AN INTRODUCTION TO ∞-CATEGORIES

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Abstract. These are notes prepared for David Ben-Zvi's 2012 workshop on Categorical Representation Theory. I assume no knowledge of simplicial sets, or of categories enriched in spaces. I do assume people know what a category is, and what limits and colimits are.

As David wrote me, "I'd really like people to accept it as a shiny black box and not get too intimidated by the machine, rather to get a feel for what it's for, how it functions in the large, what it's like to use it etc.

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1. Introduction

1.1. **Some categories have spaces of morphisms.** In algebraic topology, things become much easier when we treat the set of maps between X and Y as a *space* of maps from X to Y. For instance, given two morphisms $f_0, f_1 : X \to Y$, it makes sense to ask whether there's a path between them – that is, whether the two maps are *homotopic*. This makes the fundamental group (and homotopy groups in general) easy to work with, and gives rise to incredibly useful notions like homotopy equivalence. The notion that the set of morphisms is a *space* is useful in other categories, like the category of chain complexes.[1]

1.1.1. *Finding an appropriate language for such categories.* Perusing through Maclane [Maclane72], you see immediately that there's a ton of useful category theory for usual categories. And we know from previous talks that things like the Barr-Beck theorem help us say great things about categories. (For instance, knowing that some mystery category $\mathcal{C}$ is all of a sudden a category of modules gives you a lot of information!) It would be wonderful if all the tools from set-enriched categories could transfer over to space-enriched categories.

But to fully embrace the topological, or homotopical nature of these categories, we need a new framework. For instance, any contractible space and the one-point space are homotopy equivalent spaces. But in the language of ordinary category theory, only the one-point space is a terminal object.

1.2. **Definition of ∞-category.** I advocate that the correct framework for all this is the framework of ∞-*categories.* At first pass it's not clear that the definition will fit all our needs, but hopefully by the end of this talk, and the end of this week, you'll be convinced it's a good notion.

I'll give the definition of an ∞-category now. You won't know what a simplicial set is yet, but don't worry, I'll get there. I just want to leave this definition on the board so that, as I go through the talk, you'll see some of the motivations as we plow forward.

Definition 1.1 (∞-category)**.** Let $\mathcal{C}$ be a simplicial set. Let Λ^n_i be the simplicial set obtained from the n-simplex Δ^n by deleting the ith face and the interior. We say that $\mathcal{C}$ is an ∞-*category* if any inner horn can always be filled. That is,

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathcal{C} \\ \big\downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

for $0 < i < n$, $n \geq 2$. A *functor* $f : \mathcal{C} \to \mathcal{C}'$ is just a map of simplicial sets.

If this definition remains cryptic, don't worry. For now let's talk about what a simplicial set actually is. I'll motivate it by observing the structure of categories. Then we'll talk about how this actually recovers a notion of a *morphism space*, how to go from a topologically enriched category to an ∞-category, and how to define some of the usual notions from category theory (co/limits).

[1] For instance, homology is useful because it takes homotopies of maps of *spaces* to homotopies of maps of *chain complexes*. This is a consequence of the fact that the functor taking a space to its complex of singular chains is in fact a functor which is a continuous map on morphism spaces.

2. Categories as certain simplicial sets

The first time I was introduced to simplicial sets, the definition seemed a bit arbitrary. Let me make a few observations about categories to motivate the definition of a simplicial set.

2.1. **The simplices of a category.** Let $\mathcal{C}$ be a category. What does this mean? Well, $\mathcal{C}$ has a set of *objects*.[2] I'll denote this set as $\mathrm{Ob}\,\mathcal{C} = \mathcal{C}[0]$ and I'm going to draw the objects of $\mathcal{C}$ as vertices.

$$\mathcal{C}[0] := \mathrm{Ob}\,\mathcal{C}$$

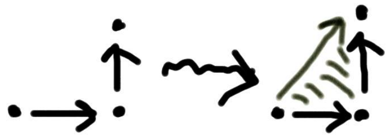
And of course for every pair of objects $X_0, X_1 \in \mathcal{C}[0]$ one also has a set of morphisms $\mathcal{C}(X_0, X_1)$. We will denote by $\mathcal{C}[1]$ the set of all morphisms:

$$\mathcal{C}[1] := \coprod_{X_0, X_1 \in \mathcal{C}[0]} \mathcal{C}(X_0, X_1)$$

And there is more structure – any category $\mathcal{C}$ has a *composition rule*, which is a map of sets

$$\mathcal{C}(X_1, X_2) \times \mathcal{C}(X_0, X_1) \to \mathcal{C}(X_0, X_2).$$

In terms of pictures, this says that the following picture with edges f_{01}, f_{12} can be filled to form a commutative triangle, with third edge $f_{02} = f_{12} \circ f_{01}$.



The composition rule.

This actually tells us two pieces of information at once – first, it tells us whether a drawing of a triangle should be considered a *commutative triangle*. Second, it tells us that one can obtain a commutative triangle from two successive edges. We now separate out these two pieces of data. First, since we know what triangles are commutative, we will set

$$\mathcal{C}[2] = \{\text{commutative triangles in } \mathcal{C}\}.$$

The composition rule is something we'll ignore for the moment – as it turns out there will be no such rule in a quasi-category. So we'll come back to the data of two successive edges giving rise to a unique triangle in just a moment. Also, for the moment let me ignore the associativity rule of a category. I'll get back to that too.

