Brieskorn's Construction of Exotic Spheres

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To Richard Friedberg, who taught me how to
pursue beauty in mathematics
with a youthful spirit.

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§1 Definitions and Results

1.1 Definitions

All manifolds in this paper will be compact, oriented, and smooth unless otherwise stated. Our general convention will be to denote closed manifolds with with the letter M and manifolds with boundary with the letters W or X. If W is a manifold with boundary, then $\mathring{W}$ will denote the noncompact manifold $W \setminus \partial W$. When not specified, the word *manifold* will mean manifold with boundary.

A *homotopy n-sphere* is a closed manifold with the homotopy type of S^n. If M is an oriented manifold, let $-M$ denote the manifold obtained by reversing the orientation on M. Two closed, oriented manifolds M_1^n and M_2^n are said to be *h-cobordant* if there exists a compact oriented manifold X^{n+1} with boundary $M_1 \coprod (-M_2)$ such that M_1 and $-M_2$ are each deformation retracts of X. Such a manifold X is called an h-cobordism between M_1 and M_2. Note that if M_1 is simply connected, and X is a manifold with boundary $M_1 \coprod (-M_2)$ that has M_1 as a deformation retract, then $H_k(X, -M_2) \cong H^{n+1-k}(X; M_1) = 0$ for all k, which implies that $-M_2$ is also a deformation retract of X. Thus for $M_1, M_2 \in \Theta_n$ with $n \geq 2$, a cobordism X between M_1 and M_2 with M_1 as a deformation retract is already an h-cobordism. Let Θ_n be the group of h-cobordism classes of oriented homotopy n-spheres under the connect sum operation. The identity element is S^n, and for all $\Sigma \in \Theta_n$, $\Sigma \# (-\Sigma)$ is h-cobordant to S^n [KM].

For $n \geq 5$, the h-cobordism theorem [M2] implies that two simply connected n-manifolds are h-cobordant if and only if they are diffeomorphic. Furthermore, it has been shown that for $n \geq 5$, every homotopy n-sphere is in fact homeomorphic to S^n [Sm], thus we can think of Θ_n as the group of C^∞ differential structures on the topological space S^n up to orientation preserving diffeomorphism. In the context of this paper, however, is it possible to think of Θ_n simply as an h-cobordism group.

Lemma 1.1.1 *A closed, simply connected manifold M is h-cobordant to S^n if and only if M bounds a contractible manifold.*

Proof: Let X^{n+1} be an h-cobordism between M and S^n. By gluing D^{n+1} to X along $S^{n+1} \subset \partial X$, we obtain a manifold W with boundary M. Because S^n is a deformation retract of X, D^{n+1} is a deformation retract of W, and therefore W is contractible.

Conversely, suppose that $M = \partial W$ with W contractible, and let X be the manifold obtained by removing a disk from W. Since W is contractible, S^n will be a deformation retract of X. Then since M is simply connected, X is an h-cobordism. □

A manifold W will be called *parallelizable* if its tangent bundle T_W is trivial. The following lemma will show that the notion of parallelizability descends to h-cobordism classes:

Lemma 1.1.2 *Suppose that two manifolds M_1 and M_2 are h-cobordant, and that M_1 bounds a parallelizable manifold W_1. Then M_2 bounds a parallelizable manifold W_2.*

Proof: Let X be an h-cobordism between M_1 and M_2, and let W_2 be the result of gluing X to W_1 along M_1. Since X is an h-cobordism, W_2 retracts onto W_1. It follows that the obstructions to trivializing T_{W_2} vanish, hence W_2 is parallelizable. Then $M_2 = \partial(-W_2)$ bounds a parallelizable manifold. □

The *boundary connect sum* of two $(n+1)$-manifolds W_1 and W_2 with nonvacuous boundaries is the $(n+1)$ manifold with boundary $\partial W_1 \# \partial W_2$ obtained by smoothing the result of gluing an n-disk in ∂W_1 to an n-disk in W_2. Let $bP_{n+1} \subset \Theta_n$ be the subset of homotopy n-spheres that bound parallelizable manifolds (well defined by Lemma 1.1.2). If W_1^{n+1} and W_2^{n+1} are parallelizable manifolds with boundaries Σ_1, $\Sigma_2 \in bP_{n+1}$, then the boundary connect sum $W_1 \# W_2$ is also parallelizable, therefore bP_{n+1} is a subgroup. The purpose of this paper will be to compute the groups bP_{n+1} for all $n \geq 5$ (Chapter 3), and to give explicit constructions of their elements (Chapter 6). The construction that we use, due originally to Brieskorn [Bk], will give us two different perspectives from which we can gain a geometric understanding for these manifolds. They will be constructed first as algebraic varieties, and then interpreted in a knot theoretic context as cyclic branched covers of the standard sphere.

1.2 Results

In the course of this paper we will give multiple interpretations of the groups bP_{n+1}. In Chapter 2 we will show that bP_{n+1} is the kernel of a homomorphism from Θ_n to a quotient of the stable n-stem $\pi_n^s = \pi_{n+k}(S^k)$ for $k \geq n+1$. The groups π_n^s are known to be finite for all $n > 0$, therefore we will conclude that Θ_n is a finite extension of bP_{n+1}. In Chapter 3 we will use surgery to show that bP_{n+1} is itself finite, therefore so is Θ_n. The following table gives Kervaire and Milnor's computation of the orders of Θ_n and bP_{n+1} for $5 \leq n \leq 18$:

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\lvert\Theta_n\rvert$	1	1	28	2	8	6	992	1	3	2	16,256	2	16	16
$\lvert bP_{n+1}\rvert$	1	1	28	1	2	1	992	1	1	1	8,128	1	2	1

In Chapter 4 we review the tools necessary to give a knot theoretic interpretation to Brieskorn's construction, which we describe in Chapter 5. Brieskorn considers polynomials of the form $f(z) = z_0^{a_0} + \ldots + z_n^{a_n}$, which have an isolated singularity at the origin of $\mathbb{C}^{n+1}$. If we intersect the zero set of f with a sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ centered at the origin, we get a manifold Σ which is called the *link* of this singularity. We will study the topology of this link by realizing it as the cyclic branched cover of S^{2n-1} along another link, and show that in many situations Σ will be a homotopy sphere. In Chapter 6, we show that the homotopy spheres that arise as links of Brieskorn singularities are exactly those that bound parallelizable manifolds. A byproduct of the knot theoretic approach will be a fourth interpretation of bP_{n+1}: we will show that an exotic sphere of dimension n bounds a parallelizable manifold if and only if it embeds into the standard sphere S^{n+2}.

§2 Vector Bundles

2.1 Homotopy Properties of Classical Groups

We begin this section by proving three lemmas about the homotopy properties of the groups SO_k, which we will use in our subsequent calculations. The results that we derive are mostly elementary, but Lemma 2.1.4 will rely on the not so elementary fact that the tangent bundle to S^k is nontrivial for $k \neq 1, 3$, or 7 (Theorem 2.1.9). At the end of the section we show that this theorem is a consequence of Bott Periodicity. It is an interesting side note that Theorem 2.1.9 can be derived independently from either real *or* complex periodicity.

Our main object of study will be the long exact sequence of the fibration $SO_k \hookrightarrow SO_{k+1} \to S^k$:

$$\pi_k(SO_{k+1}) \xrightarrow{(p_k)_*} \pi_k(S^k) \xrightarrow{\partial_k} \pi_{k-1}(SO_k) \xrightarrow{(s_k)_*} \pi_{k-1}(SO_{k+1}) \to \pi_{k-1}(S^k) = 0.$$

Lemma 2.1.1 *Let $\gamma \in \pi_k(S^k)$ be the homotopy class of the identity map $S^k \to S^k$. The boundary map ∂_k takes γ to $[T_{S^k}] \in \pi_{k-1}(SO_k)$, the obstruction to trivializing the tangent bundle to S^k.*

Proof: Fix a point $e \in S^k$, and let $\pi : (D^k, S^{k-1}) \to (S^k, e)$ be the standard projection. It is possible to lift γ to some $\tilde{\gamma} : (D^k, S^{k-1}) \to (SO_{k+1}, SO_k)$ in $\pi_k(SO_{k+1}, SO_k) \cong \pi_k(S^k)$, where $\tilde{\gamma}(q)$ is a transformation of S^k taking $\pi(q)$ to e. Away from a neighborhood of e, we can trivialize T_{S^k} by mapping $T_{S^k}|_p$ to $T_{S^k}|_e$ via the linear map $\tilde{\gamma}(\pi^{-1}p)_*$ for all p. This is poorly defined at e, because $\pi^{-1}e$ is not a single point, but rather an entire S^k. The obstruction $[T_{S^k}]$ to extending this trivialization over e is exactly $\tilde{\gamma}|_{S^{k-1}} = \partial_k(\gamma)$. □

Lemma 2.1.2 *Let $[\xi] \in \pi_k(SO_{k+1})$ be the obstruction to trivializing an oriented $(k+1)$-plane bundle ξ over S^{k+1}. Then $(p_k)_*$ takes $[\xi]$ to the Euler class $e(\xi) \in \pi_k(S^k)$.*

Proof: This is a direct consequence of the definition of the Euler class as the obstruction to sectioning ξ, which lies in $\pi_k(SO_{k+1}/SO_k) = \pi_k(S^k)$. □

Corollary 2.1.3 *The map $(p_{k-1})_* \circ \partial_k : \pi_k(S^k) \to \pi_{k-1}(S^{k-1})$ is given by multiplication by the Euler number $\chi(S^k)$, which is equal to 2 if k is even, and 0 if k is odd.*

Note that $(s_k)_*$ is surjective, and that for $N \geq k+1$, the fibration $SO_N \hookrightarrow SO_{N+1} \to S^N$ induces an isomorphism $\pi_{k-1}(SO_N) \to \pi_{k-1}(SO_{N+1})$. Thus $(s_k)_* : \pi_{k-1}(SO_k) \to \pi_{k-1}(SO)$ can be thought of as a stabilization map, with kernel L_k. Since $\pi_k(S^k)$ is generated by γ, L_k is generated by $\partial_k(\gamma)$. If k is even, then Corollary 2.1.3 tells us that $(p_{k-1})_* \circ \partial_k(\gamma)$ has infinite order, therefore $\partial_k(\gamma)$ has infinite order and $L_k \cong \mathbb{Z}$.

Lemma 2.1.4 *Suppose that k is odd. Then $L_k = 0$ if $k = 1, 3$, or 7, and $\mathbb{Z}_2$ otherwise.*

Proof: Since k is odd, $(p_k)_*$ maps $[T_{S^{k+1}}]$ to $e(T_{S^{k+1}}) = 2\gamma \in \pi_k(S^k)$, therefore $\partial_k(2\gamma) = 0$ by exactness. Since L_k is generated by $\partial_k(\gamma)$, its order is at most 2. We showed in Lemma 2.1.1 that $\partial_k(\gamma) = 0$ if and only if T_{S^k} is trivial, therefore $L_k = 0$ if and only if $k = 1, 3,$ or 7. □

Let η, ξ be vector bundles on S^n. If there exists $r, s \in \mathbb{Z}^+$ such that $\eta \oplus \epsilon^r$ is isomorphic to $\xi \oplus \epsilon^s$, then η and ξ are said to be *stably equivalent*. Bott Periodicity gives a classification of oriented real and complex vector bundles over S^n up to stable equivalence.

One way to do this is to note that an oriented real vector bundle ξ of rank k over S^n is defined by its characteristic map $f : S^{n-1} \to SO_k$, which can also be identified as a representative of the homotopy class $[\xi] \in \pi_{n-1}(SO_k)$ of the obstruction to trivializing ξ. To classify oriented real vector bundles of rank k over S^n is to classify characteristic maps up to homotopy, i.e. to compute the group $\pi_{n-1}(SO_k)$. To classify oriented real vector bundles of any rank over S^n up to stable equivalence is to compute the stable homotopy group $\pi_{n-1}(SO) = \pi_{n-1}(SO_k)$ for any $k \geq n+1$. Similarly, to classify complex vector bundles of any rank over S^n up to stable equivalence is to compute $\pi_{n-1}(U) = \pi_{n-1}(U_k)$ for any $k \geq n/2$. Thus we may give our first statement of Bott Periodicity:

Theorem 2.1.5 (Bott Periodicity) *For $n \geq 2$,*

$$\pi_{n-1}(SO) = \begin{cases} 0 & \text{if } n \equiv 3, 5, 6, \text{ or } 7 \text{ mod } 8; \\ \mathbb{Z} & \text{if } n \equiv 0 \text{ or } 4 \text{ mod } 8; \\ \mathbb{Z}_2 & \text{if } n \equiv 1 \text{ or } 2 \text{ mod } 8; \end{cases} \qquad \text{and} \qquad \pi_{n-1}(U) = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ \mathbb{Z} & \text{if } n \text{ is even.} \end{cases}$$

This is the form in which Bott originally stated the theorem [Bo]. For our applications, however, we will use a slightly stronger formulation. The statement that $\pi_{2n-1}(U) = \mathbb{Z}$ is equivalent to the statement that the reduced K group $\tilde{K}(S^{2n})$ is infinite cyclic. We want to go further and specify a generator for this group. First, note that

$$\tilde{K}(S^{2n}) = K(S^{2n}, pt) \cong K((S^2, p_1) \times \ldots \times (S^2, p_n)) \cong \tilde{K}(S^2) \otimes \ldots \otimes \tilde{K}(S^2).$$

The stronger version of Bott Periodicity asserts that the generator ξ_n of $\tilde{K}(S^{2n})$ can be identified with the tensor product $\xi_1 \otimes \ldots \otimes \xi_1$ of n copies of the generator of $\tilde{K}(S^2)$, where ξ_1 is the difference between the canonical line bundle on $\mathbb{C}P^1 \cong S^2$ and the rank 2 trivial bundle on S^2. The real picture is less simple, but works in a similar manner: the generator of the real reduced K group $\tilde{KO}(S^{n+8})$ can be expressed as the tensor product of the generators of $\tilde{KO}(S^n)$ and $\tilde{KO}(S^8)$. We now derive some consequences, first of complex periodicity and then of real periodicity.

Lemma 2.1.6 *Let η be a vector bundle of rank $r > 4m$ on a closed manifold M^{4m}. Let f be a trivialization of η away from a disk, and let $\alpha \in \pi_{4m-1}(SO) \cong \mathbb{Z}$ the obstruction to extending f over the disk. Then the top Pontrjagin class $p_m(\eta) \in \pi_{4m-1}(U/U_{2m-1}) \cong \mathbb{Z}$ is equal to $\pm a_m \cdot (2m-1)! \cdot \alpha$, where $a_m = 1$ or 2.*

Proof: Consider the inclusion $i : SO_r \to U_r$, and let i_* be the induced map on π_{4m-1}. By Bott Periodicity $\pi_{4m-1}(SO_r) \cong \pi_{4m-1}(U_r) \cong \mathbb{Z}$, therefore i_* is given by multiplication by some integer a_m. To see that $\pm a_m = 1$ or 2, consider the inclusion $j : U_r \to SO_{2r}$ given by forgetting the complex structure. The composition $j \circ i$ takes a matrix A to $A \oplus A$, therefore $(j \circ i)_*$ is given by multiplication by 2. It follows that i_* is given by multiplication by ± 1 or ± 2. Bott [Bo] shows that $a_m = 1$ if m is even, and 2 if m is odd.

Consider the relative K group $K(M, M \setminus \mathring{D})$, where $\mathring{D}$ is an open disk neighborhood of p. We can identify this group with $\tilde{K}(S^{4m})$ by excision, therefore the complex virtual bundle $\eta \otimes \mathbb{C} - \epsilon^r$ of rank 0 can be identified with $q \cdot \xi_{2m}$ for some $q \in \mathbb{Z}$. Explicitly, q is equal to the obstruction $[\eta \otimes \mathbb{C}] = i_*[\eta] = a_m \cdot \alpha$ to trivializing $\eta \otimes \mathbb{C}$. Then $p_m(\eta) = \pm c_{2m}(\eta \otimes \mathbb{C}) = \pm a_m \cdot \alpha \cdot c_{2m}(\xi_{2m})$, thus we have reduced Lemma 2.1.6 to the statement that $c_{2m}(\xi_{2m}) = (2m-1)!$.

We will in fact prove the slightly more general statement that $c_k(\xi_k) = (-1)^k(k-1)!$ for any $k \in Z^+$, even or odd. Since we are on the sphere S^{2k}, the total Chern class $c(\xi_k)$ is equal to $1 + c_k(\xi_k)$. By the Splitting Principle, ξ_k splits into a direct sum of complex line bundles $L_1, \ldots, L_k$ over a space whose cohomology ring contains $H^*(S^{2k})$ as a subring. Then the Chern polynomial $t^k + c_k(\xi_k)$ factors as $(t + \alpha_1) \ldots (t - \alpha_k)$, where $\alpha_i = c_1(L_i)$ for all i. Note that when we evaluate at $t = -\alpha_i$, we get $\alpha_i^k = (-1)^k c_k(\xi_k)$ for all i.

Consider the Chern character

$$ch(\xi_k) = \sum_{i=1}^{k} e^{\alpha_i} = \sum_{j=0}^{\infty} \frac{1}{j!} \sum \alpha_i^j = \sum_{j=0}^{\infty} \frac{(-1)^k}{(j-1)!} c_j(\xi_k) \in H^*(S^k; \mathbb{Q}).$$

Evaluating on the fundamental homology class u_k of S^{2k}, we get

$$\frac{(-1)^k}{(k-1)!} \langle c_k(\xi_k), u_k \rangle = \langle ch(\xi_k), u_k \rangle = \left\langle ch(\otimes^k \xi_1), \otimes^k u_1 \right\rangle = \langle ch(\xi_1), u_1 \rangle^k = \langle c_1(\xi_1), u_1 \rangle^k = 1,$$

therefore $c_k(\xi_k) = (-1)^k(k-1)!$. This completes the proof of Lemma 2.1.6. □

We will now use Bott Periodicity for oriented real vector bundles to show that the tangent bundle to S^k is nontrivial for $k \neq 1, 3, 7$. We begin with a pair of lemmas about Steifel-Whitney classes.

