Categorical valuations for polytopes and matroids

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Part I:

- Introduce the notion of valuations for polytopes or matroids
- Highlight four examples: the trivial invariant (polytopes), the Poincaré polynomial (matroids), the Chow and augmented Chow polynomials (matroids).
- State a theorem and a conjecture about these invariants.

Part II:

- Introduce the notion of **categorical** valuations for polytopes or matroids
- Highlight four examples: the trivial invariant (polytopes), the Orlik-Solomon algebra (matroids), the Chow and augmented Chow rings (matroids).
- State two conjectures about these categorical invariants.

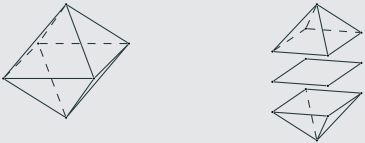
Let $\mathbb{V}$ be a real vector space. A **polytope** in $\mathbb{V}$ is the convex hull of a finite, nonempty set of points in $\mathbb{V}$.

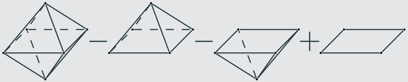
$\mathrm{Pol}(\mathbb{V}) :=$ free abelian group with basis given by polytopes in $\mathbb{V}$.

Let $\mathrm{I}(\mathbb{V}) \subset \mathrm{Pol}(\mathbb{V})$ be the kernel of the homomorphism

$$\mathrm{Pol}(\mathbb{V}) \to \text{Functions on } \mathbb{V}$$
$$P \mapsto 1_P.$$

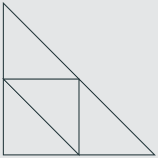
Example





$\in \mathrm{I}(\mathbb{V})$

Example



big triangle

$-$ 4 small trangles

$+$ 3 internal edges

$\in \mathsf{I}(\mathbb{V})$

Let's generalize this way of finding elements of $\mathsf{I}(\mathbb{V})$.

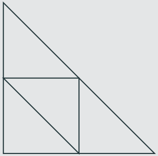
A **decomposition** of a polytope P is a collection $\mathcal{Q}$ of polytopes such that:

- If $Q \in \mathcal{Q}$, then every nonempty face of Q is in $\mathcal{Q}$.
- If $Q, Q' \in \mathcal{Q}$, then $Q \cap Q'$ is a (possibly empty) face of both Q and Q'.
- We have $P = \bigcup_{Q \in \mathcal{Q}} Q$.

Elements of $\mathcal{Q}$ are called **faces** of the decomposition. A face $Q \in \mathcal{Q}$ is **internal** if Q is not contained in the boundary of P.

Let $\mathcal{Q}_k$ denote the set of internal faces of dimension k.

Example



$$\mathcal{Q}_2 = \{\text{small triangles}\}$$

$$\mathcal{Q}_1 = \{\text{internal edges}\}$$

$$\mathcal{Q}_0 = \emptyset$$

Theorem (Volland 1957, Sallee 1968, Groemer 1978, Ardila–Fink–Rincón 2010)

We have

$$P - \sum_k (-1)^{\dim P - k} \sum_{Q \in \mathcal{Q}_k} Q \in \mathsf{I}(\mathbb{V}),$$

and $\mathsf{I}(\mathbb{V})$ *is spanned by elements of this form.*

Let A be an abelian group. A homomorphism $\varphi : \mathrm{Pol}(\mathbb{V}) \to A$ is determined by specifying $\varphi(P) \in A$ for every polytope P.

Such a homomorphism is a **valuation** if $\mathrm{I}(\mathbb{V})$ is contained in the kernel. That means

$$\varphi(P) = \sum_k (-1)^{\dim P - k} \sum_{Q \in \mathcal{Q}_k} \varphi(Q),$$

for any decomposition $\mathcal{Q}$ of a polytope P.

Example

The homomorphism $\tau : \mathrm{Pol}(\mathbb{V}) \to \mathbb{Z}$ with $\varphi(P) = 1$ for every polytope P is a valuation.

This is not hard, but also not completely obvious! We'll see one proof later.

Let's consider a particular class of polytopes.

Let E be a finite set. A **matroid** on E is a nonempty collection of finite subsets of E, called **bases** with the following property:

If B and B' are bases and $e \in B$, then there exists $e' \in B'$ such that $B \setminus \{e\} \cup \{e'\}$ is a basis.