By now you probably see a pattern. I drew $\mathcal{C}[0]$ as the set of objects in a category, which I think of as a collection of *0-simplices*. I defined $\mathcal{C}[1]$ as the set of morphisms,

[2] I'll assume $\mathcal{C}$ is small, since most categories we work with are essentially small.

which we've always drawn as arrows. So I've drawn elements of $\mathcal{C}[1]$ as a directed *1-simplex*, and likewise an element of $\mathcal{C}[2]$ is a *2-simplex* representing a commutative triangle. So I'm going to define

$$\mathcal{C}[n] = \{\text{commutative } n\text{-simplices in } \mathcal{C}\}.$$

Now I claim that some morphisms among the sets $\mathcal{C}[n]$, along with a property similar to the defining property of ∞-categories (Definition 1.1), completely characterize the idea of a category.

2.2. **The simplicial relations.** Let's say you have a commutative n-simplex. I'll write it as a string

$$X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \ldots \xrightarrow{f_{(n-1)n}} X_n.$$

One consequence of the associativity condition is that this string is enough for you to determine the n-simplex entirely. (As it turns out this will not be true for ∞-categories.)

I want to note that there's a few natural ways to get a commutative $(n-1)$-simplex from this n-simplex. Namely, pick an object, and forget it, composing morphisms if necessary. For instance, forgetting the 0th object, we get

$$X_1 \xrightarrow{f_{12}} X_2 \ldots \xrightarrow{f_{(n-1)n}} X_n.$$

while forgetting the ith object for $0 < i < n$, we get

$$X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} \ldots X_{i-1} \xrightarrow{f_{i(i+1)} \circ f_{(i-1)i}} X_{i+1} \ldots \xrightarrow{f_{(n-1)n}} X_n.$$

Let's denote by d_i the map 'forget the ith object.'

$$d_i : \mathcal{C}[n] \to \mathcal{C}[n-1].$$

This exists for any $0 \leq i \leq n$, so long as $n \geq 1$. (There is no set $\mathcal{C}[-1]$.) We'll call it *the ith face map*. This is because if you draw the n-simplex, d_i precisely picks out the face opposite the ith vertex. We can summarize these maps in the following diagram:

$$\mathcal{C}[0] \leftleftarrows \mathcal{C}[1] \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; \mathcal{C}[2] \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}\; \cdots$$

Going the other way, given an n-simplex, there's actually a way to canonically get an $(n+1)$-simplex. Namely, you insert the identity morphism after the ith object:

$$X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} \ldots \xrightarrow{f_{(i-1)i}} X_i \xrightarrow{\mathrm{id}_{X_i}} X_i \xrightarrow{f_{i(i+1)}} \ldots \xrightarrow{f_{(n-1)n}} X_n.$$

We'll call this map

$$s_i : \mathcal{C}[n] \to \mathcal{C}[n+1]$$

and call it the *ith degeneracy map*. Note there is a map for $0 \leq i \leq n$. These maps can all be drawn as follows:

$$\mathcal{C}[0] \longrightarrow \mathcal{C}[1] \rightrightarrows \mathcal{C}[2] \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots$$

Moreover, if you think about what the maps s_i and d_i do, you can prove the following relations. I'll leave the verification as an exercise; it's very straightforward.

$$(1) \qquad \begin{cases} d_i d_j = d_{j-1} d_i & i \leq j \\ s_i s_j = s_{j+1} s_i & i \leq j \\ s_i d_j = d_{j+1} s_i & i < j \\ s_i d_j = d_j s_{i+1} & i \geq j \end{cases}$$

These are called the *simplicial relations.*

Definition 2.1 (Simplicial set)**.** A collection of sets $\mathcal{D}[n]$, $n \in \mathbb{Z}_{\geq 0}$, together with maps

$$s_i : \mathcal{D}[n] \to \mathcal{D}[n+1], \qquad d_i : \mathcal{D}[n] \to \mathcal{D}[n-1]$$

for $0 \leq i \leq n$, satisfying the simplicial relations, is called a *simplicial set.*

Exercise 2.2 (The category of simplicial sets)**.** *Let Δ be the category whose objects are finite, non-empty, ordered sets, and whose morphisms are weakly order-preserving maps.*

(1) *Show that a simplicial set is the same thing as a functor*

$$\Delta^{\mathrm{op}} \to \mathsf{Sets}.$$

We let sSet *denote the category of simplicial sets, where morphisms are given by natural transformations.*

We denote by Δ^n the functor represented by the ordered set $[n] = \{0 < 1 < \ldots < n\}$. This means

$$\Delta^n[m] = \{\textit{order-preserving maps } [m] \to [n]\}.$$

*We call this the n-*simplex*.*

(2) *Show that there is a natural isomorphism of sets $\mathcal{D}[n] \cong \mathsf{sSet}(\Delta^n, \mathcal{D})$.*
(3) *Write down the simplicial set corresponding to the horn Λ^n_i.*
(4) *Let $\mathcal{C}, \mathcal{D}$ be simplicial sets. We define a collection of sets $(\mathcal{C} \times \mathcal{D})[n] := \mathcal{C}[n] \times \mathcal{D}[n]$. Show that $(\mathcal{C} \times \mathcal{D})[\bullet]$ naturally has the structure of a simplicial set.*
(5) *Show that* sSet *is enriched over itself. For instance, show that*

$$\mathsf{sSet}(\mathcal{C}, \mathcal{D})[n] := \mathsf{sSet}(\mathcal{C} \times \Delta^n, \mathcal{D})[0]$$

where $\mathsf{sSet}(-,-)[0]$ is the set of natural transformations, defines a simplicial set $\mathsf{sSet}(\mathcal{C}, \mathcal{D})$ with the obvious face and degeneracy maps.

