **GKM space**. To be explicit, write

$$X_1 = C_1 \cup \ldots \cup C_r,$$

where each C_i is a compact connected 2-manifold. Let $K_i \subset T$ be the stabilizer of a generic point on C_i, and let $T_i = T/K_i$. By definition of X_1, T_i is 1-dimensional, and it acts effectively on C_i. Furthermore, C_i is a connected component of X^{K_i}, and is therefore a symplectic submanifold. Let $\mu : X \to \mathfrak{t}^*$ be the moment map. Then the restriction of μ to C_i is a moment map for the action of T on C_i. Since T_i acts effectively, Delzant's theorem tells us that $C_i \cong S^2$, and that $\mu(C_i)$ is an interval parallel to the line $\mathfrak{t}_i^* \subset \mathfrak{t}^*$. Furthermore, each C_i contains two fixed points, which we will call p_i and q_i.

Lemma 5.64. *Suppose we are given an element $\alpha = (f, g) \in \operatorname{Sym} \mathfrak{t}^* \oplus \operatorname{Sym} \mathfrak{t}^* \cong H_T^*(\{p_i, q_i\})$. Then α extends to $H_T^*(C_i)$ if and only if $f - g \in \mathfrak{t}_i^*$.*

Proof. We imitate the proof of Proposition 5.22, which dealt with the case of $U(1)$ acting on S^2. Let $U = C_i \smallsetminus \{q_i\}$ and $V = C_i \smallsetminus \{p_i\}$, and consider the exact sequence

$$\ldots \to H_T^k(C_i) \to H_T^k(U) \oplus H_T^k(V) \to H_T^k(U \cap V) \to \ldots.$$

Since U and V retract onto p_i and q_i, the lemma may be interpreted as a statement about the image of the first map. By exactness, this is equal to the kernel of the second map. Since K_i acts trivially, we have

$$H_T^*(U \cap V) \cong H_{K_i}^*(*) \otimes H_{T_i}^*(U \cap V) \cong H_{K_i}^*(*) \cong \operatorname{Sym} \mathfrak{k}_i^*,$$

and the map is given by taking the difference and projecting from $\mathfrak{t}_i^*$ to $\mathfrak{k}_i^*$. Thus the kernel consists of pairs (f, g) whose difference lies in the kernel of this projection, which is equal to $\mathfrak{t}_i^*$. ☐

Exercise 5.65. *Show that an element of $H_T^*(X^T)$ extends to $H_T^*(X_1)$ if and only if it extends individually over each C_i.*

The following theorem, which is an immediate consequence of Theorem 5.63, Lemma 5.64, and Exercise 5.65, is known as the GKM theorem [GKM, 1.2.2].

Theorem 5.66. *Let X be a GKM space. Then the image of the restriction map*

$$H_T^*(X) \hookrightarrow H_T^*(X^T) \cong \bigoplus_{p \in X^T} \operatorname{Sym} \mathfrak{t}^*$$

is equal to the subring consisting of classes $(f_p)_{p \in X^T}$ such that, for all $i \in \{1, \ldots, r\}$, $f_{p_i} - f_{q_i} \in \mathfrak{t}_i^*$.

Example 5.67. Do $\mathbb{CP}^2$ as a toric variety. Show the three guys whose product is zero.

Example 5.68. Do $\operatorname{Flag}(\mathbb{C}^3)$, and more generally $\operatorname{Flag}(\mathbb{C}^n)$. We find that

$$H_{T^n}^*\big(\operatorname{Flag}(\mathbb{C}^n); \mathbb{Q}\big) \cong \Big\{ (f_\sigma) \mid f_\sigma - f_{s_{ij}\sigma} \text{ is a multiple of } u_i - u_j \Big\} \subset \bigoplus_{\sigma \in S_n} \mathbb{Q}[u_1, \ldots, u_n].$$

5.11 Cohomology of polygon spaces

This section was added as an afterthought to fill the last lecture. It doesn't use too much of the technology that we have been developing, but the answer is very pretty. I will be loosey-goosey throughout.

Recall that, given an n-tuple of positive real numbers $(r_1, \ldots, r_n)$, we define

$$\operatorname{Pol}(r_1, \ldots, r_n) := S^2 \times \ldots \times S^2 /\!\!/ SO(3),$$

where the moment map

$$\mu : S^2 \times \ldots \times S^2 \to \mathfrak{so}(3)^* \cong \mathbb{R}^3$$

is given by adding n vectors of length $(r_1, \ldots, r_n)$. It is smooth (and therefore symplectic) if and only if

$$\sum_{i \in S} r_i \neq \sum_{j \in S^c} r_j$$

for every subset $S \subset \{1, \ldots, n\}$; we will assume this to be the case throughout the section.

To understand the cohomology ring of $\mathrm{Pol}(r_1, \ldots, r_n)$, we will use a the following awesome theorem, known as **Kirwan surjectivity** [Ki].

Theorem 5.69. *Suppose that a compact connected Lie group G acts on a compact symplectic manifold X with moment map $\mu : X \to \mathfrak{g}^*$, and that G acts freely on $\mu^{-1}(0)$. Then the natural map*

$$H^*_G(X) \to H^*_G(\mu^{-1}(0)) \cong H^*(\mu^{-1}(0)/G) \cong H^*(X /\!\!/ G)$$

is surjective.

