On the enumeration of series-parallel matroids

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Abstract. By the work of Ferroni and Larson, Kazhdan–Lusztig polynomials and Z-polynomials of complete graphs have combinatorial interpretations in terms of quasi series-parallel matroids. We provide explicit formulas for the number of series-parallel matroids and the number of simple series-parallel matroids of a given rank and cardinality, extending results of Ferroni–Larson and Gao–Proudfoot–Yang–Zhang.

1 Introduction

Given a graph, a **series extension** is a graph obtained by subdividing an edge, and a **parallel extension** is a graph obtained by adding a new edge parallel to an existing one. A graph is called **series-parallel** if it can be constructed from a 2-cycle by a sequence of series and parallel extensions. By convention, a single edge and a single loop are also considered series-parallel graphs. A matroid associated with a series-parallel graph is called a **series-parallel matroid**. A series-parallel matroid is **simple** if and only if it comes from a graph with no loops or parallel edges.

A (possibly empty) direct sum of series-parallel matroids is called **quasi series-parallel**; this is the same as taking matroids associated with disjoint unions of series-parallel graphs. A quasi series-parallel matroid is simple if and only if each of its components is simple. Quasi series-parallel matroids are characterized by the property of having no minors equal to the uniform matroid of rank 2 on 4 elements or the matroid associated with the complete graph K_4 [FL24, Proposition 2.1]. The **rank** of a quasi series-parallel matroid is equal to the number of vertices minus the number of connected components of the corresponding graph.

Consider the following quantities:

$C_{n,k}$ = the number of series-parallel matroids on $[n]$ of rank k	[OEIS, A140945]
$E_{n,k}$ = the number of simple series-parallel matroids on $[n]$ of rank k	[OEIS, A361355]
$A_{n,k}$ = the number of quasi series-parallel matroids on $[n]$ of rank k	[OEIS, A359985]
$S_{n,k}$ = the number of simple quasi series-parallel matroids on $[n]$ of rank k	[OEIS, A361353]

Remark 1.1. The letter A stands for **All** quasi series-parallel matroids, S stands for **Simple** quasi series-parallel matroids, and C stands for **Connected** quasi series-parallel matroids, which are the same as series-parallel matroids (with the convention that the empty matroid is not connected). The letter E does not stand for anything, but it means simple and connected. In [FL24], the quantity $E_{2k,k+1}$ is denoted E_k.

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Remark 1.2. The original motivation for studying these quantities is that $A_{n,k}$ (respectively $S_{n,k}$) is equal to the coefficient of t^{n-k} in the Z-polynomial (respectively Kazhdan–Lusztig polynomial) of the matroid associated with the complete graph K_{n+1} [FL24, Theorem 1.1]. This is the only known combinatorial description of these coefficients.

Remark 1.3. Note that the number of series-parallel matroids on $[n]$ is not the same as the number of series-parallel graphs with edge set $[n]$, because different graphs can induce the same matroid. For example, there are three different ways (up to isomorphism) to label the edges of the 4-cycle with the labels $\{1, 2, 3, 4\}$, but they all induce the uniform matroid of rank 3.

Consider the following generating functions:

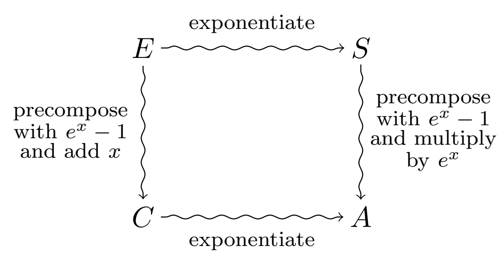
$$E(x,y) := \sum_{n=1}^{\infty}\sum_{k=0}^{n} E_{n,k}\, y^k \frac{x^n}{n!}, \qquad S(x,y) := \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_{n,k}\, y^k \frac{x^n}{n!}$$

$$C(x,y) := \sum_{n=1}^{\infty}\sum_{k=0}^{n} C_{n,k}\, y^k \frac{x^n}{n!}, \qquad A(x,y) := \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{n,k}\, y^k \frac{x^n}{n!}.$$

Note that the two generating functions on the left begin with $n = 1$, while the two on the right begin with $n = 0$; this is because the empty matroid is quasi series-parallel but not series-parallel. The combinatorial relationships between these numbers can be expressed in terms of their generating functions.