Example 2.3 (BG)**.** Let G be a discrete group. Let BG be the simplicial set such that $BG[k] = G^k$. The face maps are given by

$$d^0 : (g_1, \ldots, g_n) \mapsto (g_2, \ldots, g_n), \qquad d^n : (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{n-1}),$$

$$d^i : (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) \qquad (0 < i < n).$$

The degeneracy maps are given by inserting id_G in various places. Note that a functor $BG \to \mathcal{C}$ picks out an object in $\mathcal{C}$ together with a group action.

2.3. **Categories as a simplicial set with a horn-filling condition.** By design every category $\mathcal{C}$ gives rise to a simplicial set $\mathcal{C}[\bullet]$. This simplicial set in fact satisfies something very similar to the condition in Definition 1.1. First, let's let Λ^1_2 be the diagram one obtains by taking a 2-simplex, then deleting the interior and the face opposite the 1st vertex of the 2-simplex. (This is the same data as two successive edges.) Then if the simplicial set $\mathcal{C}[\bullet]$ arises from a category, we can complete this 'horn' into a full-blown 2-simplex: one can fill in the diagram

$$\begin{array}{ccc} \Lambda^2_1 & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow_{\exists !} & \\ \Delta^2 & & \end{array} .$$

Moreover this filling is unique, as indicated. This is due to the *composition law* – after all, given two morphisms f_{01}, f_{12}, there is only one way to compose them.

We can go further. Let Λ^i_n denote the diagram one obtains by taking an n-simplex, and deleting the face opposite the ith vertex (we delete the interior of the n-simplex as well). Then an easy exercise using the *associativity law* shows that we can fill such a horn in uniquely too, provided $0 < i < n$:

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow_{\exists !} & \\ \Delta^n & & \end{array} .$$

Staring at this commutative diagram and the diagram in Definition 1.1, you see that the only difference is in the uniqueness of this filling.

2.4. **What's it like to work in this setting?** I want to take a break at this point. The only difference between an ordinary/usual category and an ∞-category is – I am not kidding you – this uniqueness property of how one can fill in horn. That's it. I'll go into the details of what this means for us in a second, but David wanted me to say what it's like to work in this setting, so let me comment on that in two ways.

2.4.1. *As someone who just uses category theory.* As I motivated earlier, we want to formulate classical theorems, and be able to prove them, for categories which have more homotopical flavor. There are plenty of ways in which one can encapsulate the notion of a homotopical category, like model structures, enriching over spaces, or Segal spaces. But the question is: Can we write a Maclane-type tome of results using those other frameworks? No doubt we one day will, but we haven't. The only setting in which all the useful results from classical category theory have been translated into, and proven in, a homotopical setting is in the ∞-category setting. What does this mean for you, practically? In a sense it means you can retain all the algebraic intuitions you had from classical categories and apply them to the ∞-category setting. Adjunctions are more or less what you think they are, as are Kan extensions and all that.

2.4.2. *As someone who wants to prove something in category theory.* That being said, what's it like to work with an ∞-category? The point is that simplices are just a nice way to encode homotopy-commutative diagrams, just as simplices are a nice way to encode some portion of a topological space. A category is a massive thing, just as a topological space is, but we can reduce the proof of most categorical statements to proofs about how the combinatorics of simplices behave.

Moreover, the lack of uniqueness means we often don't have a composition law. Given two successive edges $X_0 \to X_1, X_1 \to X_2$, all we know is that there exists some edges $X_0 \to X_2$ which we'd like to call the composition, but there is no 'the' composition. This is the same issue that arises when one tries to define the fundamental group – there is no canonical way to compose two paths sharing an endpoint. Indeed, the willingness to depart from a strict composition law is one way in which this whole business becomes easier to deal with.[3]

2.5. **Kan complexes are ∞-groupoids are spaces.** So what does this 'lack of uniqueness' mean for us? The interpretation I'll promote is that the simplices of an ∞-category $\mathcal{C}$ are not to be interpreted as *commutative* diagrams anymore – they should be interpreted as *homotopy commutative* diagrams, together with prescribed homotopies between the various simplices of the diagram. Here's an example.

Exercise 2.4. *Let W be a topological space. Let*

$$\mathrm{Sing}(W)[n] = \mathrm{Maps}(|\Delta^n|, W)$$

where $|\Delta^n|$ is the topological space corresponding to the standard n-simplex, and Maps *is the* set *of continuous maps. Show that* $\mathrm{Sing}(W)[n]$ *defines a simplicial set, where the face maps d_i are given by restriction of maps.*

I claim that not only is $\mathrm{Sing}(W)$ a simplicial set, it's also an ∞-category. Here's an illustration in the most basic case. It's easy to see that the map $\Lambda^2_1 \to \mathrm{Sing}(W)$ corresponds to a pair of maps $a, b : |\Delta^1| \cong [0,1] \to W$ such that $a(1) = b(0)$. And to find a map $\Delta^2 \to \mathrm{Sing}(W)$ extending this is the same thing as finding a map $|\Delta^2| \to W$ of spaces which restricts to a and b on the 0th and 2nd faces, respectively.