Lemma 2.1.7 *Let ξ be an SO_m bundle over S^m and η an SO_n bundle over S^n, where n is a power of 2. Then the SO_{mn} bundle $\xi \otimes \eta$ over $S^m \times S^n$ has total Steifel-Whitney class $w(\xi \otimes \eta) = \big(1 + w_n(\eta)\big)^m + w_m(\xi)^n$.*

Proof: By the Splitting Principle, we can pass to a space where we have $P_\xi(t) = t^m + w_m(\xi) = (t + \alpha_1) \ldots (t + \alpha_m)$ and $P_\eta(t) = t^n + w_n(\eta) = (t + \beta_1) \ldots (t + \beta_n)$. Then

$$w(\xi \otimes \eta) = \prod_{i,j} (1 + \alpha_i + \beta_j) = \prod_i P_\eta(1 + \alpha_i) = \prod_i \big((1 + \alpha_i)^n + w_n(\eta)\big).$$

Since n is a power of 2, this reduces to

$$\prod_i (1+\alpha_i^n + w_n(\eta)) = \prod_i \left((1+w_n(\eta)) + \alpha_i^n\right) = \left(1+w_n(\eta)\right)^m + w_m(\xi)^n,$$

because for every i, α_i^n is a root of the polynomial $t^m - w_m(\xi)^n$. □

Lemma 2.1.8 *Let ξ and η be as above. If $m > 2$ is a power of 2, or if $w(\xi) = w(\eta) = 1$, then $w(\xi \otimes \eta) = 1$.*

Proof: We will use the expression for $w(\xi \times \eta)$ derived in Lemma 2.1.7. If $w(\xi) = w(\eta) = 1$, then it is immediate that $w(\xi \otimes \eta) = 1$. Now suppose that $m > 2$ is a power of 2, with no assumptions about $w(\xi)$ or $w(\eta)$. Then $w(\xi \otimes \eta) = 1 + w_m(\xi)^n + w_n(\eta)^n$. Recall that $(\xi - \epsilon^{\mathrm{rk}\,\xi}) \otimes (\eta - \epsilon^{\mathrm{rk}\,\eta})$ is an element of $\tilde{KO}(S^{m+n})$, therefore only $w_{m+n}(\xi \otimes \eta)$ can be nontrivial. Since $m > 2$, mn is not equal to $m+n$, therefore $w(\xi \otimes \eta) = 1$. □

Theorem 2.1.9 *The tangent bundle to S^k is nontrivial for $k \neq 1, 3, 7$.*

Proof: Suppose that $v_1, \ldots, v_k$ is a set of orthonormal sections of T_{S^k}. The the map $S^k \to SO_{k+1}$ taking x to the frame $(x, v_1(x), \ldots, v_k(x))$ defines a rank $k+1$ vector bundle ξ on S^{k+1}. By the definition of the Euler class of a bundle as the index of a generic section, we see that $e(\xi) = 1$, therefore $w_{k+1}(\xi) = 1$. Thus we can reduce Theorem 2.1.9 to the claim that the top Steifel-Whitney class of every SO_{k+1} bundle on S^{k+1} vanishes for $k \neq 1, 3, 7$. We will proceed by a 16-fold induction on k.

Consider an SO_N bundle γ on the sphere S^N with $9 \leq N \leq 16$. By Bott Periodicity, the class $\gamma - \epsilon^N \in \tilde{KO}(S^N)$ can be expressed as the tensor product of virtual bundles $\xi - \epsilon^{\mathrm{rk}\,\xi}$ on S^8 and $\eta - \epsilon^{\mathrm{rk}\,\eta}$ on S^q, with $q = 1, 2, 4$, or 8. (If N is not congruent mod 8 to a power of 2, then Bott Periodicity says that γ is stably trivial.) Since the stabilization map $(s_k)_* : \pi_{k-1}(SO_k) \to \pi_{k-1}(SO)$ is surjective, we may take ξ and η such that $\mathrm{rk}\,\xi = 8$ and $\mathrm{rk}\,\eta = q$. Then we can conclude by Lemma 2.1.8 that $w(\gamma) = w(\gamma - \epsilon^N) = 1$. The same holds if $17 \leq N \leq 24$, because any rank 0 virtual bundle on S^N can be expressed as the tensor product of virtual bundles on S^{16} and S^q. Then using the second statement of Lemma 2.1.8, we can conclude that any bundle γ of rank $N+16$ on S^{N+16} has $w(E) = 1$. This provides an inductive proof of Theorem 2.1.9. □

Remark 2.1.10 It is also possible to prove Theorem 2.1.9 using Lemma 2.1.6, which is derived from complex periodicity. For an outline of this proof, see [BM].

2.2 Stable Bundle Theory

In this section we will exploit the machinery that was developed and stated in Section 2.1. A vector bundle ξ over a manifold W is called *stably trivial* if there exists $r \geq 0$ such that $\xi \oplus \epsilon^r$ is trivial. A manifold W will be called stably parallelizable if its tangent bundle T_W is stably trivial. As an application of Bott Periodicity, along with a difficult theorem of Adams that we will not prove, we will show that homotopy spheres are stably parallelizable. We will not need this result in our computations, but it provides a second interpretation of the groups bP_{n+1} in terms of the Pontrjagin-Thom construction and the classical J-homomorphism $\pi_{n-1}(SO) \to \pi_n^s$.

Lemma 2.2.1 *Let W be a manifold of dimension n, and let ξ be a vector bundle on W of rank $k > n$. Then ξ is trivial if and only if $\xi \oplus \epsilon^1$ is trivial. If ∂W is nonvacuous, then the same result holds for all ξ of rank $k \geq n$.*

Proof: Let $f : W \to BSO_k$ classify ξ, and consider the fibration $S^k \hookrightarrow BSO_k \xrightarrow{\pi} BSO_{k+1}$. Then $\xi \oplus \epsilon^1$ is trivial if and only if $\pi \circ f$ null-homotopic, which implies that f is homotopic to a map into the fiber S^k. Since $n < k$ (or $n = k$ and ∂W is nonvacuous), any map of W into S^k is null-homotopic, therefore ξ itself is trivial. □

Corollary 2.2.2 *If W is a manifold with nonvacuous boundary, then W is stably parallelizable if and only if it is parallelizable. In particular, if $\Sigma \in \Theta_n$ bounds a stably parallelizable manifold, then $\Sigma \in bP_{n+1}$.*

As a demonstration of the usefulness of the notion of stable parallelizability, we give the following Lemma, which we will apply in Sections 3.2 and 3.4.

Lemma 2.2.3 *The intersection form on a $(2m-1)$-connected stably parallelizable manifold W^{4m} is even.*

Proof: By the Hurewicz theorem, every $\lambda \in H_{2m}(W)$ is represented by a spherical immersion $f : S^{2m} \to W$, which we may assume has only transverse double points. Let ν be the normal bundle on S^{2m} induced by f. Then a parallel copy of $\lambda = [f]$ intersects λ once for every zero of a generic section of ν, plus twice near each double point of f. We therefore have $\lambda \cdot \lambda \equiv e(\nu) \mod 2$. By Lemma 2.1.2, $e(\nu) = (p_{2m})_*[\nu]$, where $[\nu] \in \pi_{2m}(SO_{2m-1})$ is the obstruction to trivializing ν. But ν is stably trivial by stable triviality of T_{S^k} and T_W, therefore $[\nu] \in \mathrm{Ker}(s_{2m})_* = \mathrm{Im}(\partial_{2m})$. Then by Lemma 2.1.1, $[\nu]$ is a multiple of $[T_{S^{2m}}]$, and $e(\nu)$ is a multiple of $e(T_{S^{2m}}) = \chi(S^{2m}) = 2$. Thus $\lambda \cdot \lambda \equiv 0 \mod 2$. □

Let W be a submanifold of S^N with tangent bundle T_W and normal bundle ν. Since the tangent bundle to S^N can be trivialized away from a point, its restriction $T_W \oplus \nu$ to W is trivial. It follows that T_W is stably trivial if and only if ν is stably trivial. Then Lemma 2.2.1 tells us that any embedding of a stably parallelizable manifold W^n into S^{2n+1} induces a trivial normal bundle. This result generalizes to the statement that in large enough codimension, the normal bundle is independent of embedding:

Proposition 2.2.4 *Let W^n be any manifold, not necessarily stably parallelizable. Then for any $N \geq 2n+1$, any two embeddings $f, g : W \hookrightarrow S^N$ induce isomorphic normal bundles.*

Proof: Let $h : W \times [0,1] \to S^N$ be a homotopy between f and g, and let $H : W \times [0,1] \to S^N \times [0,1]$ take (x,t) to $(h(x,t), t)$. Since $N \geq 2n+1$, H can be homotoped to an immersion $\tilde{H}$ without moving the boundary [W2]. Then the normal bundle to $W \times [0,1]$ in $S^N \times [0,1]$ is an isotopy between the normal bundle to f and the normal bundle to g. □

Suppose that we are given a closed submanifold $M^n \subset S^{n+k}$, along with a trivialization σ of the normal bundle to M. We can think of this trivialization as a function f_σ from a closed tubular neighborhood X of M in S^{n+k} to the unit disk D^k, such that the boundary of the tubular neighborhood is mapped to the boundary of the disk. Let $g_\sigma : X \to S^k$ be the composition of f_σ with the map $D^k \to S^k$ that contracts the boundary of the disk to a point. Finally, let $G_\sigma : S^{n+k} \to S^k$ be the extension of g_σ obtained by sending the entire complement of X to a single point - the image of the boundary of D^k. The association $(M, \sigma) \mapsto [G_\sigma] \in \pi_{n+k}(S^k)$ is called the Pontrjagin-Thom construction [M3]. It descends to a homomorphism from the framed cobordism group $\Omega^{\mathrm{fr}}_{n,k}$ to $\pi_{n+k}(S^k)$, and in the stable range $k \geq n+1$ the Pontrjagin-Thom construction gives an isomorphism between the stable groups Ω^{fr}_n and π^s_n [Po].

Consider a trivial embedding $S^{n-1} = \partial D^n \subset S^{n+k-1}$. Let ν be a trivialization of the normal bundle to D^n (all choices of trivialization are homotopic), and let σ be the normal frame S^{n-1} given by the restriction of ν to S^{n-1}, along with the outward normal vector to $S^{n-1} \subset D^n$. For any $\alpha \in \pi_{n-1}(SO_k)$, we can define a new trivialization σ_α of the normal bundle to S^{n-1} by twisting σ. Explicitly, this means that we put $\sigma_\alpha|_p = \alpha(p) \cdot \sigma|_p$ for all $p \in S^{n-1}$. Composing with the Pontrjagin-Thom construction, we get a homomorphism $J_n : \pi_{n-1}(SO_k) \to \pi_{n-1}(S^{n+k-1})$, which in the stable range $k \geq n$ maps $\pi_{n-1}(SO)$ to π^s_{n-1}.

Consider an element $\alpha \in \pi_{n-1}(SO)$. Because the Pontrjagin-Thom construction is an isomorphism in the stable range, $J_n(\alpha) = 0$ if and only if the framed manifold (S^{n-1}, σ_α) is null-cobordant. This observation can be restated as follows:

Lemma 2.2.5 *$J_n(\alpha) = 0$ if and only if there exists a closed manifold M^n and a trivialization f of the (stable) normal bundle to M away from a point p such that α is the obstruction to extending f over p.*

Theorem 2.2.6 (Kervaire-Milnor) *Homotopy spheres are stably parallelizable.*

Proof: The only obstruction to trivializing the stable tangent bundle to a homotopy sphere $\Sigma \in \Theta_n$ is a class $\mathfrak{o}_n(\Sigma) \in H^n(\Sigma;\ \pi_{n-1}(SO)) = \pi_{n-1}(SO)$. We now break the proof up into cases corresponding to the residue class of n mod 8.

Case 1: $n \equiv 3, 5, 6,$ or 7 mod 8. $\quad \pi_{n-1}(SO) = 0 \Rightarrow \mathfrak{o}_n(\Sigma) = 0$.

Case 2: $n \equiv 1$ or 2 mod 8. Here we rely on Adams' analysis of the kernel of the J-homomorphism in the stable range [Ad]:

Theorem 2.2.7 (Adams) 1) *If $n \equiv 1$ or 2 mod 8, then J_n is injective in the stable range.*
2) *If $n \equiv 0$ or 4 mod 8, then we have $n = 4m$, and $\mathrm{Im}(J_{4m})$ has order $j_m = \mathrm{denominator}(\frac{B_m}{4m})$ in the stable range, where B_m is the m^{th} Bernoulli number.*

Remark 2.2.8 Note that J_n could not possibly be injective for $n \equiv 0$ or 4 mod 8, because $\pi_{n-1}(SO)$ is infinite and π_n^s is always finite. If $n \equiv 3, 5, 6,$ or 7 mod 8, then $\pi_n(SO) = 0$ and J_n is trivial in the stable range.

By Lemma 2.2.5, $J_n(\mathfrak{v}_n(\Sigma)) = 0$, in which case Theorem 2.2.7 tells us that $\mathfrak{v}_n(\Sigma) = 0$.

Case 3: $n \equiv 0$ or 4 mod 8. Let $n = 4m$, and apply Lemma 2.1.6 to the manifold Σ with its stable tangent bundle. This lemma says that $p_m[\Sigma] = \pm a_m \cdot (2m-1)! \cdot \mathfrak{v}_n(\Sigma)$, but the Hirzebruch Signature Formula tells us that $p_m[\Sigma]$ is proportional to $\sigma(\Sigma) = 0$, therefore $\mathfrak{v}_n(\Sigma) = 0$. This completes the proof of Theorem 2.2.6. □

Theorem 2.2.6 gives us a new interpretation of the groups bP_{n+1}. Using the Pontrjagin-Thom construction, a homotopy sphere Σ^n and a trivialization τ of its stable normal bundle determine an element G_τ of the stable n-stem π_n^s. For any different trivialization τ', (Σ, τ') will be framed cobordant to $(\Sigma, \tau)\#(S^n, \sigma)$ for some trivialization σ of the stable normal bundle to S^n, thus G_τ and $G_{\tau'}$ differ by an element of $\mathrm{Im}(J)$. If Σ_1 is h-cobordant to Σ_2 and τ is a stable normal frame of Σ_1, then τ extends over the h-cobordism to a stable normal frame of Σ_2 that determines the same element of π_n^s. We can therefore define a homomorphism $\Theta_n \to \pi_n^s/\mathrm{Im}(J)$ that takes Σ to the image of element of π_n^s determined by any stable normal framing of Σ. A homotopy sphere Σ is in the kernel of this map if and only if it bounds a manifold W with a trivial (stable) normal bundle. We have shown that this is equivalent to bounding a stably parallelizable manifold, and by Corollary 2.2.2 every stably parallelizable manifold with nonvacuous boundary is parallelizable, hence the kernel is precisely bP_{n+1}. Furthermore Θ_n/bP_{n+1} is isomorphic to the image of this homomorphism, which is a subgroup of the finite group $\pi_n^s/\mathrm{Im}(J)$. Hence we can conclude that for all n, Θ_n is a finite extension of bP_{n+1}.

§3 Computation of the Groups bP_{n+1}

3.1 Surgery

In this section we will develop the techniques of surgery required for our study of exotic spheres, closely following Kervaire and Milnor's exposition in [KM]. Let M^n be a possibly noncompact manifold without boundary, with $n = p + q + 1$. Let $f : S^p \times D^{q+1} \to M$ be a differentiable embedding, and let X be the space $M \times [0,1] \cup_f D^{p+1} \times D^{q+1}$, where f is thought of as identifying $S^p \times D^{q+1} \subset D^{p+1} \times D^{q+1}$ with its image in $M \cong M \times \{0\} \subset M \times [0,1]$. X can be smoothed into a manifold with boundary, where $\partial X = M \coprod M'$ for some closed manifold M'. We call $M' = \chi(M, f)$ the result of surgery[1] on M along f, and we call X the surgery cobordism between M and M'. If N can be obtained from M by a finite sequence of surgeries, we say that M and N are χ-equivalent.

Proposition 3.1.1 *χ-equivalence is an equivalence relation.*

Proof: Let $M' = \chi(M, f)$, $f : S^p \times D^{q+1} \hookrightarrow M$. There is a copy of $D^{p+1} \times S^q$ sitting inside M', coming from the part of the boundary of $D^{p+1} \times D^{q+1}$ that is not glued to $M \times \{0\}$. Define $f' : S^q \times D^{p+1} \to M'$ by identifying $S^q \times D^{p+1}$ with $D^{p+1} \times S^q \subset M'$. Then $\chi(M', f') \cong M$. □

The definitions of surgery and χ-equivalence can be easily extended to manifolds with boundary. Let W^n be a manifold with boundary, $n = p + q + 1$, and let $f : S^p \times D^{q+1} \to \mathring{W}$ be a differentiable embedding. Then $W' = \chi(W, f)$ is defined by taking $\chi(\mathring{W}, f)$ and gluing back the boundary. Then χ-equivalence is an equivalence relation on the set of manifolds with boundary. The motivation for defining the technique of surgery is that it can be used to kill homotopy groups of manifolds without altering the cobordism class (for closed manifolds) or the boundary (for manifolds with boundary).

Proposition 3.1.2 (Milnor) *Let $f : S^p \times D^{q+1} \to \mathring{W}$ take a generator of $\pi_p(S^p \times D^{q+1})$ to $\beta \in \pi_p(W)$, and let $W' = \chi(W, f)$. Then $\pi_i(W') = \pi_i(W)$ for all $i < \mathrm{Min}(p, q)$, and if $p < q$, then $\pi_p(W') \cong \pi_p(W)/B$ for some subgroup B containing β.*

Proof: Let $X = W \times [0,1] \cup D^{p+1} \times D^{q+1}$. X has $W \cup (D^{p+1} \times \{0\})$ as a deformation retract, hence $\pi_i(W) \to \pi_i(X)$ is an isomorphism for $i < p$, and a surjection for $i = p$. Furthermore, $\beta \in \ker(\pi_p W \to \pi_p X)$.

Similarly, X has $W' \cup (D^{q+1} \times \{0\})$ as a deformation retract, and $\pi_i(W') \to \pi_i(X)$ is an isomorphism for $i < q$. This completes the proof. □

Let W be a $(p-1)$-connected stably parallelizable manifold of dimension $n > 2p$, and let $\beta \in \pi_p(W)$ given. In order to surger W along β, we must be able to represent β by an embedding $f : S^p \times D^{q+1} \to \mathring{W}$. By the Hurewicz theorem β is spherical, and because $2p < n$, β is represented by an embedded sphere. The normal bundle to a sphere is stably trivial, and by Lemma 2.2.1,

[1] Also called spherical modification [KM].

$q+1>p$ implies that the normal bundle is in fact trivial. Therefore β is always represented by an embedding $f: S^p \times D^{q+1} \to \mathring{W}$.

This is not quite enough to establish that our procedure for killing homotopy groups is effective. We must know that we can always surger in such a way as to preserve stable parallelizability, otherwise we may be able to apply Proposition 3.1.2 only once. It is in fact possible to preserve stable parallelizability, which we will demonstrate by proving an even stronger result. We will show that given any homotopy class $\lambda \in \pi_p(W)$ and *any* stable trivialization τ of the tangent bundle to W, we can choose $f: S^p \times D^{q+1} \to \mathring{W}$ representing λ in such a way so that surgery along f does not destroy τ. We will make this precise with the notion of *framed surgery.*

Let (W^n, τ) be a manifold along with a trivialization of the stable normal tangent $T_W \oplus \epsilon^1$, and let $f: S^p \times D^q \to M$ be an embedding with $n = p+q+1$. Suppose that there exists a trivialization σ of the normal bundle to the surgery cobordism X between $\mathring{W}$ and $\mathring{W}' = \chi(\mathring{W}, f)$, such that σ restricts to τ on $\mathring{W}$. The trivialization σ also restricts to a trivialization $\sigma|_{\mathring{W}'}$ of $T_{\mathring{W}'} \oplus \epsilon^1$, which extends (up to homotopy) to a trivialization μ of $T_{W'} \oplus \epsilon^1$. We say that the framed manifold (W', μ) is framed χ-equivalent to (W, τ).