We write $\operatorname{rk} M := |B|$ for any B.

Example

Suppose that $\{v_e \mid e \in E\}$ is a collection of vectors in a complex vector space V, spanning all of V.

The collection of subsets $B \subset E$ such that $\{v_e \mid e \in B\}$ is a basis for V is a matroid on E, of rank equal to $\dim V$.

Such a matroid is called **realizable** over $\mathbb{C}$.

Each matroid determines a polytope: Let $\mathbb{R}^E = \mathbb{R}\{x_e \mid e \in E\}$. For any $S \subset E$, let

$$x_S = \sum_{e \in S} x_e \in \mathbb{R}^E.$$

Let

$$P(M) := \mathrm{Conv}\{x_B \mid B \text{ a basis}\} \subset \mathbb{R}^E.$$

We define $\mathrm{Mat}(E)$ to be the subgroup of $\mathrm{Pol}(\mathbb{R}^E)$ generated by matroid polytopes, and

$$\mathrm{I}(E) := \mathrm{Mat}(E) \cap \mathrm{I}(\mathbb{R}^E).$$

Theorem (Derksen–Fink 2010)

The subgroup $\mathrm{I}(E) \subset \mathrm{Mat}(E)$ *is generated by elements associated with decompositions of matroid polytopes into matroid polytopes.*

A homomorphism $\varphi : \mathrm{Mat}(E) \to A$ is a **valuation** if $\mathrm{I}(E)$ is contained in the kernel.

Let's discuss some examples with $A = \mathbb{Z}[t]$.

Let $\mathrm{OS}(M)$ denote the **Orlik–Solomon algebra** of M, which is can be defined by explicit generators and relations.

Example

Suppose that M is the matroid associated with a collection of vectors $\{v_e \mid e \in E\}$ that span a complex vector space $\mathbb{C}$. Let

$$H_e = v_e^{\perp} \subset V^* \quad \text{and} \quad U = V^* \setminus \bigcup_{e \in E} H_e.$$

Then $\mathrm{OS}(M) \cong H^*(U; \mathbb{Q})$.

We define the **Poincaré polynomial** $\pi_M(t) = \sum_{i=0}^{\operatorname{rk} M} t^i \dim \mathrm{OS}^i(M)$.

This is closely related to the **characteristic polynomial**

$$\chi_M(t) = (-t)^{\operatorname{rk} M} \pi_M(-t^{-1}).$$

For motivation, here is a well-known theorem about $\pi_M(t)$.

Theorem (Adiprasito–Huh–Katz 2017)

The coefficients of $\pi_M(t)$ form a log concave sequence. That is, if $\pi_M(t) = \sum a_i t^i$, then $a_i^2 \geq a_{i-1}a_{i+1}$ for all i.

Theorem (Speyer 2008)

The Poincaré polynomial is valuative. That is, the homomorphism $\mathrm{Mat}(E) \to \mathbb{Z}[t]$ taking M to $\pi_M(t)$ is a valuation.

Note #1: I'm writing M rather than $P(M)$ to denote a basis element of $\mathrm{Mat}(E)$.

Note #2: Speyer actually proved that the Tutte polynomial is valuative, and $\pi_M(t)$ is an evaluation of the Tutte polynomial.

Let $\underline{\mathrm{CH}}(M)$ denote the **Chow ring** of M, also defined by explicit generators and relations.

Example

When M is realizable over $\mathbb{C}$ and U is the corresponding hyperplane complement, $\underline{\mathrm{CH}}(M)$ is the cohomology ring of the de Concini–Procesi **wonderful compactification** of $U/\mathbb{C}^\times$.

We define the **Chow polynomial**

$$\underline{H}_M(t) = \sum_{i=0}^{\operatorname{rk} M - 1} t^i \dim \underline{\mathrm{CH}}^i(t).$$

There is also a closely related **augmented Chow ring** $\mathrm{CH}(M)$ and **augmented Chow polynomial**

$$H_M(t) = \sum_{i=0}^{\operatorname{rk} M} t^i \dim \mathrm{CH}^i(t).$$

Here's a fun conjecture about these two polynomials.