Filling the horn Λ^2_1.

This is easy to do, and in fact there are many continuous maps $|\Delta^2| \to W$ that fulfill this condition.

So how do we interpret $\mathrm{Sing}(W)$ as a type of category, and what does the non-uniqueness mean? The interpretation is that the 0-simplices of $\mathrm{Sing}(W)$ are the *objects.*

[3]That being said, many times our ∞-categories will arise from topologically enriched categories that do have an underlying composition, but as I've advocated, forgetting that structure and simply remembering the simplices of the category is what makes many proofs easier.

The 1-simplices – paths from one point to another – are the morphisms, and the higher simplices are *homotopies* of morphisms (and higher homotopies between these). For instance, a 2-simplex is a homotopy between two paths – the path obtained by concatenating the edges $0 \to 1$ and $1 \to 2$, and the path obtained by the edge of Δ^2 which goes from 0 to 2. (Here, the numbers $0, 1, 2$ are the vertices of Δ^2.) Similarly, a 3-simplex is a homotopy between various 2-simplices.

This ∞-category is very special, in that every morphism is invertible up to homotopy.

Exercise 2.5 (Groupoids)**.** *Suppose that $\mathcal{C}$ is not only a category, but also a* groupoid*. This means that every morphism has an inverse. Show that the associated simplicial set satisfies the filling condition*

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow_{\exists!} & \\ \Delta^n & & \end{array} .$$

for any *horn – i.e., even when $i = 0$ or $i = n$.*

Exercise 2.6 (∞-groupoids, Kan complexes)**.** *Show that* $\mathrm{Sing}(W)$ *satisfies the filling condition*

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array} .$$

for any *horn – i.e., even when $i = 0$ or $i = n$. This is the ∞-analogue of a groupoid. We call any simplicial set satisfying this filling condition an ∞-*groupoid*, or a* Kan complex*. (Both terms are used in the literature.)*

Definition 2.7. Let $\partial\Delta^n$ denote the simplicial set obtained by deleting the interior of the n-simplex. Let $\mathcal{C}$ be a Kan complex. We say that $\mathcal{C}$ is *contractible* if $\mathcal{C}$ is non-empty and one can always fill the following diagram:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array} .$$

Exercise 2.8. *Show that if W is a contractible space, then* $\mathrm{Sing}(W)$ *is a contractible Kan complex.*

Exercise 2.9. *Let* Kan *denote the full subcategory of* sSet *consisting of Kan complexes. Note that we have a functor*

$$\mathrm{Sing} : \mathsf{Spaces} \to \mathsf{Kan}.$$

Moreover, given any simplicial set, we have a functor called geometric realization *which produces a topological space. This is done by gluing together the n-simplices of a simplicial set via the face maps d_i. We denote this functor*

$$| \bullet | : \mathsf{Kan} \subset \mathsf{sSet} \to \mathsf{Spaces}.$$

Convince yourself that there is a natural homotopy equivalence $|\mathrm{Sing}\, W| \simeq W$.

Remark 2.10. We don't have time to go into the details here, but as a philosophy, "Kan complexes are the same as topological spaces." More precisely, there is a Quillen equivalence between a model category of simplicial sets and a model category of topological spaces, where the model structure on simplicial sets has Kan complexes as fibrant-cofibrant objects. From hereon, whenever I come across a Kan complex, I will pretend that it has the same mathematical worth as a topological space. I really wish I had more time to go into this, but this will have to do.

Remark 2.11 (Some bearings)**.** By definition we have the inclusions

$$\text{categories} \subset \infty\text{-categories} \subset \mathsf{sSet}. \qquad \text{groupoids} \subset \mathsf{Kan} \subset \mathsf{sSet}.$$

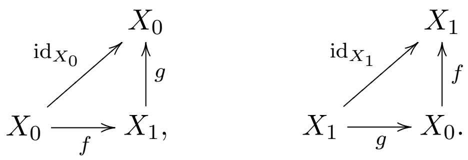
2.6. **Identity morphism and equivalences.** For every object $X \in \mathcal{C}[0]$ there is a canonical edge $s_0 X \in \mathcal{C}[1]$. We call this the *identity morphism* id_X of X. We say that a morphism $f : X_0 \to X_1$ is an *equivalence* if there is a map $g : X_1 \to X_0$ and 2-simplices $A, B \in \mathcal{C}[2]$ such that

$$d_0 A = g, \qquad d_1 A = \mathrm{id}_{X_0}, \qquad d_2 A = f$$

and

$$d_0 B = f, \qquad d_1 B = \mathrm{id}_{X_1}, \qquad d_2 B = g.$$

All this means is that there's two homotopy commutative diagrams in $\mathcal{C}$ as follows:



Note that A and B don't just tell you that this diagram is homotopy commutative, they actually specify the homotopies $f \circ g \sim \mathrm{id}_{X_1}$ and $g \circ f \sim \mathrm{id}_{X_0}$.