Consider a framed manifold (W^n, τ) and an embedding $f: S^p \times D^{q+1} \to \mathring{W}$, $n = p+q+1$. The i^{th} obstruction to trivializing the tangent bundle of the surgery cobordism X lies in the group $H^{i+1}\big(X, W \times [0,1]; \pi_i(SO_{n+1})\big) = H^{i+1}\big(D^{p+1} \times D^{q+1}, S^p \times D^{q+1}; \pi_i(SO_{n+1})\big)$. This group is trivial unless $i = p$, in which case it is isomorphic to $\pi_p(SO_{n+1})$. Let $\gamma(f) \in \pi_p(SO_{n+1})$ be the p^{th} obstruction to trivializing this bundle. Now consider a differentiable map $\alpha: S^p \to SO_{q+1}$, and define $f_\alpha: S^p \times D^{q+1} \to \mathring{W}$ by $f_\alpha(u,v) = f(u, \alpha(u) \cdot v)$. Since f_α is obtained from f by precomposing with an automorphism of D^{q+1}, f and f_α represent the same homotopy class. Thus we would like to show that we can always choose $\alpha \in \pi_p(SO_{q+1})$ such that $\gamma(f_\alpha) = 0$.

Proposition 3.1.3 *The new obstruction $\gamma(f_\alpha)$ is equal to $\gamma(f) + s_*(\alpha)$, where $s_*: \pi_p(SO_{q+1}) \to \pi_p(SO_{n+1})$ is induced by the inclusion $s: SO_{q+1} \hookrightarrow SO_{n+1}$.*

Proof: We follow the argument in [KM]. Let $t^{n+1} = e^{p+1} \times e^{q+1}$ be the standard trivialization of the tangent bundle to $D^{p+1} \times D^{q+1}$, and let $i: D^{p+1} \times D^{q+1} \to X$ be the natural inclusion. Then at every point $x \in f(S^p \times \{0\}) \subset W \subset X$, i induces a trivialization $i_*(t^{n+1})|_x$ of the tangent space $TX|_x = TW \oplus \epsilon^1|_x$. The obstruction $\gamma(f)$ is the homotopy class of the map $g: S^p \to SO_{n+1}$ obtained by comparing $\tau|_x$ to $i_*(t^{n+1})|_x$ at each point $x \in f(S^p \times \{0\})$. Passing from f to f_α has the effect of replacing $i: D^{p+1} \times D^{q+1} \to X$ with a new embedding $i_\alpha: D^{p+1} \times D^{q+1} \to X$, and we have

$$\begin{aligned} i_{\alpha *}(t^{n+1})|_x &= i_*(e^{p+1})|_x \times (f_\alpha)_*(e^{q+1})|_x \\ &= i_*(e^{p+1})|_x \times \alpha(x) \cdot f_*(e^{q+1})|_x \\ &= i_*(t^{n+1})|_x \cdot s_* \circ \alpha(x). \end{aligned}$$

The proposition follows. □

When $p < q$, both groups are stable and s_* is an isomorphism. When $p = q$, s_* is surjective by the exact sequence in Section 2.1, thus given any $f : S^p \times D^{q+1} \hookrightarrow \mathring{W}$ with $p \leq q$, there exists $\alpha \in \pi_p(SO_{q+1})$ such that surgery along f_α can be framed. In particular, this implies that if W is stably parallelizable and $\lambda \in \pi_p(W)$ is represented by an embedding $f : S^p \times D^{q+1} \to \mathring{W}$ with $p \leq q$, then we can always choose f in such a way so that $W' = \chi(W, f)$ is stably parallelizable. This gives us the following theorem:

Theorem 3.1.4 *A stably parallelizable manifold of dimension $n \geq 2k$ is χ-equivalent to a stably parallelizable, $(k-1)$-connected manifold.*

We will now investigate the possibility of killing the middle homology group of an even-dimensional manifold by (nonframed) surgery. A vector basis $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r\}$ is said to be *weakly symplectic* with respect to a given symmetric or skew-symmetric bilinear form if $\alpha_i \cdot \alpha_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{ij}$ for all i, j. A weakly symplectic basis is called *symplectic* if $\beta_i \cdot \beta_j = 0$ for all i, j.

Theorem 3.1.5 *Suppose that W is a $(k-1)$-connected manifold of dimension $n = 2k \geq 6$, and that $H_k(W)$ has a weakly symplectic basis $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r\}$ with respect to the intersection form. Suppose further that each α_i is represented by an embedded sphere with trivial normal bundle. Then W is χ-equivalent to a contractible manifold.*

Remark 3.1.6 A little diagram chasing shows that the stabilization map $\pi_k(SO_k) \to \pi_k(SO)$ is surjective for $k \neq 1, 3, 7$ [L2]. Then if $k \neq 1, 3, 7$ and W is stably parallelizable, we will in fact obtain a framed χ-equivalence.

Proof: We will proceed by induction on r. Let $f : S^k \times D^k \hookrightarrow \mathring{W}$ represent α_1. Let $W' = \chi(W, f)$, $W_0 = W \setminus f(S^k \times \mathring{D}^{k+1})$, and let $f' : S^{k-1} \times D^{k+1} \to W'$ be the map along which we can surger to reverse the surgery along f, described explicitly in Lemma 3.1.1. Consider the exact sequence

$$H_{k+1}(W, W_0) \to H_k(W_0) \to H_k(W) \xrightarrow{j_*} H_k(W, W_0) \xrightarrow{\partial} H_{k-1}(W_0) \to 0.$$

By excision,

$$\begin{aligned} H_*(W, W_0) &= H_*(S^k \times D^k, S^k \times S^{k-1}) \\ &= H_*(S^k \times (D^k, S^{k-1})) \\ &= H_*(S^k) \otimes H_*(D^k, S^{k-1}), \end{aligned}$$

therefore $H_k(W, W_0) = \mathbb{Z}$ and $H_{k+1}(W, W_0) = 0$. The $\mathbb{Z}$ of $H_k(W, W_0)$ is dual to $H_k(S^k \times D^k) = H_k\left(f(S^k \times D^k)\right)$ by the intersection pairing, therefore $H_k(W, W_0)$ is generated by an element x that has intersection number 1 with the image of $\alpha_1 = f(S^k \times \{0\})$ in $H_k\left(f(S^k \times D^k)\right)$. Then for $\lambda \in H_k(W)$, $j_*(\lambda) = \langle \lambda, \alpha_1 \rangle\, x$. It follows that $H_{k-1}(W_0) \cong \operatorname{coker}(j_*) = 0$, and $H_k(W_0) \cong \operatorname{Ker}(j_*)$ is isomorphic to $(\alpha_1, \ldots, \alpha_r, \beta_2, \ldots, \beta_r)$.

We now need to study the analagous exact sequence involving W'. Since we have shown that $H_{k-1}(W_0) = 0$, we get

$$0 \to H_{k-1}(W') \xrightarrow{j_*} H_{k-1}(W', W_0).$$

By excision,

$$\begin{aligned} H_*(W', W_0) &= H_*(S^{k-1} \times D^{k+1}, S^{k-1} \times S^k) \\ &= H_*(S^{k-1} \times (D^{k+1}, S^k)) \\ &= H_*(S^{k-1}) \otimes H_*(D^{k+1}, S^k), \end{aligned}$$

hence $H_{k+1}(W'W_0) = \mathbb{Z}$ and $H_k(W', W_0) = H_{k-1}(W', W_0) = 0$. By exactness, $H_{k-1}(W') = 0$ as well. This, along with Theorem 3.1.2, tells us that W' is $(k-1)$-connected. All that remains is to compute $H_k(W')$. We have

$$H_{k+1}(W', W_0) \xrightarrow{\partial} H_k(W_0) \to H_k(W') \to 0.$$

A generator of the infinite cyclic group $H_{k+1}(W', W_0) \cong H_{k+1}(D^{k+1}, S^k)$ is represented the map $f'|_{\{x_0\} \times D^{k+1}}$ for some $x_0 \in S^{k-1}$, therefore $\partial(1) \in H_k(W_0)$ is represented by the map $f'|_{\{x_0\} \times \partial D^{k+1}} = f|_{S^k \times \{x_0\}}$, a parallel copy of α_1. As a map to W, $f|_{S^k \times \{x_0\}}$ is homotopic to the map $f|_{S^k \times \{0\}}$ representing α_1, hence $H_k(W') = \text{coker}(\partial)$ is isomorphic to $(\alpha_2, \ldots, \alpha_r, \beta_2, \ldots, \beta_r)$.

In order to complete the induction on r we must know that $\alpha_2, \ldots, \alpha_r \in H_k(W')$ are still represented by embedded spheres with trivial normal bundles, and that the basis $\{\alpha_2, \ldots, \alpha_r, \beta_2, \ldots, \beta_r\}$ for $H_k(W')$ is symplectic. This will be immediate if the embeddings that represented our original $\alpha_2, \ldots, \alpha_r \in H_k(W)$ all landed in W_0, which is contained in W'. An equivalent condition is that for each $i > 1$, the embedded sphere representing $\alpha_i \in H_k(W)$ must be disjoint from the embedded sphere representing α_1.

This is where the hypothesis $k \geq 3$ becomes important. Because $\alpha_1 \cdot \alpha_i = 0$ for all i, it will be possible to pull α_1 apart from the other α_i's by a technique of Whitney [W1]. We will give a quick description of this procedure here:

Let M^k, N^k be submanifolds of W with algebraic intersection zero. Suppose further that M and N intersect transversely at finitely many points $p_1, q_1, \ldots, p_s, q_s$, with positive sign at each p_i and negative sign at each q_i. We will argue by induction on s that M and N can be pulled apart. Let σ be a path in M from p_1 to q_1, and let τ be a path in N from q_1 to p_1, such that σ and τ both miss all of the other double points. Since W is simply connected, the loop $\sigma\tau$ is null-homotopic in W. Since $\dim W = 2k > 4$, $\sigma\tau$ bounds an embedded disk $D^2 \subset W$. Then Whitney shows that we can pull a neighborhood of σ in M through a neighborhood of D^2 in W, thus eliminating the double points p_1 and q_1. With s applications of this technique, M and N can be deformed into disjoint submanifolds. (Note that this procedure is completely analogous to Whitney's proof of Theorem 3.3.1, which we will use in Section 3.3.)

By this argument we may assume that for all $i > 1$, α_i is represented by a sphere with trivial normal bundle embedded in W_0. This completes the inductive proof of Theorem 3.1.5. □

The problem of killing the two middle homotopy groups of an odd dimensional manifold will be studied in Section 3.2.

3.2 The Groups bP_{2k+1}

In this section we use surgery to show that $bP_{2k+1} = 0$ for all $k > 1$, following Kervaire and Milnor's exposition in [KM]. Note that by Lemma 1.1.1, this result is a consequence of the following

Theorem 3.2.1 *If W^{2k+1} is parallelizable and bounded by a homotopy sphere, then W is χ-equivalent to a contractible manifold.*

Proof: By Theorem 3.1.4, we may assume that W is $(k-1)$-connected. Then by Poincare duality and the Hurewicz theorem, it is enough to show that we can kill $H_k(W)$. By the Hurewicz theorem and generic transversality, every element of $H_k(W)$ can be represented by an embedded sphere, and by Lemma 2.2.1 these embeddings will induce trivial normal bundles on S^k. Hence every $\lambda \in H_k(W)$ can be represented by a map $f : S^k \times D^{k+1} \to \mathring{W}$.

As in Section 3.1, let $W' = \chi(W, f)$, $W_0 = W \setminus f(S^k \times \mathring{D}^{k+1})$. Let $\lambda \in H_k(W)$ be the element represented by $f|_{S^k\times\{0\}} : S^k \to \mathring{W}$, and let $\lambda' \in H_k(W')$ be the element represented by the surgery along $f' : S^k \times D^{k+1} \to \mathring{W}'$ that reverses f.

Consider the exact sequence of the pair (W, W_0). By excision,

$$H_*(W, W_0) = H_*(S^k \times D^{k+1}, S^k \times S^k) = H_*(S^k \times (D^{k+1}, S^k) = H_*(S^k) \otimes H_*(D^{k+1}, S^k),$$

therefore $H_k(W, W_0) = 0$ and $H_{k+1}(W, W_0) = \mathbb{Z}$. Because our set-up is symmetric in the dimensions of the surgeries along f and f', we also have $H_k(W', W_0) = 0$ and $H_{k+1}(W', W_0) = \mathbb{Z}$. This gives us exact sequences

$$H_{k+1}(W) \xrightarrow{\cdot\lambda} \mathbb{Z} \xrightarrow{\varepsilon'} H_k(W_0) \xrightarrow{i} H_k(W) \to 0 \tag{1}$$

and

$$H_{k+1}(W') \xrightarrow{\cdot\lambda'} \mathbb{Z} \xrightarrow{\varepsilon} H_k(W_0) \xrightarrow{i'} H_k(W') \to 0, \tag{2}$$

where $\cdot\lambda$ takes $\alpha \in H_{k+1}(W)$ to $\alpha \cdot \lambda$, and $\cdot\lambda'$ takes $\alpha' \in H_{k+1}(W')$ to $\alpha' \cdot \lambda'$.

The infinite cyclic group $H_{k+1}(W, W_0) \cong H_{k+1}(D^{k+1}, S^k)$ is generated by the map $f|_{\{x_0\}\times D^{k+1}}$ for some $x_0 \in S^k$ (compare to the proof of Theorem 3.1.5). The image $\varepsilon'(1)$ of this generator is represented by the map $f|_{\{x_0\}\times\partial D^{k+1}}$, which we can think of as a meridian of f. By symmetry, $\varepsilon(1)$ is represented by $f'|_{\{x_0\}\times\partial D^{k+1}} = f|_{S^k\times\{x_0\}}$, a parallel copy of λ. We will denote $\varepsilon(1)$ and $\varepsilon'(1)$ simply by ε and ε', respectively. In a similar abuse of notation, define $\lambda = i \circ \varepsilon : H_k(W', W_0) \to H_k(W)$, and $\lambda' = i' \circ \varepsilon : H_k(W, W_0) \to H_k(W')$. We justify this abuse by noting that $\lambda(1) = i(\varepsilon) = \lambda$, because inside W the map $f|_{S^k\times\{x_0\}}$ representing ε can be homotoped to $f|_{S^k\times\{0\}}$ by simply pulling x_0 toward the origin (compare again to the proof of Theorem 3.1.5). By an identical argument, $i'(\varepsilon') = \lambda'$.

Lemma 3.2.2 $H_k(W)/\lambda \cong H_k(W')/\lambda'$.

Proof: By the exact sequences (1) and (2), both are congruent to $H_k(W_0)/(\varepsilon, \varepsilon')$. □

Call $\alpha \in H_k(W)$ *primitive* if there exists $\beta \in H_{k+1}(W)$ such that $\alpha \cdot \beta = 1$. If λ is primitive, then $\cdot\lambda$ is surjective, therefore $\varepsilon' = 0$. Then $\lambda' = i'(\varepsilon') = 0$, and by Lemma 3.2.2, $H_k(W') \cong H_k(W)/\lambda$. When ∂W is a homotopy sphere, $H_{k+1}(\partial W) = H_k(\partial W) = 0$ and $H_{k+1}(W) \cong H_{k+1}(W, \partial W)$. Then by Poincare duality, the free part of $H_k(W)$ is generated by primitive elements. We can thus reduce Theorem 3.2.1 to the case where $H_k(W)$ is torsion. We will proceed by induction on the size of $H_k(W)$.

At this point we will need some more tools. In the following discussion we will deal with homology manifolds, throwing away any smooth structure (this will be important because our manifold W is bounded by a homology sphere, but not *a priori* a smooth sphere). Let F be any field, and let M^{2r-1} be a closed homology manifold. We define the semi-characteristic $e^*(M; F)$ as follows:

$$e^*(M; F) \equiv \sum_{i=0}^{r-1} \operatorname{rk} H_i(M; F) \mod 2.$$

Lemma 3.2.3 (Kervaire-Milnor) *For any compact homology manifold X^{2r}, the rank of the intersection pairing on $H_r(X; F)$ is congruent mod 2 to $e^*(\partial X; F) + e(X)$, where $e(X)$ is the Euler characteristic of X.*

Proof: Consider the exact sequence

$$H_r(X) \xrightarrow{h} H_r(X, \partial X) \to \ldots \to H_0(X, \partial X) \to 0$$

with coefficients in F. Replacing $H_r(X)$ with $H_r(X)/\ker(h)$, exactness tells us that

$$\begin{aligned} \operatorname{rk}(h) &\equiv \sum_{i=0}^{r-1} \operatorname{rk} H_i(\partial X) + \sum_{i=0}^{r} \operatorname{rk} H_i(X, \partial X) + \sum_{i=0}^{r-1} \operatorname{rk} H_i(X) \\ &\equiv \sum_{i=0}^{r-1} \operatorname{rk} H_i(\partial X) + \sum_{i=0}^{r} \operatorname{rk} H_i(X) \qquad \text{by Poincare duality} \\ &\equiv e^*(\partial X; F) + e(X) \mod 2. \end{aligned}$$

Since the rank of h is exactly the rank of the intersection pairing on $H_r(X; F)$, we are done. □

We will now restrict our attention to proving Theorem 3.2.1 for the case k even.

Lemma 3.2.4 *Let W^{2k+1} be $(k-1)$-connected. If k is even, then surgery along $f : S^k \times D^k \to \mathring{W}$ necessarily changes the k^{th} Betti number of W.*

Proof: Let M^{2k+1} be the closed homology manifold obtained from W by coning over the boundary ∂W, which is a homology sphere. Similarly, let M' be the homology manifold obtained by coning over the boundary of W', and let X^{2k+2} be the compact homological manifold $M \times [0,1] \cup D^{k+1} \times D^{k+1}$ that arises as the surgery cobordism between M and M'. Then X has the homotopy type of the cell complex $M \cup e^{k+1}$, therefore $e(X) = e(M) + (-1)^{k+1}$. Since the dimension of M is odd, $e(M) = 0$, and $e(X) \equiv 1 \mod 2$. Since k is even, the intersection pairing on $H_{k+1}(X)$ is skew-symmetric, and therefore of even rank. Then by Lemma 3.2.3, $e^*(M \coprod M'; \mathbb{Q}) + 1 \equiv 0 \mod 2$, therefore $\operatorname{rk} H_k(W;\mathbb{Q}) \equiv e^*(M;\mathbb{Q}) + 1 \neq e^*(M';\mathbb{Q}) + 1 \equiv \operatorname{rk} H_k(W';\mathbb{Q}) \mod 2$. □

Recall Lemma 3.2.2, in which we showed that $H_k(W)/\lambda \cong H_k(W')/\lambda'$. If $H_k(W)$ is torsion, then Lemma 3.2.4 tells us that $H_k(W')$ is not, therefore λ' must have infinite order. Consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\lambda'} H_k(W') \to H_k(W')/\lambda' \to 0.$$

Since λ' has infinite order in $H_k(W')$, the torsion in $H_k(W')$ must inject into the torsion in $H_k(W')/\lambda' \cong H_k(W)/\lambda$. It follows that the torsion subgroup of $H_k(W')$ is strictly smaller than $H_k(W)$. The free part of $H_k(W)$ will be generated by a primitive element (in fact it will be generated by λ'), and can therefore be killed by a second surgery. We thus obtain a $(k-1)$-connected manifold W'' that is χ-equivalent to W, with $H_k(W'')$ of strictly smaller order than $H_k(W)$. This completes the inductive proof of Theorem 3.2.1 when k is even.