Conjecture (Ferroni–Schröter, Stevens, Ferroni–Matherne–Stevens–Vecchi)

The polynomials $\underline{H}_M(t)$ and $H_M(t)$ have strictly interlacing real roots. That is, we have

$$H_M(t) = \prod (t - a_i) \quad \text{and} \quad \underline{H}_M(t) = \prod (t - b_i)$$

$$\text{with} \;\; a_1 < b_1 < a_2 < b_2 < \cdots < a_{\mathrm{rk}\, M-1} < b_{\mathrm{rk}\, M-1} < a_{\mathrm{rk}\, M}.$$

Theorem (Ferroni–Schröter, Ferroni–Matherne–Stevens–Vecchi)

The polynomials $\underline{H}_M(t)$ and $H_M(t)$ are both valuative.

This may or may not be helpful for proving the conjecture, but it was helpful for generating and testing it!

Let's spend some time outlining a proof that the Chow polynomial is valuative. We'll begin with a theorem in the world of polytopes.

Let $\mathbb{V}$ be a real vector space and $\psi : \mathbb{V} \to \mathbb{R}$ a linear functional.

For any polytope $P \subset \mathbb{V}$, let P_ψ be the face of P on which ψ is maximized.

Consider the homomorphism

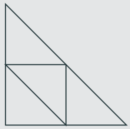
$$\begin{aligned} \delta_\psi : \mathrm{Pol}(\mathbb{V}) &\to \mathrm{Pol}(\mathbb{V}) \\ P &\mapsto P_\psi. \end{aligned}$$

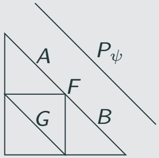
Theorem (McMullen 2009)

The homomorphism δ_ψ takes $\mathrm{I}(\mathbb{V})$ to $\mathrm{I}(\mathbb{V})$.

Example

Consider the linear functional $\psi : \mathbb{R}^2 \to \mathbb{R}$ with $\psi(a, b) = a + b$.





$$P - 4 \text{ small trangles}$$

$$+ 3 \text{ internal edges} \in \mathsf{I}(\mathbb{V})$$

Applying δ_ψ, we get

$$P_\psi - A - B - F - G + F + F + G$$

$$= P_\psi - A - B + F.$$

This is the element of $\mathsf{I}(\mathbb{V})$ corresponding to the induced decomposition of P_ψ with internal faces A, B, and F.

Now let's specialize to the matroidal setting.

Suppose that $E = E_1 \sqcup E_2$.

Given matroids M_1 and M_2 on E_1 and E_2, we can build a matroid $M_1 \sqcup M_2$ on E, whose bases are unions of bases for M_1 and M_2. We have

$$P(M_1 \sqcup M_2) = P(M_1) \times P(M_2).$$

Example

If M_1 is given by $\{v_e \mid e \in E_1\}$ in the vector space V_1

and M_2 is given by $\{v_e \mid e \in E_2\}$ in the vector space V_2,

then $M_1 \sqcup M_2$ is given by

$$\{(v_e, 0) \mid e \in E_1\} \sqcup \{(0, v_e) \mid e \in E_2\}$$

in the vector space $V_1 \oplus V_2$.

Let $\mathrm{Mat}(E_1, E_2) \subset \mathrm{Mat}(E)$ be the subgroup spanned by matroids of this form.

We have

$$\mathrm{Mat}(E_1, E_2) \cong \mathrm{Mat}(E_1) \otimes \mathrm{Mat}(E_2),$$

and

$$\mathrm{I}(E) \cap \mathrm{Mat}(E_1, E_2) \cong \mathrm{I}(E_1) \otimes \mathrm{Mat}(E_2) + \mathrm{Mat}(E_1) \otimes \mathrm{I}(E_2).$$

That is, all decompositions of matroids of this form come from decompositions of the two components.

Conversely, given a matroid M on E, we can construct a new matroid M_1 on E_1 by deleting E_2, and a new matroid M_2 on E_2 by contracting E_1.

Example

If M is given by a collection of vectors $\{v_e \mid e \in E\}$ in a vector space V, let

$$V_1 = \mathrm{Span}\{v_e \mid e \in E_1\} \quad \text{and} \quad V_2 = V/V_1.$$

Then M_1 is given by $\{v_e \mid e \in E_1\}$ in the vector space V_1, and M_2 is given by $\{[v_e] \mid e \in E_2\}$ in the vector space V_2.

A basis for $M_1 \sqcup M_2$ is a basis B for M with the property that $B \cap E_1$ is a basis for M_1, or equivalently $B \cap E_2$ is a basis for M_2. In particular, we have $P(M_1 \sqcup M_2) \subset P(M)$.