2.7. **Morphism spaces.** So how does one extract a 'space' of morphisms from two vertices $X_0, X_1 \in \mathcal{C}[0]$ in an ∞-category? There are two natural ways:

Definition 2.12 (Morphism spaces)**.** Let $X_0, X_1 \in \mathcal{C}[0]$. Then the *right morphism space* $\mathcal{C}^R(X_0, X_1)$ is defined to be the simplicial set whose n-simplices are maps $u : \Delta^{n+1} \to \mathcal{C}$ satisfying

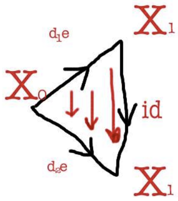
(1) $u|_{d_{n+1}\Delta^{n+1}} \mapsto X_0$.
(2) $u(n+1) = X_1$,

The *left morphism space* $\mathcal{C}^L(X_0, X_1)$ is defined to be the simplicial set whose n-simplices are maps $u : \Delta^{n+1} \to \mathcal{C}$ satisfying

(1) $u(0) = X_0$,
(2) $u|_{d_0\Delta^{n+1}} = X_1$.

Example 2.13. For example, a vertex $f \in \mathcal{C}^L(X_0, X_1) = \mathcal{C}^R(X_0, X_1)$ is an edge in $\mathcal{C}$ with initial vertex X_0 and final vertex X_1.

A 2-simplex $e \in \mathcal{C}^L(X_0, X_1)$ is a triangle whose 0th vertex is X_0, and whose face opposite X_0 is the degenerate edge $s_0 X_1$. The triangle has two other edges, $d_1 e$ and $d_2 e$, and one should interpret the 2-simplex e as a homotopy from the morphism $d_2 e$ to the morphism $d_1 e$.



An edge e in $\mathcal{C}^L(X_0, X_1)$ represents a triangle in $\mathcal{C}$ with a degenerate edge.

If you believe that Kan complexes are like morphism spaces, here's the reason I call the above a morphism space:

Proposition 2.14. *$\mathcal{C}^L(X_0, X_1)$ and $\mathcal{C}^R(X_0, X_1)$ are both Kan complexes, provided that $\mathcal{C}$ is an ∞-category. Moreover, their geometric realizations are homotopy equivalent spaces.*

The last bit of the proposition tells us that it doesn't matter which model we use, $\mathcal{C}^L$ or $\mathcal{C}^R$. Either safeguards all the homotopical information of the space of morphisms from X_0 to X_1.

Exercise 2.15. *Let $\text{Sing}(W)^R(X_0, X_0)$ be the Kan complex of right morphisms from a point $X_0 \in W$ to itself. Let ΩW be the based loop space of W based at X_0. Construct a homotopy equivalence $\text{Sing}(\Omega W) \to \text{Sing}(W)^R(X_0, X_0)$.*

Now we can define what it means to be an initial or terminal object of $\mathcal{C}$.

Definition 2.16. We say $X \in \mathcal{C}[0]$ is an *initial object* if $\mathcal{C}^R(X, Y)$ is contractible for all $Y \in \mathcal{C}[0]$. Likewise say X is a *terminal object* if $\mathcal{C}^R(Y, X)$ is contractible for all $Y \in \mathcal{C}[0]$.

Remark 2.17. In general, wherever you see the word 'unique' in classical category theory, in ∞-category jargon, you should see the word 'contractible.'

Exercise 2.18. *Show any two initial objects are equivalent. Show any two terminal objects are equivalent.*

3. Cones, limits/colimits.

I've tried to give a brief overview of the landscape of simplicial sets. Kan complexes are spaces, and ∞-categories are, well, like flimsy versions of categories. At the least they have a notion of morphism spaces, well-defined up to homotopy equivalence.

It's absurd how much of math comes down to proving a statement about limits and colimits. But it's true. And the first step in converting classical category theory to ∞-category theory is to get a good grasp on what limits and colimits really are. We embark on this task now. For this, you need to know what a *cone* is.

3.1. **Cones.** If W is a topological space, we know what its *cone* is. It's most easily described as the space $W \times [0,1]/\sim$, where the equivalence relation collapses everything every point of the form $(w, 0)$. In the world of simplicial sets, where every simplex has a direction, or an orientation, there are two kinds of cones one can make from a simplicial set K. One can form the *right cone* $K^{\rhd}$, which is a simplicial set for which the cone point $*$ receives a canonical edge from every vertex of K. One can also form the *left cone* $K^{\lhd}$, where the cone point is the source of a canonical edge *to* any given vertex of K.

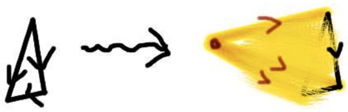
Definition 3.1 (Cones)**.** Let K be a simplicial set. Then the *left cone* $K^{\lhd}$ over K is the simplicial set defined by

$$K^{\lhd}[n] = K[n] \sqcup K[n-1] \sqcup \ldots \sqcup K[0] \sqcup \{*\}.$$

(In the above equation we declare $K[-1]$ to be the one-point set.) For any n-simplex $Y^{\lhd} \in K^{\lhd}[n]$ corresponding to an $(n-1)$-simplex Y in K, we declare

$$d_0(Y^{\lhd}) = Y, \qquad d_i(Y^{\lhd}) = (d_{i-1}Y)^{\lhd}.$$

I'll let you write down the other face maps and the degeneracy maps.