Note that by Lemma 3.1.3, the preceding argument could be carried out using framed surgeries at every step. In the case where k is odd we will once again used framed surgeries, and thus prove that every parallelizable manifold of dimension $(2k+1)$ that bounds a homotopy sphere is *framed* χ-equivalent to a contractible manifold. In this argument, however, we will do more than just rely on Lemma 3.1.3, which tells us that any element $\lambda \in H_k(W)$ can be killed by a framed surgery. We will instead exploit the fact that the result W' of the surgery depends on the choice of trivialization of the normal bundle to an embedded sphere representing λ. By choosing our trivializations carefully, we will show that it is possible to kill the torsion part of $H_k(W)$, and hence all of $H_k(W)$. We will proceed by induction on the order of $H_k(W)$.

Given a map $\alpha : S^k \to SO_{k+1}$, define $f_\alpha : S^k \times D^{k+1} \to \mathring{W}$ by the formula $f_\alpha(u,v) = f(u, \alpha(u)v)$ as in Section 3.1. We showed that α can always be chosen so that surgery along f can be framed (Proposition 3.1.3). We are free to redefine f in such a way that f itself has this property, in which case the surgery along f_α can be framed if and only if $\alpha \in \ker\big(s_* : \pi_k(SO_{k+1}) \to \pi_k(SO)\big)$.

We need to determine which of the objects that we have defined really depend on α. $W_0 = W \setminus f_\alpha(S^k \times \mathring{D}^{k+1})$ clearly does not depend on α. It follows that the homomorphism $i : H_k(W_0) \to H_k(W)$ does not depend on α, and therefore neither does ε', the generator of $\ker(i)$. On the other hand, $W'_\alpha = \chi(W, f_\alpha)$ does depend on α, as does the parallel $\varepsilon_\alpha \in H_k(W_0)$, which is represented by $f_\alpha|_{S^k \times \{x_0\}}$. Explicitly, we have $\varepsilon_\alpha = \varepsilon + j(\alpha)\varepsilon'$, where $j : \pi_k(SO_{k+1}) \to \pi_k(S^k) = \mathbb{Z}$ is induced by the standard action of SO_{k+1} on S^k.

Consider again the exact sequence (1). Since $H_k(W)$ is torsion, $H_{k+1}(W) = 0$, thus we have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\varepsilon'} H_k(W_0) \xrightarrow{i} H_k(W) \to 0,$$

in which $\varepsilon \in H_k(W_0)$ is mapped to $\lambda \in H_k(W)$. Let $l > 1$ be the order of λ. The $l \cdot \varepsilon$ is in the kernel of i, therefore there exists $l' \in \mathbb{Z}$ such that $l \cdot \varepsilon + l' \cdot \varepsilon' = 0$. Since ε' has infinite order, l' is unique. Combining this equation with our expression for ε_α, we get

$$l \cdot \varepsilon_\alpha + (l' - l \cdot j(\alpha))\varepsilon' = 0. \tag{3}$$

Let $i'_\alpha : H_k(W_0) \to H_k(W'_\alpha)$ be the map induced by inclusion, let $\lambda'_\alpha = i'_\alpha(\varepsilon')$ (recall that ε' does not depend on α), and let $l'_\alpha = |l' - l \cdot j(\alpha)|$.

Lemma 3.2.5 *The order of λ'_α is equal to l'_α (where order 0 is taken to mean infinite order).*

Proof: By applying i'_α to both sides of Equation (3), we see that the order of λ'_α divides l'_α. On the other hand, suppose that $r \cdot \lambda'_\alpha = 0$. Then $i'_\alpha(r \cdot \varepsilon') = 0$, therefore there exists s such that $r\varepsilon' + s\varepsilon_\alpha = 0$. Applying i to both sides, we see that $s = k \cdot l$ for some $k \in \mathbb{Z}$. Then since ε' has infinite order, $r = k \cdot l'_\alpha$. Thus l'_α is the order of λ'_α. □

Lemma 3.2.5 tells us that the torsion part of $H_k(W'_\alpha)$ is smaller than $H_k(W)$ if and only if $0 \le l'_\alpha < l$. We would like to be able to choose α such that this condition is satisfied.

Lemma 3.2.6 *For any integer t, there exists $\alpha \in \ker(s_*)$ such that $j(\alpha) = 2t$.*

Proof: The kernel of s_* is equal to the image of $\partial : \pi_{k+1}(S^{k+1}) \to \pi_k(SO_{k+1})$, hence we would like to know that $j \circ \partial : \pi_{k+1}(S^{k+1}) \to \pi_k(S^k)$ is given by multiplication by 2. This is precisely the statement of Corollary 2.1.3. □

By this lemma, α can be chosen so that $0 \le \lambda'_\alpha < l$ *unless* l' is an odd multiple of l. If l' is an odd multiple of l, then α can be chosen so that $\lambda'_\alpha = l$, but this is the best that we can do. Replacing f with f_α, we reduce to the case where $l' = l$. Now we once again need some more machinery.

Consider the exact sequence

$$\ldots \to H_{k+1}(W; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} H_k(W; \mathbb{Z}) \xrightarrow{i_*} H_k(W; \mathbb{Q}) \to \ldots,$$

where the map ∂ is defined by lifting $x \in H_{k+1}(W; \mathbb{Q}/\mathbb{Z})$ to $\tilde{x} \in C_{k+1}(W; \mathbb{Q})$ and taking its boundary, which lies in $H_k(W; \mathbb{Z})$. If $\tilde{x}'$ is a different lift of x, then $\tilde{x}' = \tilde{x} + y$ for some $y \in C_{k+1}(W; \mathbb{Z})$, and therefore the boundaries of $\tilde{x}$ and $\tilde{y}$ are homologous.

Let torsion elements $\alpha \in H_p(W)$ and $\beta \in H_q(W)$ be given, with $p + q = 2n$. Since α is torsion, $i_*(\alpha) = 0$, therefore there exists some $x \in H_{p+1}(W; \mathbb{Q}/\mathbb{Z})$ such that $\partial x = \alpha$. Define the *linking number* $L(\alpha, \beta) = x \cdot \beta \in \mathbb{Q}/\mathbb{Z}$. Note that if x' is a different lift of α, then $x \cdot \beta - x' \cdot \beta = (x - x') \cdot \beta = 0$, because $x - x' \in H_{p+1}(W;\mathbb{Q})$ and β is torsion. Linking numbers express the torsion version of Poincare duality, and therefore define a unimodular form on $H_p(W)$ [ST].

Lemma 3.2.7 $\pm l'/l \equiv L(\lambda, \lambda)$ *mod* 1.

Proof: Choose some $x_0 \in S^k$, and put $c' = f|_{\{x_0\}\times D^{k+1}} \in C_{k+1}(W;\mathbb{Z})$ with boundary ε'. Since $l \cdot \varepsilon + l' \cdot \varepsilon'$ is homologous to 0 in W_0, it bounds a chain $d \in C_{k+1}(W_0;\mathbb{Z})$. Then $c = (d - l' \cdot c')/l \in C_{k+1}(W;\mathbb{Q}/\mathbb{Z})$ has boundary ε, which is homologous in W to λ.

$\lambda = f(S^k \times \{0\})$ intersects c' transversely at $(x_0, 0)$, and nowhere else. Since d is contained in W_0, λ misses d completely. Then $L(\lambda, \lambda) = c \cdot \lambda = -(l'/l)c' \cdot \lambda = \pm l'/l$. □

Recall that we have reduced to the case where $l' = l$, therefore we can assume that $L(\lambda, \lambda) = 0$ for all $\lambda \in H_k(W;\mathbb{Z})$.

Lemma 3.2.8 *If $H_k(W;\mathbb{Z})$ is torsion and $L(\lambda, \lambda) = 0$ for all $\lambda \in H_k(W;\mathbb{Z})$, then $H_k(W;\mathbb{Z})$ is a direct sum of cyclic groups of order 2.*

Proof: Note that in general $L(\eta, \xi) = (-1)^{pq+1} L(\xi, \eta)$, therefore for $\eta, \xi \in H_k(W)$, $L(\eta, \xi) = L(\xi, \eta)$. Then $L(\eta + \xi, \eta + \xi) = L(\eta, \eta) + L(\xi, \xi) + 2 \cdot L(\eta, \xi)$, therefore our hypothesis implies that $L(2\eta, \xi) = 2 \cdot L(\eta, \xi) = 0$ for all $\eta, \xi \in H_k(W)$. Since the linking pairing is unimodular, we can conclude that that $2\eta = 0$ for all $\eta \in H_k(W)$. □

To summarize what we have proven so far in the case k odd, if W^{2k+1} is parallelizable and ∂W is a homotopy sphere, then W is framed χ-equivalent to a $(k-1)$-connected manifold with

$$H_k(W;\mathbb{Z}) = \bigoplus_{i=1}^{s} \mathbb{Z}_2.$$

We will now prove a lemma along the lines of Lemma 3.2.4, which we used for the case k even.

Lemma 3.2.9 $\operatorname{rk} H_k(W';\mathbb{Z}_2) \neq \operatorname{rk} H_k(W;\mathbb{Z}_2)$.

Proof: This proof is almost identical to the proof of Lemma 3.2.4, with $\mathbb{Z}_2$ coefficients substituted for $\mathbb{Q}$ coefficients. In the proof of Lemma 3.2.4, we used the fact that k was even to conclude that the intersection form on $H_{k+1}(X^{2k+2};\mathbb{Q})$ was skew-symmetric, where X was the surgery cobordism between the closed homological manifolds M and M' corresponding to W and W'. In our present context, k odd implies that the intersection form on $H_{k+1}(X;\mathbb{Z})$ is even (Lemma 2.2.3), therefore the intersection form on $H_{k+1}(X;\mathbb{Z}_2)$ is skew-symmetric. The rest of the argument is identical to that of Lemma 3.2.4. □

Now let us take another look at the effect of surgery on $H_k(W;\mathbb{Z})$. By Lemma 3.2.2, $H_k(W';\mathbb{Z})/\lambda'$ is isomorphic to $H_k(W;\mathbb{Z})/\lambda \cong (s-1)\mathbb{Z}_2$. We have assumed that λ' and λ are both of order 2, therefore

$$H_k(W';\mathbb{Z}) \cong (s-2)\mathbb{Z}_2 \oplus G,$$

where $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$. The former case would contradict Lemma 3.2.9, therefore $G \cong \mathbb{Z}_4$. Then $H_k(W')$ has the same order as $H_k(W)$, but we now have an element which is not of order 2. It follows from Lemma 3.2.8 that there exists $\mu \in H_k(W')$ such that $L(\mu,\mu) \neq 0$, which implies by Lemma 3.2.7 that $H(W')$ can be reduced in size by a further surgery. We thus have an inductive proof of Theorem 3.2.1 for k odd, and that we in fact used only framed surgeries. □

3.3 The Groups bP_{4m+2}

This section will roughly follow the exposition of Levine [L2]. Let W^{2k} be a parallelizable manifold bounded by a homotopy sphere, with $k = 2m+1$. By Theorem 3.1.2, W can be surgered into a manifold that is $(k-1)$-connected. The intersection form on $H_k(W)$ is isomorphic to that of the closed homology manifold obtained by coning over the boundary of W, therefore by Poincare duality it is unimodular. Alternatively, one could look at the homology sequence of the pair $(W, \partial W)$, and note that $H_k(\partial W) = H_{k-1}(\partial W) = 0$ implies that the inclusion $W \to (W, \partial W)$ induces an isomorphism on H_k. Since k is odd, the intersection form on $H_k(W)$ is skew-symmetric, and therefore admits a symplectic basis. By the Hurewicz theorem, every $\lambda \in H_k(W)$ is spherical. It follows from Proposition 3.1.5 that to kill the middle homotopy group $H_k(W)$, we need only represent the $\{\alpha_i\}$ by embedded spheres with trivial normal bundles.

To investigate when this is possible, we will need to use some theorems of Whitney on embeddings and immersions of S^k into manifolds of dimension $2k$. Let V^{2k} be any even dimensional manifold. Given an immersion $f : S^k \to V^{2k}$ with only transverse double points, we define the *self-intersection number* I_f of f to be the number of double points with multiplicity. If k is even, which is the case that we will consider in this proof, we will count the double points with sign according to the orientations of S^k and V, and I_f will be an integer. If k is odd, the only case that we will consider, I_f is defined to be an element of $\mathbb{Z}_2$.

Theorem 3.3.1 (Whitney) *Let $f : S^k \to V^{2k}$ be an immersion with self-intersection number zero. If V is simply connected and $k \geq 3$, then f is regularly homotopic to an embedding.*

Proof: This is proven using Whitney's double point removal technique, which is sketched in the proof of Theorem 3.1.5. For a detailed proof, see [W1] and [M1].

Corollary 3.3.2 *If V^{2k} is simply connected and $k \geq 3$, then every $\lambda \in \pi_k(V)$ can be represented by an embedded sphere.*

Proof: Represent λ by an immersed sphere, and let r be the self-intersection number of the immersion. By connect summing with $|r|$ null-homotopically immersed spheres, each with self-intersection number $-r/|r|$, we obtain an immersed sphere with self-intersection number zero that still represents λ. Then apply Theorem 3.3.1. $\square$

Theorem 3.3.3 (Whitney) *If two embeddings $f, g : S^k \to \mathring{V}^{2k}$ are homotopic, then they are concordant as immersions.*

Proof: Let $h : S^k \times [0,1] \to \mathring{V}^{2k}$ be a homotopy between f and g. Whitney shows in [W2] that the map $H : S^k \times [0,1] \to \mathring{V}^{2k} \times [0,1]$ taking (x,t) to $(h(x,t),t)$ can be smoothed into an immersion.

We now return to the problem of killing $H_k(W)$. By Corollary 3.3.2 all of the $\alpha_i \in H_k(W)$ are represented by embedded spheres, hence we only have to worry about triviality of their normal bundles. Given an immersion $f : S^k \to W$, let $\nu(f)$ be the induced normal bundle on S^k. Since W is parallelizable, $\nu(f) \oplus T_{S^k} = f^* T_W$ is trivial. Then $\nu(f)$ is stably trivial, therefore the obstruction $[\nu(f)]$ to trivializing $\nu(f)$ can be thought of as lying in $L_k = \mathrm{Ker}\big((s_k)_* : \pi_{k-1}(SO_k) \to \pi_{k-1}(SO)\big)$. In Section 2.1 we showed that $L_k = 0$ if $k = 1, 3$, or 7, and $L_k \cong \mathbb{Z}_2$ for k odd, $k \neq 1, 3, 7$ (Lemma 2.1.4). Thus if $k = 3$ or 7, the obstruction to trivializing the normal bundle to an embedded $S^k \subset W$ necessarily vanishes, and as a consequence every parallelizable manifold of dimension 6 or 14 that is bounded by a homotopy sphere in χ-equivalent to a contractible manifold. Then by Theorem 1.1.1, $bP_6 = bP_{14} = 0$.

Assume now that $k \neq 1, 3, 7$. Given an immersion $f : S^k \to W$, let $\Phi(f) = [\nu(f)] \in L_k = \mathbb{Z}_2$. If two immersions $f, g : S^k \to W$ are concordant, then the concordance gives an isotopy between the normal bundles $\nu(f)$ and $\nu(g)$, therefore $\Phi(f) = \Phi(g)$. Theorem 3.3.3 tells us that if f and g are embeddings representing the same homotopy class, then they are concordant. Thus for $\lambda \in H_k(W)$, we may define $\Phi(\lambda) = \Phi(f)$, where f is any spherical immersion representing λ.

Identifying $H_k(W;\mathbb{Z}_2)$ with $H_k(W;\mathbb{Z}) \otimes \mathbb{Z}_2$, define $\Phi_2 = \Phi \otimes \mathrm{id} : H_k(W;\mathbb{Z}_2) \to \mathbb{Z}_2$. If V is a finite dimensional vector space over $\mathbb{Z}_2$ and B is a unimodular, skew-symmetric bilinear form on V, then a *$\mathbb{Z}_2$-valued quadratic form* on V associated to B is a map $q : V \to \mathbb{Z}_2$ such that for all $x, y \in V$, $q(x+y) = q(x) + q(y) + B(x,y)$. The *Arf invariant* $\mathrm{Arf}(q)$ is defined to be the quantity $\sum q(x_i) q(y_i)$, where $\{x_i, y_i\}_i$ is any symplectic basis for V with respect to B [MH].

Proposition 3.3.4 *The map $\Phi_2 : H_k(W;\mathbb{Z}_2) \to \mathbb{Z}_2$ is a $\mathbb{Z}_2$-valued quadratic form associated to the intersection pairing.*

Proof: If $f, g : S^k \to W$ are transverse embeddings representing α and β respectively, then $\alpha + \beta$ is represented by the immersed sphere obtained by connecting $\mathrm{Im}(f)$ to $\mathrm{Im}(g)$ with a small tube. Let $f \# g : S^k \to W$ denote this immersion. The self-intersection number $I_{f\#g}$ of $f \# g$ is equal to the intersection $\alpha \cdot \beta$. Then if $\alpha \cdot \beta = 0$, we have

$$\Phi(\alpha + \beta) = \Phi(f \# g) = \Phi(f) + \Phi(g) = \Phi(\alpha) + \Phi(\beta) + \alpha \cdot \beta.$$

On the other hand, suppose that $\alpha \cdot \beta = 1$, and let $h : S^k \to W$ be a null-homotopic immersion with self-intersection number 1 such that $\mathrm{Im}(h)$ misses $\mathrm{Im}(f\#g)$. Then $I_{f\#g\#h} = I_{f\#g} + I_h = 0$, therefore

$$\Phi(\alpha+\beta) = \Phi(f\#g\#h) = \Phi(f) + \Phi(g) + \Phi(h) = \Phi(\alpha) + \Phi(\beta) + \Phi(h).$$

Thus we need to show that $\Phi(h) = 1$.

We argue as in Levine [L2]: The obstruction $\Phi(h)$ does not depend at all on the global structure of W, thus it is enough to check this equality for a null-homotopic immersion $h : S^k \to S^k \times S^k$ with self-intersection number 1. Let $a, b : S^k \to S^k \times S^k$ be the standard embeddings that represent the two generators of $H_k(S^k \times S^k)$, and let $d : S^k \to S^k \times S^k$ be the diagonal map representing the homology class $[a] + [b]$. We have $[a] \cdot [b] = 1$, therefore $\Phi(d) = \Phi(a) + \Phi(b) + \Phi(h)$ as above. The obstructions $\Phi(a)$ and $\Phi(b)$ are both evidently zero, therefore $\Phi(h) = \Phi(d) = [\nu(d)] = [T_{S^k}]$. Recall that we are considering this obstruction as an element of L_k, not of $\pi_{k-1}(SO_k)$. Since T_{S^k} is nontrivial (Theorem 2.1.9), we must have $\Phi(h) = [T_{S^k}] = 1$. $\square$

Since Φ_2 is a quadratic form, we can define its Arf invariant $c(W)$, which we will call the Kervaire[2] invariant of W. Note that c is additive with respect to boundary connect summation.