Define $\psi : \mathbb{R}^E \to \mathbb{R}$ by the formula

$$\psi\left(\sum_{e \in E} a_e x_e\right) = \sum_{e \in E_1} a_e.$$

Lemma

We have $P(M)_\psi = P(M_1 \sqcup M_2)$.

So we can think of δ_ψ as a homomorphism from $\mathrm{Mat}(E)$ to $\mathrm{Mat}(E_1) \otimes \mathrm{Mat}(E_2)$ taking M to $M_1 \otimes M_2$.

Putting everything together, we obtain the following construction.

Let

$$\varphi_1 : \mathrm{Mat}(E_1) \to \mathbb{Z}[t] \quad \text{and} \quad \varphi_2 : \mathrm{Mat}(E_2) \to \mathbb{Z}[t]$$

be valuations. Define a homomorphism

$$\varphi_1 * \varphi_2 : \mathrm{Mat}(E) \to \mathbb{Z}[t]$$

as follows:

$$\mathrm{Mat}(E) \xrightarrow{\delta_\psi} \mathrm{Mat}(E_1) \otimes \mathrm{Mat}(E_2) \xrightarrow{\varphi_1 \otimes \varphi_2} \mathbb{Z}[t] \otimes \mathbb{Z}[t] \xrightarrow{\mathrm{mult}} \mathbb{Z}[t].$$

More concretely,

$$(\varphi_1 * \varphi_2)(M) := \varphi_1(M_1) \cdot \varphi_2(M_2).$$

Theorem (Ardila–Sanchez 2022)

The homomorphism $\varphi_1 * \varphi_2$ is a valuation.

$$\mathrm{Mat}(E) \xrightarrow{\delta_\psi} \mathrm{Mat}(E_1) \otimes \mathrm{Mat}(E_2) \xrightarrow{\varphi_1 \otimes \varphi_2} \mathbb{Z}[t] \otimes \mathbb{Z}[t] \xrightarrow{\text{mult}} \mathbb{Z}[t]$$

Theorem (Ardila–Sanchez 2022)

The homomorphism $\varphi_1 * \varphi_2$ is a valuation.

Proof.

By McMullen's Theorem, δ_ψ takes $\mathrm{I}(E)$ to

$$\mathrm{I}(E_1) \otimes \mathrm{Mat}(E_2) + \mathrm{Mat}(E_1) \otimes \mathrm{I}(E_2).$$

Since φ_1 kills $\mathrm{I}(E_1)$ and φ_2 kills $\mathrm{I}(E_2)$, this is killed by $\mathrm{mult} \circ (\varphi_1 \otimes \varphi_2)$. $\square$

Now let's tie this back to the theorem about the Chow polynomial. First a little terminology:

An element $e \in E$ is a **loop** for the matroid M if it is not contained in any basis.

If $E = E_1 \sqcup E_2$, then we say that E_1 is a **flat** for the matroid M if M_2 has no loops.

For $S \subset E$ and $k \in \mathbb{N}$, consider the homomorphism $\varphi_{S,k} : \mathrm{Mat}(S) \to \mathbb{Z}$ given by

$$\varphi_{S,k}(M) = \begin{cases} 1 & \text{if } M \text{ has no loops and } \operatorname{rk} M = k \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to show that $\varphi_{S,k}$ is a valuation.

Suppose that $S_1 \sqcup S_2 \sqcup \cdots \sqcup S_r = E$, and let

$$\varphi_{\underline{S},\underline{k}} := \varphi_{S_1,k_1} * \varphi_{S_2,k_2} * \cdots * \varphi_{S_r,k_r} : \mathrm{Mat}(E) \to \mathbb{Z}.$$

By the theorem of Ardila–Sanchez, $\varphi_{\underline{S},\underline{k}}$ is a valuation.

For any matroid M on E, we have

$$\varphi_{\underline{S},\underline{k}}(M) = \begin{cases} 1 & \text{if, for all } i,\ S_1 \sqcup \cdots \sqcup S_i \text{ is a flat of rank } k_1 + \cdots + k_i \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (Feichtner–Yuzvinsky 2004)

The Chow ring $\underline{\mathrm{CH}}(M)$ has a basis indexed by chains of flats, along with some auxiliary data.