Left cone $K^{\lhd} \cong * \star K$.

The *right cone* $K^{\rhd}$ has the same set of n-simplices as $K^{\lhd}$. But the simplest degeneracy maps are defined by

$$d_n(Y^{\rhd}) = Y, \qquad d_i(Y^{\rhd}) = (d_iY)^{\rhd}$$

following the notation above. Again, it's an exercise for you to write down the other face and degeneracy maps.

Right cone $K^{\rhd} \cong K \star *$.

One can interpret $K^{\rhd}$ as an ∞-category obtained by affixing a terminal object to K, and likewise, $K^{\lhd}$ as attaching an initial object:

Exercise 3.2. *Let K be an ∞-category. Show $K^{\rhd}$ and $K^{\lhd}$ are both ∞-categories. Moreover, the cone point $* \in K^{\rhd}[0]$ is a terminal object, and $* \in K^{\lhd}[0]$ is an initial object.*

3.2. **Limits and colimits.** Let $F : D \to C$ be a functor between ordinary categories. Then one can define a category of objects living under F, and a colimit to F is defined to be an initial object in this under-category. Similarly we define a limit to be a terminal object in an over-category. We do the same thing here.

Exercise 3.3. *Let $F : K \to \mathcal{C}$ be a functor of ∞-categories. (That is, a map of simplicial sets between ∞-categories.) Then define* the over-category

$$\mathcal{C}_{/F} \subset \mathsf{sSet}(K^{\lhd}, \mathcal{C})$$

to be the simplicial set of functors $K^{\triangleleft} \to \mathcal{C}$ such that the restriction to $K \subset K^{\triangleleft}$ agrees with F. Show that $\mathcal{C}_{/F}$ is an ∞-category.

Definition 3.4. Let $F : K \to \mathcal{C}$ be a functor. We say that $X \in \mathcal{C}_{/F}[0]$ is a *limit of F* if X is a terminal object.

Similarly, we can define the *under-category* $\mathcal{C}_{F/}$ of objects under F as functors $K^{\triangleright} \to \mathcal{C}$ which agree with F on $K \subset K^{\triangleright}$. Note we are taking the right cone this time, not the left cone. A *colimit* is an initial object in this category. Let's do some examples.

Example 3.5 (Initial and terminal objects)**.** let $K = \emptyset$ be the empty simplicial set, so $K[n] = \emptyset$ for all n. Then $K^{\triangleleft} = K^{\triangleright} = \Delta^0$, the constant simplicial set also known as the 0-simplex. Then $\mathcal{C}_{\emptyset/} \cong \mathcal{C} \cong \mathcal{C}_{/\emptyset}$ so the limit of the empty diagram is a terminal object of $\mathcal{C}$, and the colimit of the empty diagram is an initial object of $\mathcal{C}$.

Exercise 3.6. *Let $K = \Delta^0$ and choose a functor $F : \Delta^0 \to \mathcal{C}$ of $\mathcal{C}$. Show that the limit and colimit of this diagram is $F(K)$.*

3.3. **Relation to model categorical ideas.** Let's let $\overline{\mathsf{C}}$ be a simplicial model category. Since the fibrant-cofibrant subcategory $\mathsf{C} \subset \overline{\mathsf{C}}$ is enriched over Kan complexes, it turns out that there is a way to construct an ∞-category $\mathcal{C}$ out of C. This is called the *simplicial nerve* of C, and we write

$$\mathfrak{N}\mathsf{C} := \mathcal{C}.$$

We don't go into the details of the construction here, but I will say the following:

Theorem 3.7. *Let $\mathcal{C}$ be the simplicial nerve of a small simplicial model category C. Then*

(1) $\mathcal{C}[0] \cong \mathrm{Ob}\,\mathsf{C}$ *as a set.*
(2) *For any two objects X_0, X_1, the morphisms spaces of $\mathcal{C}$ are homotopy equivalent to the morphism spaces of C. That is, $\mathcal{C}^R(X_0, X_1) \simeq \mathsf{C}(X_0, X_1)$.*
(3) *For any functor of simplicial categories $\mathsf{F} : \mathsf{K} \to \mathsf{C}$, one obtains a functor of quasi-categories $F : K \to \mathcal{C}$. Moreover the homotopy (co)limit of F agrees with the homotopy (co)limit of F.*

Example 3.8. Let $\mathcal{C} = \mathsf{Spaces}_*$ be the category of pointed topological spaces. Then the limit of the diagram

$$\begin{array}{ccc} & & * \\ & & \downarrow \\ * & \longrightarrow & W \end{array}$$

is homotopy equivalent to the based loop space ΩW. Note the the diagram

$$\begin{array}{ccc} \Omega W & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & W \end{array}$$

is not at all a commutative diagram; it is the data of the 1-morphisms as portrayed, and a specific homotopy from the composition $W \to * \to W$ to itself.