Theorem 3.3.5 *If $c(W) = 0$, then W is χ-equivalent to a contractible manifold.*

Proof: Suppose that $c(W) = 0$, i.e. that there exists a symplectic basis $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r\}$ for $H_k(W; \mathbb{Z}_2)$ such that $\sum_{i=1}^{r} \Phi_2(\alpha_i)\Phi_2(\beta_i) = \mathrm{Arf}(\Phi_2) = 0$. For a given i, if $\Phi_2(\alpha_i)\Phi_2(\beta_i) = 0$, then let

$$\alpha_i' = \begin{cases} \alpha_i & \text{if } \Phi_2(\alpha_i) = 0, \\ \beta_i & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \beta_i' = \begin{cases} \beta_i & \text{if } \Phi_2(\alpha_i) = 0, \\ \alpha_i & \text{otherwise,} \end{cases}$$

so that $\Phi_2(\alpha_i') = 0$. Since $\sum_{i=1}^{r} \Phi_2(\alpha_i)\Phi_2(\beta_i) = 0$, we have $\Phi_2(\alpha_i)\Phi_2(\beta_i) \neq 0$ for an even number of values i. Given a pair of such values we may assume without loss of generality that they are $i = 1$ and $i = 2$. Then put

$$\alpha_1' = \alpha_1 + \alpha_2 \qquad \beta_1' = \beta_1,$$

$$\alpha_2' = \beta_2 - \beta_1 \qquad \beta_2' = \alpha_1.$$

By this procedure we construct a new symplectic basis $\{\alpha_1', \ldots, \alpha_r', \beta_1', \ldots, \beta_r'\}$ for $H_k(W; \mathbb{Z})$ such that $\Phi(\alpha_i') = 0$ for all i. Then by Theorem 3.1.5, W is χ-equivalent to a contractible manifold. $\square$

Lemma 3.3.6 *Let $\Sigma_i = \partial W_i$ be homotopy spheres for $i = 1, 2$. If $c(W_1) = c(W_2)$, then Σ_1 is h-cobordant to Σ_2.*

[2]This invariant has also been named after Arf [Kf] and Robertello [Hz].

Proof: The Kervaire invariant $c(W_1\#(-W_2))$ is equal to $c(W_1)-c(W_2)=0$, therefore Theorem 3.3.5 tells us that $(W_1\#(-W_2))$ can be surgered into a contractible manifold. Then $\Sigma_1\#(-\Sigma_2) = \partial(W_1\#(-W_2))$ is h-cobordant to S^{4m+1}. □

In Chapter 6, we will construct a parallelizable manifold W^{2k} bounded by a homotopy sphere such that $c(W)=1$. Then by Lemma 3.3.6, we can define a surjection $b_{2k}:\mathbb{Z}_2\to bP_{2k}$ taking t to the boundary of a parallelizable manifold with Kervaire invariant t. The manifold $b_{2k}(1)$ is called the Kervaire sphere, and it is the only potentially exotic element of bP_{2k}. There still remains the question of whether or not the Kervaire sphere is h-cobordant to the standard sphere. Browder [Br] showed that $b_{2k}(1)$ is exotic whenever $k\neq 2^r-1$. It is known, however, that $b_{30}(1)$ and $b_{62}(1)$ are both diffeomorphic to standard spheres (see [MT] and [BJM]). Thus for k odd,

$$bP_{2k} = \begin{cases} \mathbb{Z}_2 & \text{if } k\neq 2^r-1; \\ 0 & \text{if } k=3,7,15, \text{ or } 31; \end{cases}$$

and is unknown in the remaining dimensions.

3.4 The Groups bP_{4m}

We will classify the elements of bP_{4m} by the signatures of the manifolds that they bound. As in Section 3.3, if $\Sigma^{4m-1}=\partial W^{4m}$ is a homotopy sphere, then the intersection form on W can be identified with the intersection form the closed homology manifold obtained by coning over $\Sigma=\partial W$. It follows that the form is unimodular, and that $\sigma(W)$ is invariant under χ-equivalence.

Proposition 3.4.1 *Let W be a stably parallelizable manifold of dimension $4m$ with $\Sigma=\partial W$ a homotopy sphere. Then W can be surgered into a contractible manifold if and only if $\sigma(W)=0$.*

Proof: One direction follows immediately from χ-invariance of signature. Now suppose that $\sigma(W)=0$. Any even, unimodular quadratic form with signature 0 admits a symplectic basis $\{\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_r\}$ [MH]. By the Hurewicz theorem and Whitney's embedding theorem (Corollary 3.3.2), all of the α_i are represented by embedded spheres. It remains to show that the $\{\alpha_i\}$ have trivial normal bundles.

Let $f_0:S^{2m}\hookrightarrow W$ represent $\lambda\in H_{2m}(W)$, and let ν denote its normal bundle. Since W is stably parallelizable, ν is stably trivial, thus $\nu\oplus\epsilon^1$ is trivial by Lemma 2.2.1.

Lemma 3.4.2 *The bundle ν is trivial if and only if the intersection number $\lambda\cdot\lambda$ is equal to zero.*

Proof: Since ν is stably trivial, the obstruction $[\nu]$ to trivialization lies in the kernel of the stabilization map $s_*:\pi_{k-1}(SO_k)\to\pi_{k-1}(SO)$. Then $[\nu]=c\cdot[T_{S^k}]$ for some integer c (Lemma 2.1.1), and therefore $\lambda\cdot\lambda=e(\nu)=c\cdot\chi(S^k)=2c$ (Lemma 2.1.2). Then $\lambda\cdot\lambda=0$ if and only if $c=0$, which is true if and only if $[\nu]=0$. □

Since $\alpha_i \cdot \alpha_i = 0$ for all i, the normal bundles to each of the α_i are trivial. Thus by Theorem 3.1.5, W can be surgered into a contractible manifold. □

Corollary 3.4.3 *Let W_1 and W_2 be parallelizable manifolds that are bounded by homotopy spheres Σ_1 and Σ_2, and suppose that $\sigma(W_1) = \sigma(W_2)$. Then Σ_1 is h-cobordant to Σ_2.*

Proof: Let W be the boundary connect sum of W_1 and $-W_2$. By the Novikov Addition Theorem, $\sigma(W) = \sigma(W_1) - \sigma(W_2) = 0$. Then by Proposition 3.4.1 W can be surgered into a contractible manifold, and by Lemma 1.1.1 $\Sigma_1 \#(-\Sigma_2)$ is h-cobordant to S^{4m-1}. □

Let N be the subgroup of $\mathbb{Z}$ consisting of signatures of $4m$-dimensional parallelizable manifolds that are bounded by homotopy spheres. Corollary 3.4.3 tells us that bP_{4m} is a quotient of N, under the map taking an integer $\sigma(W) \in N$ to $\partial W \in bP_{4m}$. We now need to determine which integers arise as signatures of parallelizable manifolds that are bounded by S^{4m-1}, i.e. which elements of N map to the identity in bP_{4m}. We will follow the exposition of [MK].

A closed manifold M will is called *almost parallelizable* if its tangent bundle can be trivialized away from a point. If W is a parallelizable manifold with boundary S^{4m-1}, then we can attach a disk to W to obtain a closed almost parallelizable C^∞ manifold M with the same signature as W. Conversely, we can puncture a closed almost parallelizable manifold to obtain a parallelizable manifold with boundary S^{4m-1}. Thus we will study the signatures of closed almost parallelizable manifolds of dimension $4m$.

Recall the stable J-homomorphism $J_{4m} : \pi_{4m-1}(SO) \to \pi_s^{4m-1}$, with j_m equal to the order of $\mathrm{Im}(J_{4m})$. Identifying $\pi_{4m-1}(SO)$ with $\mathbb{Z}$, $\alpha \in \mathrm{Ker}(J_{4m})$ if and only if $\alpha = k \cdot j_m$ for some $k \in \mathbb{Z}$. It follows from Lemma 2.2.5 that α is the obstruction to trivializing the stable normal bundle of an almost parallelizable $4m$-manifold if and only if $\alpha = k \cdot j_m$ for some $k \in \mathbb{Z}$.

Let M^{4m} be almost parallelizable. Then all Pontrjagin classes $p_i(T_M)$ for $i \le m$ vanish, and the Hirzebruch Signature Formula [MS] simplifies to

$$\sigma(M) = 2^{2m}(2^{2m-1} - 1)B_m\, p_m[M]/(2m)!,$$

where B_m is the m^{th} Bernoulli number. By Lemma 2.1.6, $p_m[M] = \pm a_m(2m-1)!\alpha$, where α is the obstruction to trivializing the stable normal bundle on M. By Theorem 2.2.7, $j_m = \text{denominator}(\frac{B_m}{4m})$. We can thus conclude the following

Corollary 3.4.4 *There exists an almost parallelizable manifold with signature n if and only if n is a multiple of $\sigma_m = 2^{2m+1}(2^{2m-1} - 1) \cdot a_m \cdot \text{numerator}(\frac{B_m}{4m})$. Thus $bP_{4m} \cong N/(\sigma_m \mathbb{Z})$.*

It remains only to compute N.

Proposition 3.4.5 *$N \subseteq 8\mathbb{Z}$.*

Proof: Consider an arbitrary unimodular symmetric bilinear form on a free $\mathbb{Z}$-module X. Let $X_{(2)}$ be the mod 2 reduction of X, and for each $x \in X$, let $\bar{x}$ be its image in $X_{(2)}$. Then $h : X_{(2)} \to \mathbb{Z}_2$ taking $\bar{x}$ to $\bar{x} \cdot \bar{x}$ is a linear functional on a $\mathbb{Z}_2$ vector space, and is thus given by inner product with some $\bar{u} \in X_{(2)}$. Let u and u' be lifts of $\bar{u}$ to X, so that $u \cdot x \equiv x \cdot x \mod 2$ for all $x \in X$, and the same holds for u'. u and u' are called characteristic elements for the induced quadratic form Q. We must have $u' = u + 2x$ for some $x \in X$. Then

$$\begin{aligned} u' \cdot u' &= (u + 2x) \cdot (u + 2x) \\ &= u \cdot u + 4u \cdot x + 4x \cdot x \\ &= u \cdot u + 4(x \cdot x + 2k) + 4x \cdot x \\ &= u \cdot u + 8(x \cdot x + k), \end{aligned}$$

therefore $u \cdot u$ is well-defined mod 8.

Every odd, indefinite, unimodular form decomposes as $\oplus^p (1) \oplus^q (-1)$ [MH]. The signature $p - q$ of this form is congruent to $u \cdot u$ mod 8, because we can take u to be the sum of the basis elements. Our form Q may not be odd and indefinite, but the form $Q \oplus (1) \oplus (-1)$ on the module $X \oplus \mathbb{Z}^2$ is. Then by additivity of signature, $u \cdot u$ is always congruent mod 8 to $\sigma(Q)$. By Lemma 2.2.3, we can choose 0 for a characteristic element of the intersection form. Then $\sigma(W) \equiv 0 \cdot 0 = 0 \mod 8$, and $N \subseteq 8\mathbb{Z}$. □

Remark 3.4.6 In Chapter 6 we will show by construction that $N = 8\mathbb{Z}$ for $m \geq 2$, and therefore bP_{4m} is cyclic of order $\sigma_m/8$. Note that these values agree with the table in Section 1.2.

§4 Techniques in Knot Theory

The classical theory of knots and links began with the study of 1-dimensional submanifolds of S^3. Many of the constructions, however, generalize to codimension 2 submanifolds of S^n for any $n \geq 3$. This will prove to be extremely helpful in our pursuit of geometric intuition for exotic spheres: we will build exotic spheres using constructions that arise as generalizations of natural constructions involving knotted circles in S^3.

4.1 Seifert Manifolds

Consider a knot $K^n \subset S^{n+2}$, by which we mean any closed, oriented submanifold of S^{n+2}.

Theorem 4.1.1 *Suppose that the normal bundle to K is trivial, and $n > 0$. Then there exists an oriented manifold $W^{n+1} \subset S^{n+2}$ with $\partial W = K$.*

Proof: Let $N(K)$ be a tubular neighborhood of K in S^{n+2}. Choose an identification $N(K) \cong K \times D^2$, and let $p : \partial N(K) \to S^1$ be the corresponding projection. Note that up to homotopy, our choice of identification is tantamount to a choice of homotopy class in $[K, S^1]$. Let $X = S^{n+2} \setminus \mathring{N}(K)$, and let $\mathfrak{o} \in H^2(X, \partial X; \pi_1(S^1))$ be the obstruction to extending p over X.

$$\begin{aligned} H^2(X, \partial X; \pi_1(S^1)) &= H_n(X; \pi_1(S^1)) \text{ by Lefschetz duality} \\ &= H^1(N(K); \pi_1(S^1)) \text{ by Alexander duality} \\ &= H^1(K; \pi_1(S^1)) \text{ because } N(K) \cong K \times D^2 \\ &= [K, S^1] \text{ because } S^1 = K(\mathbb{Z}, 1). \end{aligned}$$

If we change our choice of identification of $N(K)$ with $K \times D^2$ by an element $\alpha \in [K, S^1]$, then the obstruction to extending p will change by α. It follows that such an identification can be chosen to make $\mathfrak{o}$ vanish.

Let $\varphi : S^{n+2} \to D^2$ be defined on X by extending p, and on $N(K)$ by projection onto D^2. We can choose φ to be smooth, and by Sard's theorem φ and $\varphi|_{\varphi^{-1}(S^1)}$ have a mutual regular value $x \in S^1 = \partial D^2$. Let R be the closed radius of D^2 connecting x to the origin, and let $W = \varphi^{-1}(R)$. Then W^{n+1} is a smooth submanifold of S^{n+2} with boundary K, with orientation induced by the orientations of S^{n+2} and $S^1 = \partial D^2$. □

W is called a Seifert manifold for K. If $n+1 = 2k$, we will define a bilinear pairing θ on $H_k(W)$ called the Seifert form. In order to do this, we define the linking number of two disjoint cycles a, $b \in Z_k(S^{2k+1})$. The pairing that we define here will be analogous to what we called the linking pairing in Section 3.2, though there will be some important differences. The pairing that we define here will be defined on cycles instead of on homology classes, and it will take values in $\mathbb{Z}$ instead of $\mathbb{Q}/\mathbb{Z}$.

Let a, $b \in Z_k(S^{2k+1})$ be disjoint. Choose A, $B \in C_{k+1}(D^{2k+2})$ such that $\partial A = a$, $\partial B = b$, and A and B intersect transversely at finitely many points. Put $lk(a, b) = A \cdot B$, the number of intersections counted with multiplicity. Given a different choice $A' \in C_{k+1}(D^{2k+2})$ with $\partial A' = a$, $A - A'$ is a cycle, therefore we will have $(A - A') \cdot B = 0$. It follows that $lk(a,b)$ is independent of choice of A, and similarly independent of choice of B.

Remark 4.1.2 A more standard definition is to take $B \in C_{k+1}(S^{2k+1})$ with $\partial B = b$, and put $lk(a,b) = a \cdot B$, which looks a lot more like the definition given in Section 3.2. These definitions are in fact equivalent [Ro].

It is immediate that lk is bilinear, and that $lk(a,b) = (-1)^{k+1} lk(b,a)$. We will now use linking numbers to define the Seifert form on $H_k(W)$. Since W is oriented, we can choose a small positive normal field v to W inside of S^{2k+1}. Given a pair of cycles $x, y \in Z_k(W)$, let y^* denote the element of $Z_k(S^{n+2} \smallsetminus W)$ obtained by pushing y along v, and let y_* denote the element of $Z_k(S^{n+2} \smallsetminus W)$ obtained by pushing y along $-v$. Put $\theta(x,y) = lk(x, y^*) = lk(x_*, y)$.[3]

Lemma 4.1.3 *If x_1 is homologous to x_2 and y_1 is homologous to y_2, then $\theta(x_1,y_1) = \theta(x_2,y_2)$, thus θ descends to a bilinear pairing $H_k(W) \times H_k(W) \to \mathbb{Z}$.*

Proof: By bilinearity of linking numbers, it suffices to show that if either x or y is null-homologous, then $\theta(x,y) = 0$. Suppose that x bounds a $(k+1)$-chain $A \subset W$. We can choose a $(k+1)$-chain B with $y^* = \partial B = B \cap S^{2k+1}$. Then $\theta(x,y) = lk(x,y^*) = A \cdot B = 0$, because $A \subset S^{2k+1} \smallsetminus y$. If y bounds a $(k+1)$-chain $B \subset W$, we can push B into $S^{n+2} \smallsetminus W$ along v, and conclude by a similar argument that $\theta(x,y) = 0$. □

Proposition 4.1.4 *$\theta(x,y) + (-1)^k \theta(y,x) = (-1)^k x \cdot y$, where $x \cdot y$ is the intersection of x and y on W.*

Proof:

$$\begin{aligned} \theta(x,y) + (-1)^k \theta(y,x) &= lk(x,y^*) + (-1)^k lk(y_*, x) \\ &= lk(x,y^*) - lk(x,y_*) \\ &= lk(x, y^* - y_*). \end{aligned}$$

Choose $A \in C_{k+1}(D^{2k+2})$ with $x = \partial A = A \cap S^{2k+1}$, and let B be a band in S^{2k+1} connecting y^* and y_*, oriented so that $\partial B = y^* - y_*$. Explicitly, B is the union over all $t \in [-1,1]$ of the push-out of y along the normal field tv. Then a careful sign computation reveals that $lk(x, y^* - y_*) = A \cdot B = (-1)^k x \cdot y$. □

Finally, we would like to relate the Seifert pairing to another function that we have defined on the middle homology group of a manifold with boundary.

[3]There is a discrepancy in the literature about whether to put $\theta(x,y)$ equal to $lk(x,y^*)$ or $lk(x^*,y)$. Here we choose the convention that will make the signs in our applications the least messy.

Proposition 4.1.5 *Suppose that k is odd, $k \neq 1, 3, 7$, and that W is $(k-1)$-connected. Let $\theta_2 : H_k(W;\mathbb{Z}_2) \times H_k(W;\mathbb{Z}_2) \to \mathbb{Z}_2$ be the mod 2 reduction of the Seifert pairing. Then for any $x \in H_k(W;\mathbb{Z}_2)$, $\theta_2(x,x) = \Phi_2(x)$.*

Proof: Let $\tilde{x}$ be a lift of x to $H_k(W;\mathbb{Z})$, represented by an embedding $f : S^k \hookrightarrow W$. Extend f to a proper embedding $F : D^{k+1} \hookrightarrow D^{2k+2}$, and note that F can be thought of as an element of $C_{k+1}(D^{2k+2})$. As before, let v be a positive normal field to $W \subset S^{2n+1}$, and let $\omega \in \pi_k(V_{k+1,1}) = \pi_k(S^k) = \mathbb{Z}$ be the obstruction to extending v to a nonvanishing normal field on $D^{k+1} = \mathrm{Im}(F) \subset D^{2k+2}$. Let $\hat{v}$ be a generic extension of v to D^{k+1}, possibly vanishing at finitely many points. Then ω is equal to the number of zeros of $\hat{v}$ counted with multiplicity, which is equal to $\pm lk(\tilde{x}, \tilde{x}^*) = \pm\theta(\tilde{x}, \tilde{x})$. Thus ω reduces mod 2 to $\theta_2(x,x)$.