Corollary

The homomorphism taking M to $\underline{H}_M(t)$ is equal to a $\mathbb{Z}[t]$-linear combination of homomorphisms of the form $\varphi_{\underline{S},\underline{k}}$. In particular, it is a valuation.

A similar argument can be made for $H_M(t)$.

Recap:

The polynomials $\pi_M(t)$, $\underline{H}_M(t)$, and $H_M(t)$ are all valuative invariants of matroids. This makes them easy to compute for large classes of matroids, which is great. But the valuativity condition itself is somewhat mysterious, and the proofs are opaque. What is really going on?

Let's go back to trying to understand the homomorphism

$$\tau : \mathrm{Pol}(\mathbb{V}) \to \mathbb{Z}$$

with $\tau(P) = 1$ for all P. Why is this a valuation?

Let $\mathcal{Q}$ be a decomposition of a polytope P. Choose an orientation of every face in $\mathcal{Q}$.

The faces of $\mathcal{Q}$ are the cells in a cell complex with total space P, and we have the corresponding cellular chain complex:

$$0 \to \bigoplus_{\substack{Q \in \mathcal{Q} \\ \dim Q = \dim P}} \mathbb{Q} \to \bigoplus_{\substack{Q \in \mathcal{Q} \\ \dim Q = \dim P - 1}} \mathbb{Q} \to \cdots \to \bigoplus_{\substack{Q \in \mathcal{Q} \\ \dim Q = 1}} \mathbb{Q} \to \bigoplus_{\substack{Q \in \mathcal{Q} \\ \dim Q = 0}} \mathbb{Q} \to 0.$$

If we kill the terms corresponding to boundary faces, we obtain a cellular chain complex for the pair $(P, \partial P)$:

$$0 \to \bigoplus_{Q \in \mathcal{Q}_{\dim P}} \mathbb{Q} \to \bigoplus_{Q \in \mathcal{Q}_{\dim P - 1}} \mathbb{Q} \to \cdots \to \bigoplus_{Q \in \mathcal{Q}_1} \mathbb{Q} \to \bigoplus_{Q \in \mathcal{Q}_0} \mathbb{Q} \to 0.$$

The homology of this complex is equal to the homology of the pair $(P, \partial P)$, which is 1-dimensional and concentrated in top degree.

We can build an exact complex by adding one more copy of $\mathbb{Q}$ in the beginning:

$$0 \to \mathbb{Q} \to \bigoplus_{Q \in \mathcal{Q}_{\dim P}} \mathbb{Q} \to \bigoplus_{Q \in \mathcal{Q}_{\dim P - 1}} \mathbb{Q} \to \cdots \to \bigoplus_{Q \in \mathcal{Q}_1} \mathbb{Q} \to \bigoplus_{Q \in \mathcal{Q}_0} \mathbb{Q} \to 0.$$

Now we know that the Euler characteristic is zero, i.e.

$$\begin{aligned} 0 &= 1 - \sum_k (-1)^{\dim P - k} \sum_{Q \in \mathcal{Q}_k} 1 \\ &= \tau(P) - \sum_k (-1)^{\dim P - k} \sum_{Q \in \mathcal{Q}_k} \tau(Q). \end{aligned}$$

This is precisely the statement that τ vanishes on the generator of $\mathsf{I}(\mathbb{V})$ corresponding to $\mathcal{Q}$.

Let $\mathcal{P}(\mathbb{V})$ be the $\mathbb{Q}$-linear additive category whose objects are (formal direct sums of) polytopes in $\mathbb{V}$, and where

$$\mathrm{Hom}(P, P') = \begin{cases} \mathbb{Q} \cdot \iota_{P,P'} & \text{if } P' \subset P \\ 0 & \text{otherwise.} \end{cases}$$

Given a decomposition $\mathcal{Q}$ of P (with orientations), we obtain a complex $C_\bullet(\mathcal{Q})$ in $\mathcal{P}(\mathbb{V})$:

$$0 \to P \to \bigoplus_{Q \in \mathcal{Q}_{\dim P}} Q \to \bigoplus_{Q \in \mathcal{Q}_{\dim P - 1}} Q \to \cdots \to \bigoplus_{Q \in \mathcal{Q}_1} Q \to \bigoplus_{Q \in \mathcal{Q}_0} Q \to 0,$$

with components of maps given by $\pm\iota$ (depending on orientations).

Let $\mathcal{A}$ be an abelian category.