3.4. **Sheaves.** I want to point out now that if we have a sheaf with values in an ∞-category, the sheaf condition becomes much more homotopical. Namely, we define a presheaf $\mathcal{F}$ on a space W to be a functor

$$\mathcal{F} : \mathsf{Opens}(W)^{\mathrm{op}} \to \mathcal{C}.$$

Here we are treating the poset $\mathsf{Opens}(W)$ as an ∞-category, and this functor is a functor of ∞-categories. In particular the inclusion of open sets $U_0 \to U_1 \to U_2$ in $\mathsf{Opens}(W)$ is now went to a *homotopy* commutative diagram of restrictions

$$\begin{array}{ccc} \mathcal{F}(U_2) & \longrightarrow & \mathcal{F}(U_1) \\ & \searrow & \downarrow \\ & & \mathcal{F}(U_0). \end{array}$$

We say that $\mathcal{F}$ is a sheaf if the cosimplicial diagram defined by an open cover $\{U_i\} \to U$ has limit equal to $\mathcal{F}(U)$. Namely,

$$\mathcal{F}(U) \simeq \lim(\oplus_i \mathcal{F}(U_i) \Longleftarrow \oplus_{i,j} \mathcal{F}(U_i \cap U_j) \Lleftarrow \oplus_{i,j,k} \mathcal{F}(U_i \cap U_j \cap U_k) \ldots)$$

where the limit is taken in the sense of ∞-categories. Note that 'cosimplicial object' in the limit is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$, so is in particular the diagram is not a strictly commutative one, but is a homotopy coherent one.

4. Adjunctions

Adjunctions make the world go round. Don't get me started. Once you know a pair of functors form an adjunction you can take an enormous amount of shortcuts in proving various identities. Most of us probably know what an adjunction is, but let me review it real quick before generalizing it.

4.1. **Adjunctions for usual categories.** First, recall that if $\mathcal{D}$ is a usual category (i.e., enriched in sets), there's a canonical functor

$$\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Sets}$$

given by sending $D_0, D_1 \mapsto \mathcal{D}(D_0, D_1)$, the hom-set. Now, if you have a functor $F : \mathcal{C} \to \mathcal{D}$, we get a functor of the form

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Sets}$$

given by $(C, D) \mapsto \mathcal{D}(F(C), D)$. Likewise, a functor $G : \mathcal{D} \to \mathcal{C}$ produces another functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ to Sets by taking $\mathcal{C}(C, G(D))$. We say F, G are *adjoints*, and that F is a *left adjoint*, G a *right adjoint*, if the two functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Sets}$ defined by F and G are naturally isomorphic. As a consequence of this definition, we have natural isomorphisms

$$\mathcal{D}(F(C), D) \cong \mathcal{C}(C, G(D))$$

for any pair of objects $C \in \mathrm{Ob}\, \mathcal{C}, D \in \mathrm{Ob}\, \mathcal{D}$. Here are some examples:

(1) Hom-Tensor for vector spaces, where $F = - \otimes B$ and $G = \mathsf{Vect}(B, -)$.

$$\mathsf{Vect}(A \otimes B, C) \cong \mathsf{Vect}(A, \mathsf{Vect}(B, C)).$$

(2) Free and forget, where $F = \text{Free}(-)$ and G is the forgetful functor.

$$\mathsf{Alg}(\text{Free}(W), V) \cong \mathsf{Vect}(W, V).$$

(3) Pushing and pulling sheaves. Let $f : D \to C$ be a continuous map of topological spaces. Then

$$\mathsf{Shv}_D(f^*\mathcal{F}, \mathcal{G}) \cong \mathsf{Shv}_C(\mathcal{F}, f_*\mathcal{G})$$

for sheaves $\mathcal{G} \in \mathsf{Shv}_X$ and $\mathcal{F} \in \mathsf{Shv}_Y$.

4.2. **Adjunctions for ∞-categories.** The definition in the ∞-setting is almost identical:

Definition 4.1 (Baby version)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors between ∞-categories. We say F is *left adjoint to* G, and that G is *right adjoint to* F if the functors

$$\mathcal{D}(F-, -) : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \mathsf{Spaces}, \qquad \mathcal{C}(-, G-) : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \mathsf{Spaces}$$

are equivalent.

I say this is the baby version because it's the formulation most familiar to a classical category theorist. But the thing I'm sweeping under the carpet is how to rigorously define a functor

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathsf{Spaces}$$

of ∞-categories. You probably believe this is do-able, and it goes the way you'd expect, but I obviously don't want to prove it here. So let me just list some examples, in parallel to the examples above.

(1) Hom-Tensor for chain complexes, where $F = - \otimes_k B$ and $G = \mathsf{Chain}_k(B, -)$.

$$\mathsf{Chain}_k(A \otimes_k B, C) \cong \mathsf{Chain}_k(A, \mathsf{Chain}_k(B, C)).$$

Here we've passed from the vector space world to the dg world, of chain complexes over some field k. Note that if take k to be a ring like $\mathbb{Z}$, the above formula will not be correct at a naive level. The ways around this are not purely ∞-categorical; we must take the correct model for the ∞-category of $\mathsf{Chain}_{\mathbb{Z}}$, and we won't get into it here. Alternatively one can take the derived tensor $\otimes^L$, but this also involves some extra machinery of coherence, so I won't get into it here.

(2) Free and forget, where $F = \text{Free}(-)$ and G is the forgetful functor.

$$\mathsf{dgAlg}(\text{Free}(W), V) \cong \mathsf{Chain}_k(W, V).$$

The free functor creates the free dg algebra generated by a chain complex W.