Let's apply this in our situation. By Theorem 5.19, we have

$$H^*_{SO(3)}(S^2 \times \ldots \times S^2) \cong H^*_{U(1)}(S^2 \times \ldots \times S^2)^{\mathbb{Z}/2\mathbb{Z}}.$$

We know that

$$H^*_{U(1)^n}(S^2 \times \ldots \times S^2) \cong \mathbb{Q}[x_1, y_1, \ldots, x_n, y_n]/\langle x_i y_i \mid 1 \leq i \leq n \rangle,$$

and that the map from $\mathbb{Q}[u_1, \ldots, u_n]$ takes u_i to $x_i - y_i$. If we wanted to pass to ordinary cohomology, we would set each u_i equal to zero. We want to do something a little less extreme, namely to pass to equivariant cohomology for the diagonal subgroup. This means that instead of setting each u_i equal to zero, we should set them all equal to each other. Thus we have

$$H^*_{U(1)}(S^2 \times \ldots \times S^2) \cong \mathbb{Q}[x_1, y_1, \ldots, x_n, y_n]/\langle x_i y_i, (x_1 - y_1) - (x_i - y_i) \mid i \leq n \rangle.$$

How does $\mathbb{Z}/2\mathbb{Z}$ act? Well, it should take u to $-u$, so it's not hard to guess that it should swap x_i and y_i for all i. That means the invariant subring is generated by $x_1 + y_1, \ldots, x_n + y_n$. Let $c_i = x_i + y_i$. Note that we have

$$c_i^2 = (x_i + y_i)^2 = (x_i - y_i)^2 = u^2 \qquad \text{(because } x_i y_i = 0\text{)},$$

thus we have

$$H^*_{SO(3)}(S^2 \times \ldots \times S^2) \cong \mathbb{Q}[c_1, \ldots, c_n]/\langle c_i^2 - c_j^2 \mid i, j \leq n \rangle.$$

By Theorem 5.69, the cohomology ring $H^*(\mathrm{Pol}(r_1, \ldots, r_n))$ is a quotient of this ring by some more relations.

Let's try to guess where in $H^2(\mathrm{Pol}(r_1, \ldots, r_n))$ the classes $c_1, \ldots, c_n$ go. Degree 2 cohomology classes arise as Euler classes of line bundles, so it is reasonable to look for n natural line bundles on $\mathrm{Pol}(r_1, \ldots, r_n)$.

Fix an index i, and define

$$U_i := \{(v_1, \ldots, v_n) \in \mathbb{R}^{3n} \mid |v_j| = r_j \ \forall j, \ \sum v_j = 0, \text{ and } v_i \text{ points up}\}.$$

There is a free action of $U(1)$ on U_i given by rotation around the vertical axis, and we have a canonical diffeomorphism $U_i/U(1) \cong \mathrm{Pol}(r_1, \ldots, r_n)$. Let

$$L_i := (U_i \times \mathbb{C})/U(1);$$

this is a complex line bundle over $\mathrm{Pol}(r_1, \ldots, r_n)$.

Proposition 5.70. *The Kirwan map takes c_i to $e(L_i)$.*

Exercise 5.71. *Prove Proposition 5.70.*

Though I won't prove Proposition 5.70, I will establish that it is plausible by convincing you that $e(L_i)^2 = e(L_j)^2$ for all i, j. My strategy is simple: showing that $e(L_i)^2 = e(L_j)^2$ is the same as showing that $(e(L_i) + e(L_j)) \cdot (e(L_i) - e(L_j)) = 0$. Thus I will try to represent both $e(L_i) + e(L_j)$ and $e(L_i) - e(L_j)$ by submanifolds and then show that those submanifolds don't intersect.

Let's begin with

$$e(L_i) - e(L_j) = e(L_i) + e(L_j^*) = e(L_i \otimes L_j^*) = e(\mathcal{H}om(L_j, L_i)).$$

To represent this with a submanifold, I want to find a homomorphism from L_j to L_i, and then take the vanishing locus (with some multiplicity). I will do this by trying to define a smooth, $U(1)$-equivarinant map from U_j to U_i. This map will fail to be defined over some elements of $\mathrm{Pol}(r_1, \ldots, r_n)$, and that will be my vanishing locus. The "map" is defined in the most natural way, that is, by rotating along the shortest path. This is equivariant, and it is well-defined as long as v_i and v_j do not point in opposite directions. Thus we have sort of proven the following statement.