Consider the exact sequence

$$\pi_k(SO_k) \xrightarrow{i_*} \pi_k(SO_{k+1}) \xrightarrow{p_*} \pi_k(S^k) \xrightarrow{\partial} \pi_{k-1}(SO_k)$$

induced by the fibration $SO_k \hookrightarrow SO_{k+1} \to S^k$. By exactness, $\partial(\omega) = 0$ if and only if $\omega \in \mathrm{im}(p_*)$. This is the case if and only if there exists a trivialization σ of the normal bundle to f, in which case ω is the image of the obstruction to extending the frame (v, σ) over D^{k+1}. Thus $\partial(\omega) = [\nu(f)]$, the obstruction to trivializing the normal bundle to f. Then by definition of Φ, ω reduces mod 2 to $\Phi_2(x)$. □

Remark 4.1.6 Taken together, Propositions 4.1.4 and 4.1.5 provide an alternate proof of the fact that Φ_2 is a quadratic form (Proposition 3.3.4) in the special case where W is embedded in a sphere of one greater dimension.

4.2 Cyclic Branched Covers

Consider a knot $K^n \subset S^{n+2}$ with trivial normal bundle, and let $N(K)$ be a tubular neighborhood of K. We showed in Section 4.1 that there exists a map $\varphi : S^{n+2} \to D^2$ with zero set K that restricts to a projection on $N(K)$.

Consider the map $\lambda_a : D^2 \to D^2$ taking z to $-z^a$, where D^2 is being identified with the closed unit disk in $\mathbb{C}$, and let $\tilde{\lambda}_a : M_a(S^{n+2}, K) \to S^{n+2}$ be the pull-back of λ_a along φ. We call $M_a(S^{n+2}, K)$ the *a-fold cyclic branched cover* of S^{n+2} along K. Applications of this definition in Section 5.1 will reveal why we prefer to define $\lambda_a(z) = -z^a$ instead of z^a. Note that over K, $\tilde{\lambda}_a$ is a diffeomorphism, and away from $\tilde{\lambda}_a^{-1}(K)$, $\tilde{\lambda}_a$ is an a-fold covering with automorphism group $\mathbb{Z}_a$. This property, along with the triviality of the normal bundle to $\tilde{\lambda}_a^{-1}(K)$ in $M_a(S^{n+2}, K)$, characterizes $M_a(S^{n+2}, K)$ up to orientation, and may in fact be used as a definition. Thus $M_a(S^{n+2}, K)$ does not depend essentially on choice of φ.

Consider the map $\psi : D^{n+3} \to D^2$ obtained by taking the cone over φ. We have $\psi^{-1}(0) = CK$, and we would like to perturb ψ a little bit so that the inverse image of 0 is smooth. There exists $p \in D^2$ near the origin such that $W = \psi^{-1}(p)$ is a smooth manifold with boundary $\partial W \subset S^{n+2}$

a parallel copy of K. We can think of W as a Seifert manifold for K pushed into D^{n+3} so that it is properly embedded. Let Ψ be the result of composing ψ with a diffeomorphism of D^2 that fixes $S^1 = \partial D^2$ and takes 0 to p, so that $W = \Psi^{-1}(0)$. Let $\hat{\lambda}_a : N_a(D^{n+3}, W) \to D^{n+3}$ be the pull-back of λ_a along Ψ. We call $N_a(D^{n+3}, W)$ the a-fold cyclic branched cover of D^{n+3} along W. Its construction is completely analogous to that of $M_a(S^{n+2}, K)$, and similarly $N_a(D^{n+3}, W)$ depends only on W, not on choice of φ or Ψ.

Remark 4.2.1 We have defined $M_a(S^{n+2}, K)$ and $N_a(D^{n+3}, W)$ in such a way that $M_a(S^{n+2}, K)$ is isomorphic to the boundary of $N_a(D^{n+3}, W)$. Since

$$N_a(D^{n+3}, W) = \{(x, z) \in D^{n+3} \times D^2 \mid \Psi(x) + z^a = 0\}$$

comes with a natural embedding in $D^{n+5} = D^{n+3} \times D^2$, we get a corresponding embedding of $M_a(S^{n+2}, K) = \partial N_a(D^{n+3}, W)$ in $S^{n+4} = \partial D^{n+5}$.

§5 Brieskorn's Construction

5.1 Preliminaries

Consider a polynomial map $f : \mathbb{C}^{n+1} \to \mathbb{C}$ of the form

$$f(z) = z_0^{a_0} + \ldots + z_n^{a_n},$$

where $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$ and $a_i \geq 2$ for all i. Let $V = V(a_0, \ldots, a_n) = f^{-1}(0)$, and $\Sigma = \Sigma(a_0, \ldots, a_n) = V \cap S^{2n+1}$, where S^{2n+1} is the sphere of radius 1 about the origin in $\mathbb{C}^{n+1}$. We call f a Brieskorn polynomial, V the corresponding Brieskorn variety, and Σ the link associated to f or V.

Let $\phi : \mathbb{C}^{n+1} \setminus V \to S^1$ take z to $f(z)/|f(z)|$, and let φ be the restriction of ϕ to $S^{2n+1} \setminus \Sigma$. We use the symbol φ for a reason: one consequence of the theorem that follows is that outside of a tubular neighborhood of Σ, φ shares the properties of the map φ constructed in the proof of Theorem 4.1.1.

Theorem 5.1.1 (Milnor's Fibration Theorem) *The maps ϕ and φ are both smooth bundle projections. Furthermore, φ restricts to a trivialization of the boundary of a tubular neighborhood of Σ.*

Remark 5.1.2 Milnor proves this theorem in greater generality [M4], allowing f to be any analytic function. We will give an argument modeled on Kauffman's [Kf] that is specific to the case of weighted homogeneous polynomials (see Remark 5.1.3), which include Brieskorn polynomials.

Proof: We first show that $f : \mathbb{C}^{n+1} \setminus V \to \mathbb{C}^*$ is a fibration. In order to locally trivialize this projection, we define the following actions of $\mathbb{R}^+$ and $\mathbb{R}$ on $\mathbb{C}^{n+1}$: For $\rho \in \mathbb{R}^+$, $\theta \in \mathbb{R}$, let $\rho * z = \left(\rho^{\frac{1}{a_0}} z_0, \ldots, \rho^{\frac{1}{a_n}} z_n\right)$, and let $\theta \star z = \left(e^{\frac{i\theta}{a_0}} z_0, \ldots, e^{\frac{i\theta}{a_n}} z_n\right)$. Note that $f(\rho * z) = \rho \cdot f(z)$ and $f(\theta \star z) = e^{i\theta} \cdot f(z)$. For the sake of later applications we have defined the actions $*$ and $\star$ separately, but in this situation they should be thought of as a single action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$. Indeed, $\rho \circ \theta : f^{-1}(\alpha) \to f^{-1}(\rho e^{i\theta}\alpha)$ is a smooth family of isomorphisms that locally trivializes $f|_{\mathbb{C}^{n+1} \setminus V}$.

Remark 5.1.3 A weighted homogeneous polynomial of type $(a_0, \ldots, a_n)$ is precisely a polynomial such that $\rho \circ \theta$ maps $f^{-1}(\alpha)$ to $f^{-1}(\rho e^{i\theta}\alpha)$ for all α.

We obtain the map ϕ by composing $f|_{\mathbb{C}^{n+1} \setminus V}$ with the standard projection of $\mathbb{C}^*$ onto S^1, hence ϕ is a smooth bundle projection. Let X be the noncompact manifold $f^{-1}(S^1_\delta)$, where δ is any positive real number and S^1_δ is the circle of radius delta centered at the origin in $\mathbb{C}$. We will use the action $*$ to define a map $\Gamma : \mathbb{C}^{n+1} \setminus \{0\} \to S^{2n+1}$ as follows: for $z \in \mathbb{C}^{n+1} \setminus \{0\}$, put $\Gamma(z) = \rho_z * z$, where ρ_z is the unique element of $\mathbb{R}^+$ that sends z to S^{2n+1}.

Lemma 5.1.4 *The restriction of Γ to X is a diffeomorphism from X to $S^{2n+1} \setminus \Sigma$.*

Proof: Given $p \in S^{2n+1} \setminus \Sigma$, let $\rho = \frac{\delta}{|f(p)|}$, and put $z = \rho * x \in X$. Then $\Gamma(z) = \rho^{-1} * \rho * p = p$, therefore Γ is surjective. To see that $\Gamma|_X$ is injective, note that if $\rho_w * w = \rho_z * z$, then $f(w) = \frac{\rho_z}{\rho_w} f(z)$. Since $w, z \in X$, this implies that $\rho_w = \rho_z$, therefore $w = z$. □

Since ϕ is a smooth bundle projection and $\Gamma|_X$ is a diffeomorphism, $\varphi = \phi \circ \Gamma|_X^{-1}$ is a smooth bundle projection. To see that φ restricts to a trivialization of the boundary of a tubular neighborhood of Σ, consider the neighborhood $N = \{z \in S^{2n+1} \mid |f(z)| \leq \delta\}$, where $\delta \in \mathbb{R}^+$ is chosen sufficiently small. For any $e^{i\theta} \in S^1$, $\varphi^{-1}(e^{i\theta}) \cap N = f^{-1}(\delta e^{i\theta})$ is a single parallel copy of $\Sigma = f^{-1}(0)$. It follows that the restriction of φ to any given slice of ∂N is a diffeomorphism, and therefore that φ trivializes ∂N. □

Let F be the fiber $\varphi^{-1}(1) \subset S^{2n+1} \setminus \Sigma$, and let $\bar{F}$ be its closure inside of S^{2n+1}. The orientations of S^{2n+1} and S^1 induce an orientation on $\bar{F}$, hence $\bar{F}$ is a compact, oriented manifold with boundary Σ. Since $\bar{F}$ is an orientable, codimension 1 submanifold of S^{2n+1}, its normal bundle is trivial, and it follows from Section 2.2 that $\bar{F}$ is parallelizable. For $\delta \in R^+$, let $V_\delta = f^{-1}(\delta)$. By restricting the diffeomorphism of Lemma 5.1.4, we obtain a diffeomorphism $\Gamma : V_\delta \to \varphi^{-1}(1) = F$. In particular, $F \cong V_1$, the variety defined by the equation $z_0^{a_0} + \ldots z_n^{a_n} - 1 = 0$. If we choose δ small enough, then $W_\delta = V_\delta \cap D^{2n+1} \cong \bar{F}$ will be a smooth Seifert manifold for a parallel copy of Σ, properly embedded in D^{2n+2}.

Because it will be difficult to keep track of all of the different manifolds that in some sense represent the fiber of φ, we review what we have so far: the fiber is by definition F, which is diffeomorphic to V_δ for any $\delta > 0$, and in particular to V_1. This manifold is the interior of a manifold $\bar{F} \cong W_\delta$, which has boundary Σ. *All* of the manifolds F, V_δ, V_1, $\bar{F}$, and W_δ are homotopy equivalent. These algebraic descriptions of the link and the fiber are complemented by the following knot theoretic interpretation of Σ and W_δ as cyclic branched covers:

Proposition 5.1.5 *Let $\Sigma_k = \Sigma(a_0, \ldots, a_n, k)$. Let $W = W_\delta$ be as defined above, and let W_k be the corresponding manifold for Σ_k. Then $\Sigma_k \cong M_k(S^{2n+1}, \Sigma)$, and $W_k \cong N_k(D^{2n+2}, W)$.*

Proof: The first statement follows from the second by taking the boundary of each side. To see the second, note that

$$W_k \cong \{(x, y) \in D^{2n+2} \times D^2 \mid f(x) + y^k = \delta\} = \{(x, y) \in D^{2n+2} \times D^2 \mid f(x) - \delta = \lambda_k(y)\}.$$

The map $f - \delta$ can be deformed into a smooth map $\Psi : D^{2n+2} \to D^2$ such that $\Psi|_{S^{2n+1}} = \varphi$ and $\Psi(0, \ldots, 0) = -\delta$, inducing an isotopy between W_k and $\{(x, y) \in D^{2n+2} \times D^2 \mid \Psi(x) = \lambda_k(y)\} = N_k(D^{2n+2}, W)$. □

This proposition gives an interpretation of the link associated to a Brieskorn variety as the result of a tower of cyclic branched covers of spheres. To trace the tower back to the classical case of knots and links in S^3, observe that $\Sigma(a_0, a_1)$ is embedded in S^3 as the torus link of type (a_0, a_1).

The two standard projections of the torus onto S^1 realize $\Sigma(a_0, a_1)$ as a covering space of S^1 with either a_0 or a_1 sheets. Taking the projection with a_1 sheets, we get a description of $\Sigma(a_0, a_1)$ as the cyclic branched cover of S^1 along the empty set $\Sigma(a_0)$, a submanifold of codimension 2.

5.2 The Geometry of the Fiber

In this section we compute the homology of $\bar{F}$, as well as the Seifert form on $\bar{F}$, considered as a Seifert manifold for Σ. This information will become important in Section 6.2, when we will need to compute the signature of the intersection form on $\bar{F}$. We conclude by computing the monodromy of the bundle φ, which will be important tool for understanding the geometry of Σ.

Let $\Omega_{a_j} \subset \mathbb{C}$ denote the group of a^{th} roots of unity, generated by $\varepsilon_j = e^{\frac{2\pi i}{a_j}}$. Consider the space

$$\begin{aligned} J &= \left\{ (z_0, \ldots, z_n) \;\middle|\; f(z) = 1 \text{ and } z_j^{a_j} \in \mathbb{R}^+ \cup \{0\}\ \forall j \right\} \\ &= \left\{ (t_0 \varepsilon_0^{k_0}, \ldots, t_n \varepsilon_n^{k_n}) \;\middle|\; \sum t_j^{a_j} = 1 \text{ and } t_j \in \mathbb{R}^+ \cup \{0\}\ \forall j \right\} \subset V_1, \end{aligned}$$

and note that J can be identified with the join $\Omega_{a_0} * \ldots * \Omega_{a_n}$. Recall that $\bar{F}$ is homotopy equivalent V_1. The following lemma shows that $\bar{F}$ is homotopy equivalent to J.

Lemma 5.2.1 *J is a deformation retract of V_1.*

Proof: For any $z \in V_1$, move z along a path $z(t)$, with $z = z(0)$, such that for all j, $z_j(t)^{a_j}$ moves on a straight line to the real axis. Then z moves to some z' such that $(z'_j)^{a_j} = \text{real}(z_j^{a_j})$ for all j. If all of the component paths are parametrized so that each $z_j^{a_j}$ moves at a constant speed, then we have $f(z(t)) = \text{real}\, f(z) + (1-t)\, \text{im}\, f(z) = 1$ for all t, hence $z(t)$ stays in V_1.

Next, for each j such that $(z'_j)^{a_j} < 0$, move $(z'_j)^{a_j}$ in a straight line to 0 along the real axis, while simultaneously scaling the positive $(z'_j)^{a_j}$'s so as to remain within V_1. This path ends at some $z'' \in V_1$ such that each z''_j has the form $t_j \varepsilon_j^{k_j}$ for some $t_j \geq 0$, $k_j \in \mathbb{Z}$; in other words $z'' \in J$. Since J was fixed throughout this sequence of deformations, we are done. □

Consider the simplest possible Brieskorn link, $\Sigma(a) \subset S^1$. As a set $\Sigma(a)$ is empty, and its complement S^1 fibers over S^1 with fiber $F = V_1 = \Omega_a$. Indeed, Ω_a is a Seiftert manifold for the empty knot in S^1. Lemma 5.2.1 is in fact a special case of a more general phenomenon: for any knot $K^n \subset S^{n+2}$ with Seifert manifold W, Kauffman and Neumann show that the knot $M_a(K) \subset S^{n+4}$ has a Seiftert manifold with $W * \Omega_a$ as a deformation retract [KN]. This in turn is a special case of a still more general operation studied by Kauffman and Neumann.

Call a link $L^m \subset S^{m+2}$ *fibered* if there exists a fibration $\varphi : S^{m+2} \smallsetminus L^m \to S^1$ that restricts to a trivialization of a tubular neighborhood of L. Thus Theorem 5.1.1 says that $\Sigma(a_0, \ldots, a_n)$ is a fibered link, and in its more general form [M4] it states that the link associated to any analytic singularity is fibered. Given a pair of knots $K^n \subset S^{n+2}$ and $L^m \subset S^{m+2}$ along with a fibration φ of L (K need not be fibered), Kaufman and Neumann define a knot product $K \otimes L \subset S^{n+m+5}$. As a special case, $M_a(K)$ is the knot product of K and the empty knot $\Sigma(a)$, which is fibered by λ_a. Though we will not develop the general construction in this paper, we will proceed with the philosophy suggested by Kauffman and Neumann's approach, which is that $\Sigma(a_0, \ldots, a_n) = \Sigma(a_0) \otimes \ldots \otimes \Sigma(a_n)$ should be thought of as built up from a bunch of empty knots.

Lemma 5.2.2 (Milnor) *Let A, B be topological spaces such that H_*A has no torsion. Then*

$$\tilde{H}_{k+1}(A * B) \cong \sum_{i+j=k} \tilde{H}_i(A) \otimes \tilde{H}_j(B).$$

Proof: The space $A * B$ can be described as $(CA \times B) \bigcup_{A \times B} (CB \times A)$, where CA and CB denote the cones over A and B, respectively. Consider the Meyer-Vietoris sequence

$$H_{k+1}(A * B) \to H_k(A \times B) \to H_k(CA \times B) \oplus H_k(CB \times A) \xrightarrow{\varphi} H_k(A * B).$$

Note that the inclusion $CA \times B \hookrightarrow A * B$ is null-homotopic (first retract to $B \times$ (apex of CA), then retract to the apex of CB). Similarly the inclusion $CB \times A \hookrightarrow A * B$ is null-homotopic, therefore $\varphi = 0$. Since $H_k(CA \times B) \cong H_k(B)$ and $H_k(CB \times A) \cong H_k(A)$, we are left with short exact sequences

$$0 \to H_{k+1}(A * B) \to H_k(A \times B) \to H_k(A) \oplus H_k(B) \to 0,$$

and we can conclude that

$$H_{k+1}(A * B) = \mathrm{Ker}\Big(H_k(A \times B) \to H_k(A) \oplus H_k(B)\Big).$$

By the Kunneth formula and the fact that H_*A has no torsion, $H_{k+1}(A * B)$ is isomorphic to $\sum_{i+j=k} H_i(A) \otimes H_j(B)$. Then $\tilde{H}_{k+1}(A * B) = H_{k+1}(A * B)$ is the kernel of the map

$$\sum_{i+j=k} H_i(A) \otimes H_j(B) \to H_k(A) \oplus H_k(B),$$

which is precisely equal to $\sum_{i+j=k} \tilde{H}_i(A) \otimes \tilde{H}_j(B)$. □

Recall that the fiber F of the bundle projection φ is diffeomorphic to V_1, which retracts onto J. Thus F, $\bar{F}$, V_1, and J all have the same homology, which we can describe explicitly with the following

Corollary 5.2.3 *The groups $\tilde{H}_k(\bar{F}) = 0$ for $k < n$, and $H_n(\bar{F}) \cong \tilde{H}_0(\Omega_{a_1}) \otimes \ldots \otimes \tilde{H}_0(\Omega_{a_n})$ is free of rank $\mu = \prod_{j=0}^{n}(a_j - 1)$.*

Consider the basis $\{x^k \mid 0 \le k \le a-2\}$ for $\tilde{H}_0(\Omega_a)$, where $x^k = [\varepsilon^k] - [\varepsilon^{k+1}]$, and $\varepsilon = e^{\frac{2\pi i}{a}}$. Let θ_a be the Seifert form for $\Sigma(a)$ with Seifert manifold Ω_a, considered as a Seifert manifold for the empty knot in S^1. With respect to this basis, θ_a is represented by the $(a-1)\times(a-1)$ matrix

$$\Lambda_a = \begin{pmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & -1 & 1 \end{pmatrix}.$$

The following proposition asserts that the Seifert form on $\bar{F}$, a Seifert manifold for Σ, is represented by a tensor product of matrices of this form.