Definition

A functor $\Phi : \mathcal{P}(\mathbb{V}) \to \mathcal{A}$ is a **categorical valuation** if $\Phi(C_\bullet(\mathcal{Q}))$ is exact.

Note: Let A be the Grothendieck group of $\mathcal{A}$. If Φ is a categorical valuation, then the induced homomorphism

$$\mathrm{Pol}(\mathbb{V}) = K(\mathcal{P}(\mathbb{V})) \to K(\mathcal{A}) = A$$

is an ordinary valuation.

Example

Consider the trivial functor $T : \mathcal{P}(\mathbb{V}) \to \mathrm{Vec}_{\mathbb{Q}}$ taking every polytope to $\mathbb{Q}$ and every inclusion to the identity map. This is a categorical valuation, categorifying the valuation τ.

Similarly, we define $\mathcal{M}(E)$ to be the subcategory of $\mathcal{P}(\mathbb{R}^E)$ generated by matroid polytopes, and we say that a functor

$$\Phi : \mathcal{M}(E) \to \mathcal{A}$$

is a categorical valuation if $\Phi(C_\bullet(\mathcal{Q}))$ is exact for any decomposition of a matroid polytope into matroid polytopes.

Example

If $P(M') \subset P(M)$, there is a natural ring homomorphism $\mathrm{OS}(M) \to \mathrm{OS}(M')$, thus OS is a functor from $\mathcal{M}(E)$ to the category of graded vector spaces, categorifying the Poincaré polynomial $\pi_M(t)$.

Let $\mathcal{A}$ be the category of graded vector spaces.

Theorem (E–M–P–V)

- *The functor* $\mathrm{OS} : \mathcal{M}(E) \to \mathcal{A}$ *is a categorical valuation.*
- *There exist categorical valuations*

$$\underline{\mathrm{CH}}, \mathrm{CH} : \mathcal{M}(E) \to \mathcal{A}$$

taking a matroid M *to the Chow ring* $\underline{\mathrm{CH}}(M)$ *and the augmented Chow ring* $\mathrm{CH}(M)$.

The proof of the first statement has a similar flavor to the proof that the trivial functor T is a categorical valuation.

For the second statement, the key step is the categorical analogue of McMullen's theorem.

Let $\mathbb{V}$ be a real vector space and $\psi : \mathbb{V} \to \mathbb{R}$ a linear functional. For any polytope $P \subset \mathbb{V}$, let P_ψ be the face of P on which ψ is maximized.

Consider the functor

$$\Delta_\psi : \mathcal{P}(\mathbb{V}) \to \mathcal{P}(\mathbb{V})$$

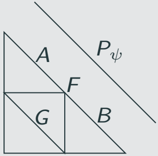
given as follows:

- For any polytope P, $\Delta_\psi(P) = P_\psi$.
- If $P' \subset P$ and $P'_\psi \subset P_\psi$, then $\Delta_\psi(\iota_{P,P'}) = \iota_{P_\psi,P'_\psi}$.
- If $P' \subset P$ and $P'_\psi \not\subset P_\psi$, then $\Delta_\psi(\iota_{P,P'}) = 0$.

Theorem (E–M–P–V)

Suppose that $\mathcal{Q}$ is a decomposition of P, and let $\mathcal{Q}_\psi$ be the induced decomposition of P_ψ. Then $\Delta_\psi(C_\bullet(\mathcal{Q}))$ is homotopy equivalent to a shift of $C_\bullet(\mathcal{Q}_\psi)$.

Example



$C_\bullet(\mathcal{Q})$:

$P \to \bigoplus_{\substack{\text{small}\\\text{triangles}}} \to \bigoplus_{\substack{\text{internal}\\\text{edges}}}$

$\Delta_\psi(C_\bullet(\mathcal{Q}))$:

$$\begin{tikzcd}
 & G \arrow[r] & G \\
 & A \arrow[r] & F \\
P_\psi \arrow[ur] \arrow[r] \arrow[dr] & F \arrow[ur] \arrow[r] & F \\
 & B \arrow[ur] &
\end{tikzcd}$$

$\Delta_\psi(C_\bullet(\mathcal{Q}))$ is homotopy equivalent to this:

$$\begin{tikzcd}
 & A \arrow[dr] & \\
P_\psi \arrow[ur] \arrow[dr] & & F \\
 & B \arrow[ur] &
\end{tikzcd}$$

This is isomorphic to $C_\bullet(\mathcal{Q}_\psi)$ shifted one degree to the left.