Lemma 5.72. *The degree 2 cohomology class $e(L_i) - e(L_j)$ is some multiple of the class defined by the codimension 2 submanifold consisting of polygons for which the i and j edges point in opposite directions.*

To do something similar for $e(L_i) + e(L_j)$, we need the "down" version of our construction. That is, let

$$D_i := \{(v_1, \ldots, v_n) \in \mathbb{R}^{3n} \mid |v_j| = r_j \ \forall j, \ \sum v_j = 0, \text{ and } v_i \text{ points down}\}$$

and

$$M_i := (D_i \times \mathbb{C})/U(1).$$

Lemma 5.73. *The line bundles L_i and M_i are dual to each other.*

Proof. It's enough to find a diffeomorphism $\varphi_i : U_i \to D_i$ over $\mathrm{Pol}(r_1, \ldots, r_n)$ that is $U(1)$-anti-equivariant. That is, we want $\varphi_i(t \cdot u) = t^{-1} \cdot \varphi_i(u)$ for all $u \in U_i$, $t \in U(1)$. Such a map is given by 180 degree rotation around the x-axis. □

By Lemma 5.73, we have

$$e(L_i) + e(L_j) = e(L_i \otimes L_j) = e(\mathcal{H}om(M_j, L_i).$$

We can try to define a map from D_j to U_i as above, and we will fail exactly when v_i and v_j point in the same direction. Thus we obtain the following analogue of Lemma 5.72.

Lemma 5.74. *The degree 2 cohomology class $e(L_i) + e(L_j)$ is some multiple of the class defined by the codimension 2 submanifold consisting of polygons for which the i and j edges point in the same direction.*

Combining Lemmas 5.72 and 5.74, it is clear that

$$e(L_i)^2 - e(L_j)^2 = (e(L_i) + e(L_j)) \cdot (e(L_i) - e(L_j)) = 0.$$

Finally, I would like to talk about the kernel of the Kirwan map. That is, we know that the cohomology ring of $\mathrm{Pol}(r_1, \ldots, r_n)$ is generated by n degree 2 classes $c_1, \ldots, c_n$, all of which square to the same thing. But what are the other relations?

We say that a subset $S \subset \{1, \ldots, n\}$ is **short** if $\sum_{i \in S} r_i < \sum_{j \in S^c} r_j$, and **long** if the opposite inequality holds. Note that every subset is either short or long. Suppose that S is long, and choose an element $i \in S$. It is clear from the preceding discussion that the element

$$\prod_{j \in S \smallsetminus \{i\}} (c_i + c_j)$$

is in the kernel of the Kirwan map, since it is impossible to have every vector in a long subset pointing in the same direction. Amazingly, these are the only extra relations [HK, §7].

Theorem 5.75. *The Kernel of the Kirwan map*

$$\mathbb{Q}[c_1, \ldots, c_n]/\langle c_i^2 - c_j^2 \mid i, j \leq n \rangle \twoheadrightarrow H^*(\mathrm{Pol}(r_1, \ldots, r_n))$$

is generated by expressions of the form

$$\prod_{j \in S \smallsetminus \{i\}} (c_i + c_j)$$

where S is a long subset and $i \in S$.

Example 5.76. The cohomology ring of the equilateral pentagon space is

$$\mathbb{Q}[c_1, \ldots, c_5]/\langle c_i^2 - c_j^2, (c_i + c_j)(c_i + c_k)\rangle.$$

References

[AB] M. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology* 23 (1984) no. 1, 1–28.

[Au] M. Audin. *Torus actions on symplectic manifolds.* Birkhäuser, 2004.

[BL] L. Billera and C. Lee. Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes. *Bull. Amer. Math. Soc.* 2 (1980), no. 1, 181-185.

[CdS] A. Cannas da Silva. *Lectures on symplectic geometry.* Lecture Notes in Mathematics 1764, Springer-Verlag, 2001.

[CS] T. Chang and T. Skjelbred. The topological Schur lemma and related results. *Ann. of Math.* 100 (1974), 307-321.

[CG] N. Chriss and V. Ginzburg. *Representation theory and complex geometry.* Birkhäuser, 1997.

[Da] V. Danilov The geometry of toric varieties. (Russian) *Uspekhi Mat. Nauk* 33 (1978), no. 2(200), 85-134, 247.

[De] T. Delzant. Hamiltoniens périodiques et image convexe de l'application moment. *Bull. Soc. Math. France* 116 (1988), 315-339.

[GKM] M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.* 131 (1998), no. 1, 25-83.

[Ha] A. Hatcher. *Algebraic Topology.* Cambridge, 2002.

[HK] J-C. Hausmann, A. Knutson. The cohomology ring of polygon spaces. *Ann. Inst. Fourier, Grenoble* 48 (1998), no. 1, 281-321.

[Ki] F. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry.* Mathematical Notes 31, Princeton University Press, 1984.

[McC] J. McCleary. *A user's guide to spectral sequences.* Cambridge Studies in Advanced Mathematics 58, Cambridge University Press, 2000.

[McM] P. McMullen. The numbers of faces of simplicial polytopes. *Israel J. Math.* 9 (1971), 559-570.

[Mi] J. Milnor. *Morse theory.* Annals of Mathematical Studies 51, Princeton University Press, 1963.

[Pa] R. Palais. Morse theory on Hilbert manifolds. *Topology* 2 (1963), 299-340.

[Pr] N. Proudfoot. Lectures on toric varieties. http://pages.uoregon.edu/njp/tv.pdf.

[St] R. Stanley. The number of faces of a simplicial convex polytope. *Adv. in Math.* 35 (1980), no. 3, 236-238.