Proposition 5.2.4 *If θ is the Seifert form on $\bar{F}$, then $\theta = (-1)^{n(n+1)/2}\cdot\theta_{a_0}\otimes\ldots\otimes\theta_{a_n}$.*

Proof: By Corollary 5.2.3 and the Hurewicz theorem, the elements of $H_n(\bar{F})$ are spherical. By Whitney's embedding theorem (Corollary 3.3.2), they are represented by embedded spheres. Then Lemma 5.2.2 allows us to reduce to the following statement:

Let $r = 2p+1$, $t = 2q+1$. Suppose that $\alpha, \beta \in Z_p(S^r)$, and $\alpha', \beta' \in Z_q(S^t)$. Then

$$lk_{S^{r+t+1}}(\alpha * \alpha', \beta * \beta') = (-1)^{(p+1)(q+1)}\cdot lk_{S^r}(\alpha,\beta)\cdot lk_{S^t}(\alpha',\beta').$$

To prove this, choose $A, B \in C_{p+1}(D^{k+1})$ with $\partial A = \alpha$ and $\partial B = \beta$, so that

$$lk_{S^{r+t+1}}(\alpha * \alpha', \beta * \beta') = \left\langle A * \alpha', B * \beta' \right\rangle.$$

Note that $A*\alpha' = (A\times C\alpha')\cup(CA\times\alpha')$ and $B*\beta' = (B\times C\beta')\cup(CB\times\beta')$, and the only intersection comes from the pieces $A\times C\alpha'$ and $B\times C\beta'$. Note also that since α' and β' are spherical, $C\alpha'$ and $C\beta'$ can be smoothed. Then

$$\begin{aligned} lk_{S^{r+t+1}}(\alpha * \alpha', \beta * \beta') &= \left\langle A * \alpha', B * \beta' \right\rangle \\ &= \left\langle A\times C\alpha', B\times C\alpha' \right\rangle \\ &= (-1)^{(p+1)(q+1)}\cdot\langle A, B\rangle\cdot\left\langle C\alpha', C\beta' \right\rangle \\ &= (-1)^{(p+1)(q+1)}\cdot lk_{S^r}(\alpha,\beta)\cdot lk_{S^t}(\alpha',\beta'). \end{aligned}$$

This proves Proposition 5.2.4. □

To conclude the section on the geometry of F, we will study the monodromy of the bundle $\varphi : S^{2n+1}\smallsetminus\Sigma \to S^1$. If $E\to S^1$ is a fiber bundle with fiber X, then E is obtained from $X\times[0,1]$ by identifying $X\times\{0\}$ with $X\times\{1\}$ via some diffeomorphism $h : X\to X$. For $\theta\in\mathbb{R}$, $z\in S^{2n+1}\smallsetminus\Sigma$, put $h_\theta(z) = \theta\star z$ as defined in Section 5.1. Since this action was used to prove the local triviality of the fibration φ, $h = h_{2\pi} : F\to F$ is the monodromy of the bundle φ. Restricting h to J, we get a diffeomorphism $r_{a_0} * \ldots * r_{a_n} : J \to J$, where $r_{a_j} : \Omega_{a_j}\to\Omega_{a_j}$ is given by multiplication by ε_j.

Let $h_* = r_{a_0 *} \otimes \ldots \otimes r_{a_n *}$ be the induced automorphism of $H_n(F) = H_n(J)$, and let $\Delta(t)$ be its characteristic polynomial.

Remark 5.2.5 Levine [L1] gives an interpretation of $\Delta(t)$ as the generalized Alexander polynomial of the knot $\Sigma \subset S^{2n+1}$.

Lemma 5.2.6 $\Delta(t) = \displaystyle\prod_{1 \leq k_j \leq a_j - 1 \ \forall j} \left(t - \varepsilon_0^{k_0} \ldots \varepsilon_n^{k_n}\right).$

Proof: By our description of h_* as the tensor product of maps $r_{a_j *}$, we need only show that for each j, the complexification $r_{a_j *} \otimes \mathbb{C} : \tilde{H}_0(\Omega_{a_j}; \mathbb{C}) \to \tilde{H}_0(\Omega_{a_j}; \mathbb{C})$ has eigenvalues $\Omega_{a_j} \setminus \{1\}$. Indeed, $r_{a_j} \otimes \mathbb{C}$ takes the vector

$$v_k = \sum_{1 \leq i \leq a_j - 1} \varepsilon_j^{-ik} [\varepsilon_j^i] \qquad \text{to} \qquad \varepsilon_j^k \cdot v_k.$$

The $\{v_k \mid 1 \leq k \leq a_j - 1\}$ form a basis for $\tilde{H}_0(\Omega_{a_j}; \mathbb{C})$, thus we are done. □

5.3 The Geometry of the Link - When is Σ a Homotopy Sphere?

Lemma 5.3.1 *If $n \geq 3$, then Σ is $(n-2)$-connected.*

Proof: First we follow Hirzebruch's argument [Hz] to show that $\pi_1(\Sigma)$ is abelian. We will then complete the proof by showing that $\tilde{H}_i(\Sigma) = 0$ for $k \leq n - 2$.

Recall that we have $f(z) = z_0^{a_0} + \ldots + z_n^{a_n}$, and $V = V(a_0, \ldots, a_n) = f^{-1}(0)$. Let $\tilde{V}$ be the space obtained from V by removing those elements with $z_n = 0$, and consider the inclusion $\tilde{V} \hookrightarrow V \setminus \{0\}$. Since the set $\{z_n = 0\} \subset V$ has codimension 2, this inclusion induces a surjection on fundamental groups. The map $\Gamma : V \setminus \{0\} \to \Sigma$ is a deformation retraction, hence $\pi_1(\Sigma) = \pi_1(V \setminus \{0\})$ is the homomorphic image of $\pi_1(\tilde{V})$. It is therefore enough to prove that $\pi_1(\tilde{V})$ is abelian.

Define $\psi : \tilde{V} \to \mathbb{C}^*$ taking $z = (z_0, \ldots, z_n)$ to z_n. This is a bundle projection with fiber $V_\delta(a_0, \ldots, a_{n-1}) = \left\{z \mid z_0^{a_0} + \ldots + z_{n-1}^{a_{n-1}} = \delta\right\}$. Lemma 5.2.1 tells us that $V_\delta(a_0, \ldots, a_{n-1})$ is homotopy equivalent to the join $\Omega_{a_0} * \ldots * \Omega_{a_{n-1}}$, which by Lemma 5.2.2 is simply connected for $n \geq 3$. It follows that $\pi_1(\tilde{V}) = \pi_1(\mathbb{C}^*) = \mathbb{Z}$, therefore $\pi_1(\Sigma)$ is abelian.

Consider the homology sequence of the pair $(\bar{F}, F)$:

$$H_{k+1}(\bar{F}) \to H_{k+1}(\bar{F}, F) \to H_k(F).$$

By Alexander duality [Ma], $H_{k+1}(\bar{F}, F) \cong H^{2n-k-1}(\Sigma) \cong H_k(\Sigma)$. Then Corollary 5.2.3 tells us that $\tilde{H}_k(\Sigma) = 0$ for $k \leq n - 2$. □

Proposition 5.3.2 *The link Σ is a homotopy sphere if and only if $\Delta(1) = \pm 1$.*

Proof: Consider the Wang sequence of the bundle $\varphi : S^{2n+1} \setminus \Sigma \to S^1$:

$$0 \to H_{n+1}(S^{2n+1} \setminus \Sigma) \to H_n(F) \overset{I_*-h_*}{\longrightarrow} H_n(F) \to H_n(S^{2n+1} \setminus \Sigma) \to 0.$$

The map $I_* - h_*$ is an isomorphism if and only if $\Delta(1) = \det(I_* - h_*) = \pm 1$. By Alexander duality, $H_{k+1}(S^{2n+1} \setminus \Sigma) \cong H^{2n-k-1}(\Sigma) \cong H_k(\Sigma)$ for all k, hence $I_* - h_*$ is an isomorphism if and only if the groups $H_{n+1}(S^{2n+1} \setminus \Sigma) \cong H_n(\Sigma)$ and $H_n(S^{2n+1} \setminus \Sigma) \cong H_{n-1}(\Sigma)$ are both trivial. By Lemma 5.3.1 and Poincare duality, this condition is equivalent to Σ being a homotopy sphere. $\square$

We now use Proposition 5.3.2 to give some specific examples of homotopy spheres that arise as links of Brieskorn polynomials.

Corollary 5.3.3 *For $n \geq 3$ odd, $\Sigma(3, 2, ..., 2)$ is a homotopy sphere.*

Proof: $\Delta(1) = (1 - e^{\frac{2\pi i}{3}}) \cdot (1 - e^{\frac{4\pi i}{3}}) = 2 - e^{\frac{2\pi i}{3}} - e^{\frac{4\pi i}{3}} = 2 - 1 = 1.$ $\square$

Corollary 5.3.4 *For p, q odd and relatively prime, $n > 3$ even, $\Sigma(p, q, 2, ..., 2)$ is a homotopy sphere.*

Proof:

$$\Delta(1) = \prod_{\substack{1 \leq j \leq p-1 \\ 1 \leq k \leq q-1}} (1 + e^{2\pi i(\frac{j}{p}+\frac{k}{q})}) = \left(\prod_{\zeta \in \Omega_{pq} \setminus \{1\}} (1+\zeta) \right) \cdot \left(\prod_{\zeta \in \Omega_p \setminus \{1\}} (1+\zeta) \quad \cdot \prod_{\zeta \in \Omega_q \setminus \{1\}} (1+\zeta) \right)^{-1}.$$

For any odd number r,

$$\prod_{\zeta \in \Omega_r \setminus \{1\}} (1+\zeta) \quad \cdot \prod_{\zeta \in \Omega_r \setminus \{1\}} (1-\zeta) = \prod_{\zeta \in \Omega_r \setminus \{1\}} (1-\zeta^2) = \prod_{\zeta \in \Omega_r \setminus \{1\}} (1-\zeta).$$

Since $\prod_{\zeta \in \Omega_r \setminus \{1\}} (1-\zeta)$ is nonzero, $\prod_{\zeta \in \Omega_r \setminus \{1\}} (1+\zeta)$ must equal 1. Hence $\Delta(1) = \frac{1}{1 \cdot 1} = 1$. $\square$

§6 Construction of the Groups bP_{n+1}

6.1 Construction of bP_{4m+2}

In this section we give three different proofs that for $n = 2m+1 \geq 3$, $\Sigma(3,2,\ldots,2)$ is in fact the Kervaire sphere, the only potentially nontrivial element of bP_{4m+2}.

Let $\varepsilon = e^{\frac{2\pi i}{3}}$. By Corollary 5.2.3, $H_n(\bar{F};\mathbb{Z}_2) \cong \tilde{H}_0(\Omega_3;\mathbb{Z}_2) = \{0, x^0, x^1, x^0+x^1\}$, where

$$x^0 = [1] - [\varepsilon] \quad \text{and} \quad x^1 = [\varepsilon] - [\varepsilon^2].$$

Any basis for a rank 2 $\mathbb{Z}_2$-vector space is symplectic with respect to any skew-symmetric form, hence we may use the basis $\{x^0, x^1\}$ to compute $c(\bar{F})$. The isomorphism h_* induced by the monodromy h takes x^0 to x^1 and x^1 to $x^0 + x^1$, therefore we must have $\Phi_2(x^0) = \Phi_2(x^1) = \Phi_2(x^0 + x^1)$. Since Φ_2 is a quadratic form, these quantities cannot all be zero, hence $\Phi_2(x^0) = \Phi_2(x^1) = 1$ and $c(\bar{F}) = 1$. This proves that $\Sigma(3,2,\ldots,2)$ is the Kervaire sphere.

Alternatively, we can see that $c(\bar{F}) = 1$ using the knot theoretic approach. We saw in Proposition 4.1.5 that $\Phi_2 = \theta_2$, the reduced Seifert form. We know by Proposition 5.2.4 that θ_2 is represented by the matrix Λ_3 with respect to this basis, hence $\theta_2(x^0) = \theta_2(x^1) = 1$.

Yet a third approach, also derived from knot theory, is to make use of a theorem of Levine [L1] which asserts that $c(\bar{F}) = 1$ if and only if $\Delta(-1) \equiv \pm 3 \mod 8$. In our case, $\Delta(-1) = (-1 - e^{\frac{2\pi i}{3}}) \cdot (-1 - e^{\frac{4\pi i}{3}}) = 3$, therefore $c(\bar{F}) = 1$. For a more general formulation and proof of Levine's theorem, see [Lu].

6.2 Construction of bP_{4m}

In this section we will use Brieskorn's construction with $n = 2m \geq 4$ to construct the elements of bP_{2m}. The results of Section 3.4 tell us that we should study the intersection form on $\bar{F}$, and compute its signature. We will be able to do this using Lemma 5.2.4 and Proposition 4.1.4, which relates the intersection form to the Seifert pairing. We will compute the intersection form on $\bar{F}$ for general n, and the restrict to the case $n = 2m$ even to compute its signature.

Let G_{a_j} denote the cyclic group of order a_j, isomorphic to Ω_{a_j}, with generator w_j corresponding to $\varepsilon_j \in \Omega_{a_j}$. We will use the notation G_{a_j} when we want to think of this group abstractly, and Ω_{a_j} when we want to think of a subset of the complex numbers. Let $G = G_{a_0} \times \ldots \times G_{a_n}$. We will also think of w_j as the element of G, representing the product of $w_j \in G_{a_j}$ with the identity $e \in G_{a_i}$ for all $i \neq j$. The reason for the extra notation is that we will eventually want to consider representations of G in which an element $w_0^{k_0} \ldots w_n^{k^n} \in G$ is mapped to $\varepsilon_0^{x_0 k_0} \ldots \varepsilon_n^{x_n k_n} \in \mathbb{C}$, where $(x_0, \ldots, x_n)$ is some $(n+1)$-tuple of integers. To avoid confusion, we *must* distinguish between w_j and ε_j.

Recall that by Lemma 5.2.1, $\bar{F}$ has the same homology as J, an n-dimensional simplicial complex with n-simplices corresponding bijectively to elements of G. Let x_e be the simplex corresponding to the identity element of G. Since G acts freely on the set of n-simplices, $C_n(J;\mathbb{Z}) = \mathbb{Z}(G)x_e$, where $\mathbb{Z}(G)$ is the group ring of G. Let $\eta = \prod_{j=0}^{n}(e - w_j) \in \mathbb{Z}(G)$, and let $h = \eta x_e \in C_n(J;\mathbb{Z})$.

Lemma 6.2.1 *The n^{th} homology group $H_n(J;\mathbb{Z})$ is congruent to the additive group $\mathbb{Z}(G)\eta \subset \mathbb{Z}(G)$.*

Proof: Since J has no cells in dimension greater than n, we have $H_n(J;\mathbb{Z}) = Z_n(J;\mathbb{Z}) \subset C_n(J;\mathbb{Z})$. By definition of h, $\mathbb{Z}(G)h \cong \mathbb{Z}(G)\eta$ as an additive group. Thus we need to show that $Z_n(J;\mathbb{Z}) = \mathbb{Z}(G)h$.

Consider the face operator $\partial_j : C_n(J;\mathbb{Z}) \to C_{n-1}(J;\mathbb{Z})$ taking a simplex to the face opposite the vertex corresponding to the G_{a_j} factor of G. Since changing a vertex will not change the opposing face, we have the composition $\partial_j \circ w_j = \partial_j$, where w_j denotes multiplication by $w_j \in G \subset \mathbb{Z}(G)$. It follows that $\partial_j(h) = 0$ for all j, therefore h is a cycle, and we have $\mathbb{Z}(G)h \subset Z_n(J;\mathbb{Z})$. By Lemma 5.2.3, $Z_n(J;\mathbb{Z}) = H_n(J;\mathbb{Z})$ is free of rank $(a_0 - 1)\ldots(a_n - 1)$. Since $\mathbb{Z}(G)h$ is a free subgroup of rank $(a_0 - 1)\ldots(a_n - 1)$ over $\mathbb{Z}$, and its generators $\{g\eta \mid g \in G\}$ are indivisible over $\mathbb{Z}$, we must have $Z_n(J;\mathbb{Z}) = \mathbb{Z}(G)h$. □

Consider the basis $\left\{x_j^k = [\varepsilon_j^k] - [\varepsilon_j^{k+1}] \mid 0 \le k \le a_j - 2\right\}$ for $\tilde{H}_0(\Omega_{a_j})$, where $\varepsilon_j = x^{\frac{2\pi i}{a_j}}$. In Section 5.2 we showed that with respect to this basis, the Seifert form θ_{a_j} has matrix Λ_{a_j}. Now consider a group element $g = w_0^{k_0}\ldots w_n^{k_n} \in G$, with $0 \le k_j \le a_j - 1$ for all j. The corresponding cycle

$$g\eta = \prod_{j=0}^{n}(w_j^{k_j} - w_j^{k_j+1}) \in H_n(\bar{F};\mathbb{Z}) \cong H_n(J;\mathbb{Z})$$

can be identified with the tensor product $x_0^{k_0} \otimes \ldots \otimes x_n^{k_n}$ of elements of $\tilde{H}_0(\Omega_{a_j})$, using the isomorphism $H_n(\bar{F}) \cong \tilde{H}_0(\Omega_{a_1})\otimes\ldots\otimes\tilde{H}_0(\Omega_{a_n})$ of Corollary 5.2.3. We must be careful here: $\{x_j^0, \ldots, x_j^{a_j-2}\}$ is a basis for $\tilde{H}(\Omega_{a_j})$. The element that we call $x_j^{a_j-1}$ is not in this basis, but can be expressed as the sum $-\sum_{k=0}^{a_j-2} x_j^k$. This is important to keep in mind in the computation that follows.