What can we do with a categorical valuation that we could not do with an ordinary one?

Some categorical valuations of matroids, including OS, $\underline{\mathrm{CH}}$, and CH, are functorial not only with respect to inclusions of polytopes, but also with respect to permutations of E.

If Γ acts on E by permutations and acts on a decomposition $\mathcal{Q}$ of the polytope $P(M)$ into other matroid polytopes, then the exact sequence

$$\mathrm{OS}(C_\bullet(\mathcal{Q}))$$

gives us a relation involving $\mathrm{OS}(M)$ and $\mathrm{OS}(N)$ for various matroids N with $P(N) \in \mathcal{Q}$, regarded as representations of Γ.

Same for $\underline{\mathrm{CH}}$, CH, and many other categorical valuations.

These are much richer invariants than the polynomials!

Let me conclude by stating two open conjectures about $\mathrm{OS}(M)$, $\underline{\mathrm{CH}}(M)$, and $\mathrm{CH}(M)$, as representations of the group Γ of symmetries of M.

Conjecture (Gedeon–P–Young 2017)

The Orlik–Solomon algebra $\mathrm{OS}(M)$ *is equivariantly log concave. That is, there exists an inclusion*

$$\mathrm{OS}^i(M) \otimes \mathrm{OS}^i(M) \supset \mathrm{OS}^{i-1}(M) \otimes \mathrm{OS}^{i+1}(M)$$

as representations of Γ*.*

This categorifies the theorem of Adiprasito–Huh–Katz.

To state the last conjecture, let me first remind you of the conjecture that I stated earlier.

Conjecture (Ferroni–Schröter, Stevens, Ferroni–Matherne–Stevens–Vecchi)

$\underline{H}_M(t)$ and $H_M(t)$ have strictly interlacing real roots.

Let's reinterpret this conjecture as a positivity statement.

Suppose that $f(t), g(t) \in \mathbb{R}[t]$ with $\deg f(t) = 1 + \deg g(t)$. The **Bézout matrix** $B(f, g)$ is the symmetric matrix whose (i, j) entry is equal to the coefficient of $x^i y^j$ in the polynomial

$$\frac{f(x)g(y) - f(y)g(x)}{x - y}.$$

Theorem (Krein–Naimark 1981)

The polynomials $g(t)$ and $f(t)$ have strictly interlacing real roots if and only if $B(f, g)$ is positive definite.

Given graded representations

$$V = \bigoplus_{i=0}^{d} V^i \quad \text{and} \quad W = \bigoplus_{i=0}^{d-1} W^i$$

of Γ, we can define

$$f_V(t) := \sum_i t^i V^i \quad \text{and} \quad f_W(t) := \sum_i t^i W^i,$$

and define $B(f_V, f_W)$ as above, using tensor product for multiplication. The result will be a symmetric matrix whose entries are virtual representations of Γ.

We say that W **strictly interlaces** V if all of the principal minors of $B(f_V, f_W)$ are nonzero honest (rather than virtual) representations.

Conjecture (Nasr–P)

$\underline{\mathrm{CH}}(M)$ *strictly interlaces* $\mathrm{CH}(M)$.

Example

Suppose that M is the boolean matroid on $\{1,2,3\}$, meaning that $\{1,2,3\}$ is the unique basis. Then S_3 acts on M, and the Bézout matrix $B(f_{\mathrm{CH}(M)}, f_{\underline{\mathrm{CH}}(M)})$ looks like this:

$$\left(\begin{array}{c|c|c} V_{[2,1]} \oplus V_{[3]} & V_{[2,1]}^{\oplus 2} \oplus V_{[3]}^{\oplus 2} & V_{[3]} \\ \hline V_{[2,1]}^{\oplus 2} \oplus V_{[3]}^{\oplus 2} & V_{[1,1,1]}^{\oplus 2} \oplus V_{[2,1]}^{\oplus 7} \oplus V_{[3]}^{\oplus 6} & V_{[2,1]} \oplus V_{[3]}^{\oplus 2} \\ \hline V_{[3]} & V_{[2,1]} \oplus V_{[3]}^{\oplus 2} & V_{[3]} \end{array}\right)$$

One can check that all of the principal minors (in fact, *all* of the minors!) are nonzero honest representations.