(3) Pushing and pulling sheaves. Let $f : D \to C$ be a continuous map of topological spaces. Then

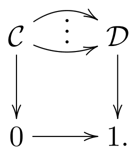
$$\mathsf{Shv}_D(f^*V, W) \cong \mathsf{Shv}_C(V, f_*W)$$

for sheaves $W \in \mathsf{Shv}_D$ and $V \in \mathsf{Shv}_C$. Now by Shv we actually mean sheaves of chain complexes.

4.3. **Optional: The definition in Higher Topos Theory.** If you look, you'll notice that the above definition is not 'verbatim' same as the one in Lurie's Higher Topos Theory [Lurie09]. But I promise it's the same once you pass through some grunt-work. This section gives a very brief outline as to why.

Definition 4.2 (Definition of adjunction in Lurie [Lurie09])**.** Let $p : \mathcal{M} \to \Delta^1$ be a map of ∞-categories such that $p^{-1}(0) =: \mathcal{C}$ and $p^{-1}(1) =: \mathcal{D}$ are both ∞-categories. We say that $\mathcal{M}$ defines *an adjunction between* $\mathcal{C}$ *and* $\mathcal{D}$ if p is both a Cartesian and a coCartesian fibration.

Wow, what does that even mean, right? The following is a rough drawing of the map $\mathcal{M} \to \Delta^1$:



Note there aren't any morphisms starting at $\mathcal{D}$ and ending in $\mathcal{C}$, as there's no morphism $1 \to 0$ in Δ^1.

Exercise 4.3. *Show that two sub-∞-categories $\mathcal{M}_0, \mathcal{M}_1 \subset \mathcal{M}$ define a functor of ∞-categories*

$$\mathcal{M}_0^{\text{op}} \times \mathcal{M}_1 \to \mathsf{Kan}$$

given on objects by setting

$$(M_0, M_1) \mapsto \mathcal{M}(M_0, M_1).$$

As we discussed earlier, this morphism space $\mathcal{M}(C, D)$ isn't well-defined, but fix a model (say $\mathcal{M}^R$), or be content that it's well-defined up to homotopy equivalence.[4]

All the remains is to explain the words *Cartesian* and *co-Cartesian.* There is definitely no time to explain these, but this just means that the edge $(0 \to 1) \in \Delta^1[1]$ defines a functor $G : \mathcal{D} \to \mathcal{C}$, and a functor $F : \mathcal{C} \to \mathcal{D}$, such that

$$\mathcal{C}(C, G(D)) \simeq \mathcal{M}(C, D) \simeq \mathcal{D}(F(D), D).$$

You might think this definition is kind of strange, and that it's sweeping all the difficulties under the notions of (co)Cartesian fibrations, and you're right.

5. Barr-Beck-Lurie

A good reference for this is [G]. Any adjoint pair $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ defines a monad $A = G \circ F : \mathcal{C} \to \mathcal{C}$. The classical Barr-Beck theorem tells us that if G satisfies two conditions, then in fact $\mathcal{D}$ is equivalent to the category of modules in over A. Dually, if F satisfies two conditions, then $\mathcal{C}$ is equivalent to the category of comodules over the comonad $A^\vee = F \circ G$.

Proposition 5.1. *If $G : \mathcal{D} \to \mathcal{C}$ is conservative and preserves geometric realizations, then $\mathcal{D} \simeq A\mathsf{Mod}$. If $F : \mathcal{C} \to \mathcal{D}$ is conservative and preserves totalizations, then $\mathcal{C} \simeq A^\vee\mathsf{Comod}$.*

[4]We haven't talked about opposite ∞-categories. Well, given any simplicial set $\mathcal{C}$, one can define a simplicial set $\mathcal{C}^{\text{op}}$ by setting its degeneracy and face maps to be $d_i^{\mathcal{C}^{\text{op}}} = d_{n-i}^{\mathcal{C}}, s_i^{\mathcal{C}^{\text{op}}} = s_{n-i}^{\mathcal{C}}$.

Recall that the A-module structure is given by

$$A(G(D)) = (G \circ F) \circ G(D) \simeq G(F \circ G(D)) \to GD$$

where the last map is the counit of the adjunction. There are technicalities involved in defining the correct notion of a monad in the ∞-categorical setting, so let me get away with saying that unit maps, counit maps, and all the usual structures of adjunctions hold in an ∞ sense. For details you can see [LuDAG2]. As in the classical category theory setting, *conservative* means $G(f)$ is an equivalence in $\mathcal{C}$ if and only if f is an equivalence in $\mathcal{D}$. To preserve geometric realization means that for any functor $K : \Delta^{op} \to \mathcal{D}$, we have the canonical equivalence $\operatorname{colim}(G \circ K) \simeq G(\operatorname{colim} K)$. Likewise, preserving totalizations means $\lim(F \circ K) \simeq F(\lim K)$.

Example 5.2 (G-modules are comodules over functions on G)**.** Let G be some group scheme. There's the usual projection map of stacks

$$\pi : \operatorname{Spec} k = EG \to BG.$$

This gives an adjunction

$$\pi^* : QC(BG) \rightleftarrows QC(\operatorname{Spec} k) : \pi_*.$$

Here $QC(BG)$ is just the category of complexes of modules over k with a G action.

Exercise 5.3 ([BZ-N])**.** *Show there is an equivalence*

$$QC(BG) \simeq \mathcal{O}(G) - \mathsf{Comod}.$$

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