By Proposition 4.1.4,

$$\begin{aligned}\left\langle w_0^{k_0}\ldots w_n^{k_n}\eta, \eta\right\rangle &= \theta(w_0^{k_0}\ldots w_n^{k_n}\eta, \eta) + \theta(\eta, w_0^{k_0}\ldots w_n^{k_n}\eta) \\ &= (-1)^{n(n+1)/2}\prod_{j=0}^{n}\theta_{a_j}(x_j^{k_j}, x_j^0) + (-1)^{n(n+1)/2}\prod_{j=0}^{n}\theta_{a_j}(x_j^0, x_j^{k_j}).\end{aligned}$$

Looking at the matrix Λ_{a_j},

$$\theta_{a_j}(x_j^{k_j}, x_j^0) = \begin{cases} 1 & \text{if } k_j = 0; \\ -1 & \text{if } k_j = 1; \\ 0 & \text{otherwise.}\end{cases}$$

Hence $\prod_{j=0}^{n}\theta_{a_j}(x_j^{k_j},x_j^0)=0$ unless $k_j\in\{0,1\}\ \forall j$, in which case

$$\prod_{j=1}^{n}\theta_{a_j}(x_j^{k_j},x_j^0)=(-1)^r,\ \text{ where } r=\#\{j\mid k_j=1\}.$$

Similarly,

$$\theta_{a_j}(x_j^0,x_j^{k_j})=\begin{cases}1 & \text{if } k_j=0;\\ -1 & \text{if } k_j=a_j-1;\\ 0 & \text{otherwise.}\end{cases}$$

Hence $\prod_{j=0}^{n}\theta_{a_j}(x_j^0,x_j^{k_j})=0$ unless $k_j\in\{0,a_j-1\}\ \forall j$, in which case

$$\prod_{j=1}^{n}\theta_{a_j}(x_j^0,x_j^{k_j})=(-1)^s,\ \text{ where } s=\#\{j\mid k_j=a_j-1\}.$$

We have thus proven the following

Theorem 6.2.2 *Let $g=w_0^{k_0}\dots w_n^{k_n}\in G$. Then $\langle g\eta,\eta\rangle=(-1)^{n(n+1)/2}\big(r(g)+s(g)\big)$, where*

$$r(g)=\begin{cases}(-1)^{\#\{j\mid k_j=1\}} & \text{if } k_j\in\{0,1\}\ \forall j;\\ 0 & \text{otherwise;}\end{cases}\quad \text{and } s(g)=\begin{cases}(-1)^{\#\{j\mid k_j=a_j-1\}} & \text{if } k_j\in\{0,a_j-1\}\ \forall j;\\ 0 & \text{otherwise.}\end{cases}$$

Left multiplication by G is an isometry; that is for all $x,y\in\mathbb{Z}(G)\eta$, $g\in G$, we have $\langle gx,gy\rangle=\langle x,y\rangle$. Thus theorem 6.2.2 actually tells us everything about the intersection form on $\bar{F}$. For $x=\sum_G n_g\,g\in\mathbb{Z}(G)$, let $\bar{x}=\sum_G n_g\,g^{-1}$. Then for $x,\ y\in\mathbb{Z}(G)$, we have $\langle x\eta,y\eta\rangle=\langle x\bar{y}\eta,\eta\rangle$. Now define a function $f:\mathbb{Z}(G)\to\mathbb{Z}$ by sending g to $\langle g\eta,\eta\rangle$, and extending linearly to $\mathbb{Z}(G)$. For all $x\in\mathbb{Z}(G)$, $\langle x\eta,\eta\rangle=f(x)=\mathrm{Tr}(x\hat{f})$, where $\hat{f}=\sum_G f(g)g^{-1}$ and $\mathrm{Tr}\big(\sum n_g\,g\big)=n_e$. Thus for all $x,y\in\mathbb{Z}(G)\eta$,

$$\langle x\eta,y\eta\rangle=\langle x\bar{y}\eta,\eta\rangle=\mathrm{Tr}(x\bar{y}\hat{f}).$$

Note that this form is symmetric if n is even, and skew-symmetric if n is odd. This is true because it is the intersection form on a manifold of dimension $2n$, but we can also verify it directly by checking that $\hat{f}$ conjugates to $(-1)^n\hat{f}$, therefore $\langle y\eta,x\eta\rangle=\mathrm{Tr}(\bar{x}y\hat{f})=\mathrm{Tr}(\overline{x\bar{y}\hat{f}})=(-1)^n\langle x\eta,y\eta\rangle$. From this point on we will assume that $n=2m$, in which case our form is symmetric, and $(-1)^{n(n+1)/2}$ simplifies to $(-1)^m$. We are interested in the signature of the intersection form over $H_{2m}(\bar{F};\mathbb{R})=\mathbb{R}(G)\eta$.

We will continue in a more general context. Let G be any abelian group, A an element of $\mathbb{R}(G)$ such that $\bar{A}=A$, and consider the G-invariant symmetric bilinear form $(x,y)=\mathrm{Tr}(x\bar{y}A)$. Let σ be the signature of this form.

Proposition 6.2.3 *If $\hat{G}$ is the set of irreducible complex representations of G, then $\sigma=\sum_{\chi\in\hat{G}}\operatorname{sign}\chi(A)$.*

Proof: For any $\chi \in \hat{G}$, put

$$s_\chi = \sum_{g \in G} \chi(g)\, g^{-1}, \quad t_\chi = s_\chi + s_{\bar{\chi}}, \quad \text{and} \quad u_\chi = -i\,(s_\chi - s_{\bar{\chi}})\,,$$

all elements of the complex group ring $\mathbb{C}(G)$. When we consider $\mathbb{C}(G)$ as a representation of G, it decomposes as $\bigoplus_{\chi \in \hat{G}} \mathbb{C}(G)_\chi$, where

$$\mathbb{C}(G)_\chi = \{\alpha \in \mathbb{C}(G) \mid \forall g \in G,\ g\alpha = \chi(g)\alpha\}$$

is the complex one-dimensional subspace of $\mathbb{C}(G)$ spanned by s_χ. This leads to the decomposition

$$\mathbb{R}(G) = \bigoplus_{\text{pairs } \chi, \bar{\chi}} (\mathbb{C}(G)_\chi + \mathbb{C}(G)_{\bar{\chi}}) \cap \mathbb{R}(G),$$

where $(\mathbb{C}(G)_\chi + \mathbb{C}(G)_{\bar{\chi}}) \cap \mathbb{R}(G)$ is spanned by t_χ and u_χ. For $x \in \mathbb{C}(G)_{\chi_1}$, $y \in \mathbb{C}(G)_{\chi_2}$, and $g \in G$, $(x, y) = (gx, gy) = \chi_1(g)\chi_2(g)(x, y)$. If χ_1 is not equal to $\bar{\chi}_2$, then we can pick $g \in G$ such that $\chi_1(g)\chi_2(g) \neq 1$, therefore $(x, y) = 0$. Thus our bilinear form is orthogonal with respect to the above decomposition of $\mathbb{R}(G)$, therefore to calculate its signature we can add up the signatures σ_χ of the restrictions to each piece $(\mathbb{C}(G)_\chi + \mathbb{C}(G)_{\bar{\chi}}) \cap \mathbb{R}(G)$.

First look at the pieces where $\chi = \bar{\chi}$. These pieces are spanned by t_χ alone, and $(t_\chi, t_\chi) = \mathrm{Tr}(t_\chi \bar{t_\chi} A) = \frac{1}{|G|}\,\mathrm{Tr}(t_\chi A)$, therefore $\sigma_\chi = \mathrm{sign}\,\mathrm{Tr}(t_\chi A)$.

Now look at the pieces where $\chi \neq \bar{\chi}$. Since G is an abelian group acting irreducibly on a two dimensional vector space, it must act by rotation. A rotation-invariant inner product on $\mathbb{R}^2$ is unique up to scalar, hence our form must be a multiple of the standard inner product. It follows that $\sigma_\chi = 2\,\mathrm{sign}(t_\chi, t_\chi) = 2\,\mathrm{sign}\,\mathrm{Tr}(t_\chi A) = \mathrm{sign}\,\mathrm{Tr}(t_\chi A) + \mathrm{sign}\,\mathrm{Tr}(t_{\bar{\chi}} A)$. Thus $\sigma = \sum_{\chi \in \hat{G}} \sigma_\chi = \sum_{\chi \in \hat{G}} \mathrm{sign}\,\mathrm{Tr}(t_\chi A)$.

Put $A_g = \mathrm{Tr}(gA)$, the coefficient of g^{-1} in A. Since $A = \bar{A}$, $A_g = A_{g^{-1}}$, therefore

$$\begin{aligned}
\mathrm{Tr}(t_\chi A) &= \sum_G (\chi(g) + \bar{\chi}(g))\, A_g \\
&= \sum_{g \in G} (\chi(g) A_g + \bar{\chi}(g) A_g) \\
&= \sum_{g \in G} \left(\chi(g^{-1}) A_g + \chi(g^{-1}) A_g\right) \quad \text{because } A_{g^{-1}} = A_g \\
&= 2 \cdot \sum_{g \in G} \chi(g^{-1}) A_g \\
&= 2 \cdot \chi(A).
\end{aligned}$$

Then $\mathrm{sign}\,\mathrm{Tr}(t_\chi A) = \mathrm{sign}\,\chi(A)$, and $\sigma = \sum_{\hat{G}} \mathrm{sign}\,\chi(A)$. □

We will now apply this theorem to $G = G_{a_0} \times \ldots \times G_{a_n}$ and $A = \hat{f}$. The irreducible characters of G are indexed by $(n+1)$-tuples $(x_0, \ldots, x_n)$, $0 \le x_k < a_k$, taking w_k to $\varepsilon_k^{x_k}$ for all $0 \le k \le n$.

Lemma 6.2.4 *If any $x_k = 0$, then $\chi(\hat{f}) = 0$.*

Proof: Suppose that $x_0 = 0$. Then χ descends to a function χ' on $G' = G/G_{a_0}$, and

$$\begin{aligned}\chi(\hat{f}) &= \sum_{g \in G} f(g)\chi(g^{-1}) \\ &= \sum_{h \in G'} \sum_{g_0 \in G_{a_0}} f(g_0 h)\chi'(h^{-1}) \\ &= \sum_{h \in G'} \chi'(h^{-1}) \sum_{g_0 \in G_0} f(g_0 h).\end{aligned}$$

Either $f(g_0 h) = 0$ for all $g_0 \in G_{a_0}$, or $f(g_0 h) = +1$ for exactly one value of g_0, -1 for exactly one value of g_0, and 0 for all other values of g_0. Hence $\sum_{g_0 \in G_0} f(g_0 h) = 0$ for all $h \in G'$, and therefore the total sum $\chi(\hat{f})$ vanishes. □

Proposition 6.2.5 *Let χ be the irreducible character corresponding to the $(n+1)$-tuple $(x_0, \ldots, x_n)$, with $0 < x_k < a_k$. Then $\chi(\hat{f}) < 0$ if and only if $0 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 1 \mod 2\mathbb{Z}$, and $\chi(\hat{f}) > 0$ if and only if $1 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 2 \mod 2\mathbb{Z}$.*

Proof: Evaluating χ on $\hat{f}$, and plugging in the values of f computed in Theorem 6.2.2, we have

$$\begin{aligned}\chi(\hat{f}) &= \sum_{0 \le k_j < a_j \forall j} \varepsilon_0^{-x_0 k_0} \ldots \varepsilon_n^{-x_n k_n} f(w_0^{k_0} \ldots w_n^{k_n}) \\ &= (-1)^m \sum_{0 \le k_j < a_j \forall j} \varepsilon_0^{-x_0 k_0} \ldots \varepsilon_n^{-x_n k_n} \left(r(w_0^{k_0} \ldots w_n^{k_n}) + s(w_0^{x_0 k_0} \ldots w_n^{x_n k_n}) \right) \\ &= (-1)^m \sum_{k_j \in \{0,1\} \forall j} \varepsilon_0^{-x_0 k_0} \ldots \varepsilon_n^{-x_n k_n} (-1)^{\#\{k_j \neq 0\}} \\ &\qquad + (-1)^m \sum_{k_j \in \{0,-1\} \forall j} \varepsilon_0^{-x_0 k_0} \ldots \varepsilon_n^{-x_n k_n} (-1)^{\#\{k_j \neq 0\}}.\end{aligned}$$

Each of these sums then factors as a product, and we have

$$\begin{aligned}\chi(\hat{f}) &= (-1)^m \prod_{j=0}^{n} \left(1 - \varepsilon_j^{-x_j}\right) + (-1)^m \prod_{j=0}^{n} \left(1 - \varepsilon_j^{x_j}\right) \\ &= 2 \cdot (-1)^m \operatorname{real}\left[\prod_{j=0}^{n} \left(1 - \varepsilon_j^{x_j}\right) \right].\end{aligned}$$

We now use the identity $1 - e^{2\theta} = -2ie^{\theta}\sin(\theta)$ to obtain

$$\begin{aligned}\chi(\hat{f}) &= 2\cdot(-1)^m \operatorname{real}\left[\prod_{j=0}^{n}\left(-2ie^{\pi i\frac{x_j}{a_j}}\sin\left(\pi\frac{x_j}{a_j}\right)\right)\right] \\ &= 2\cdot i^n\cdot(-1)^m \operatorname{real}\left[\prod_{j=0}^{n}\left(2\sin\left(\pi\frac{x_j}{a_j}\right)\right)e^{\pi i\left(\frac{1}{2}+\sum\frac{x_j}{a_j}\right)}\right] \\ &= 2\cdot\operatorname{real}\left[\prod_{j=0}^{n}\left(2\sin\left(\pi\frac{x_j}{a_j}\right)\right)e^{\pi i\left(\frac{1}{2}+\sum\frac{x_j}{a_j}\right)}\right], \text{ since } i^n = (-1)^m.\end{aligned}$$

Since we are taking the sign of this expression, we can drop the positive number $2\cdot\prod_{j=0}^{n}\left(2\sin\left(\pi\frac{x_j}{a_j}\right)\right)$, and we are left with

$$\chi(\hat{f}) = \operatorname{sign}\operatorname{real}\left[\exp\left(\pi i\left(\frac{1}{2}+\sum\frac{x_j}{a_j}\right)\right)\right],$$

which proves Proposition 6.2.5. □

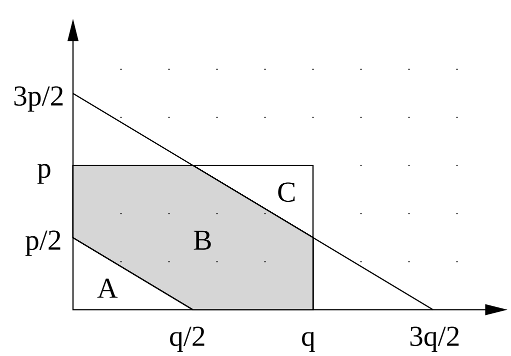
We now specialize to the case $\Sigma = \Sigma(p, q, 2, ..., 2)$, where p, q are odd and relatively prime. Let $N_{p,q} = \#\left\{1 \le x \le \frac{p-1}{2} \mid 1 \le qx \le \frac{p-1}{2} \mod p\right\}$.

Proposition 6.2.6 *The signature σ of the Brieskorn link $\Sigma(p, q, 2, ..., 2)$ is equal to the quantity* $(-1)^m\left[\frac{1}{2}(p-1)(q-1) + 2(N_{p,q} + N_{q,p})\right]$.

Proof: Assume that m is even, so that $(-1)^m = 1$. The case m odd follows easily from this one. Taken together, Propositions 6.2.3 and 6.2.5 tell us that

$$\sigma^+(\bar{F}) = \#\left\{(x_0, x_1)\ \Big|\ \frac{1}{2} < \frac{x_0}{p} + \frac{x_1}{q} < \frac{3}{2}\right\}, \qquad \text{and}$$

$$\sigma^-(\bar{F}) = \#\left\{(x_0, x_1)\ \Big|\ \frac{x_0}{p} + \frac{x_1}{q} < \frac{1}{2} \text{ or } \frac{x_0}{p} + \frac{x_1}{q} > \frac{3}{2}\right\}.$$



In the above picture, $\sigma^+(\bar{F})$ is equal to the number of interior lattice points in region B, and $\sigma^-(\bar{F})$ is equal to the number of interior lattice points in regions A and C, or twice the number of interior lattice points in region A. Since $(p,q,2)=1$, there are no lattice points on the diagonal lines, hence $\sigma^+(\bar{F})+\sigma^-(\bar{F})=(p-1)(q-1)$. Then $\sigma(\bar{F})=(p-1)(q-1)-2\cdot\sigma^-(\bar{F})=(p-1)(q-1)-4\cdot T$, where T is the number of interior lattice points in region A.

$$T=\sum_{x=1}^{\frac{p-1}{2}}\left[\frac{q}{2}-\frac{qx}{p}\right]=\sum_{x=1}^{\frac{p-1}{2}}\left(\frac{q-1}{2}-\left[\frac{qx}{p}\right]+\left[\frac{1}{2}-R_{p,q}\right]\right),$$

where [_] denotes greatest integer function, and $R_{p,q}$ is the least non-negative integer representing qx mod p.

Evaluating this sum, we have

$$T=\frac{p-1}{2}\cdot\frac{q-1}{2}-N_{p,q}-\sum_{x=1}^{\frac{p-1}{2}}\left[\frac{qx}{p}\right].$$

By symmetry, we may switch p and q in the above equation. Adding the two equations together, we get

$$2\cdot T=\frac{1}{2}\cdot(p-1)(q-1)-N_{p,q}-N_{q,p}-\sum_{x=1}^{\frac{p-1}{2}}\left[\frac{qx}{p}\right]-\sum_{y=1}^{\frac{q-1}{2}}\left[\frac{py}{q}\right].$$

We can calculate the quantity $\sum_{x=1}^{\frac{p-1}{2}}\left[\frac{qx}{p}\right]+\sum_{y=1}^{\frac{q-1}{2}}\left[\frac{py}{q}\right]$ by a lattice point argument that is familiar from one of the common proofs of quadratic reciprocity. The sum $\sum_{x=1}^{\frac{p-1}{2}}\left[\frac{qx}{p}\right]$ is equal to the number of interior lattice points in the triangle with vertices at $(0,0)$, $(\frac{p}{2},0)$, and $(\frac{p}{2},\frac{q}{2})$, while $\sum_{y=1}^{\frac{q-1}{2}}\left[\frac{py}{q}\right]$ is the number of interior lattice points in the triangle with vertices at $(0,0)$, $(0,\frac{q}{2})$, and $(\frac{p}{2},\frac{q}{2})$. When we add these two sums together, we get the number of interior lattice points of the rectangle of width $\frac{p}{2}$, height $\frac{q}{2}$, and two sides along the coordinate axes, which is equal to $\frac{p-1}{2}\cdot\frac{q-1}{2}$. Then $\sigma(\bar{F})=(p-1)(q-1)-4T=\frac{1}{2}(p-1)(q-1)+2(N_{p,q}+N_{q,p})$. □

Corollary 6.2.7 *$\sigma\big(\Sigma(6k-1,3,2,...,2)\big)=(-1)^m\cdot 8k$. Thus $bP_{4m}\cong 8\mathbb{Z}/\sigma_m\mathbb{Z}\cong\mathbb{Z}_{\frac{\sigma_m}{8}}$, and all of its elements are of the form $\Sigma(p,q,2,...,2)$.*

The main results of this paper can be summarized by the following

Theorem 6.2.8 *Let Σ^n be a homotopy sphere, $n\geq 5$. The following are equivalent:*

1) *Σ is the cyclic branched cover of S^n along an oriented knot K^{n-2}*

2) *Σ embeds in S^{n+2}*

3) *$\Sigma\in bP_{n+1}$.*

Proof: 1 ⇒ 2 by Remark 4.2.1, which says that the cyclic branched cover of a sphere along a manifold of codimension 2 itself embeds in codimension 2. Now suppose that we are given an embedding of Σ in S^{n+2}. Since Σ is a homotopy sphere, $H^2(\Sigma) = 0$, therefore there is no obstruction to trivializing the normal bundle. Theorem 4.1.1 tells us that Σ bounds a Seifert manifold, which is oriented. An oriented, $(n+1)$-dimensional submanifold of S^{n+2} has a trivial normal bundle, therefore it is stably parallelizable. But this Seifert manifold has a nonvacuous boundary, therefore by Corollary 2.2.2 it is parallelizable, and $\Sigma \in bP_{n+1}$. This shows that $2 \Rightarrow 3$. Finally, $3 \Rightarrow 1$ by the computations in this and the previous section. $\square$

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