Proposition 1.4. *We have the following identities:*

$$\begin{aligned} S(x,y) &= e^{E(x,y)} \\ A(x,y) &= e^{C(x,y)} \\ C(x,y) &= E(e^x - 1, y) + x \\ A(x,y) &= S(e^x - 1, y) \cdot e^x \end{aligned}$$



Proof. A quasi series-parallel matroid on $[n]$ is given by a partition of $[n]$ along with a series-parallel matroid on each part, and it is simple if and only if each component is simple. This fact, combined with [Sta24, Corollary 5.1.6], implies the first two identities. When $n \geq 2$, a series-parallel matroid on $[n]$ is given by a partition of $[n]$ into parallel classes and a simple series-parallel matroid on the set of parallel classes. This observation, combined with [Sta24, Theorem 5.1.4], implies the third identity. (The addition of x comes from the matroid of rank 0 on the set $[1]$, which is series-parallel but not simple.) Finally, a quasi series-parallel matroid on $[n]$ is given by a set of loops, a partition of the nonloops into parallel classes, and a simple series-parallel matroid on the set of parallel classes. This statement implies the fourth identity by [Sta24, Proposition 5.1.1 and Theorem 5.1.4], with the factor of e^x corresponding to the choice of the set of loops. $\square$

We focus here on the numbers $E_{n,k}$, from which all of the others can be computed. We know that we have $E_{n,k} = 0$ when $n \geq 2k > 0$ [FL24, Proposition 2.10]. Theorem 1.5 provides formulas for $E_{2k-1,k}$ [FL24, Corollary 2.12] and $E_{2k-2,k}$ [GPYZ, Corollary 1.6].

Theorem 1.5. [FL24, GPYZ] *We have*

$$\frac{E_{2k-1,k}}{(2k-1)!!} = (2k-1)^{k-3} \qquad \text{and} \qquad \frac{E_{2k-2,k}}{(2k-3)!!} = (2k-1)^{k-2} - (2k-2)^{k-2} + \frac{2}{3}(k-2)(2k-2)^{k-3}.$$

Our goal in this note is to provide a formula for $E_{2k-r,k}$ for arbitrary k and r. Our formula becomes more complicated as r grows. It can be used to recover Theorem 1.5, and we also use it to provide an explicit closed formula for the next case $E_{2k-3,k}$ (Example 1.7).

Consider the **unsigned associated Stirling number of the first kind**

$$\left[\!\!\left[\begin{matrix} n \\ k \end{matrix}\right]\!\!\right] = (n-1)\left[\!\!\left[\begin{matrix} n-2 \\ k-1 \end{matrix}\right]\!\!\right] + (n-1)\left[\!\!\left[\begin{matrix} n-1 \\ k \end{matrix}\right]\!\!\right], \tag{1}$$

which counts the number of derangements of $[n]$ with k cycles [Com74, page 256]. This quantity vanishes when $n < 2k$, and Equation (1) implies the following formulas when n is close to $2k$:

$$\left[\!\!\left[\begin{matrix} 2k \\ k \end{matrix}\right]\!\!\right] = (2k-1)!!, \quad \left[\!\!\left[\begin{matrix} 2k+1 \\ k \end{matrix}\right]\!\!\right] = \frac{2}{3}k\,(2k+1)!!, \quad \text{and} \quad \left[\!\!\left[\begin{matrix} 2k+2 \\ k \end{matrix}\right]\!\!\right] = \frac{1}{9}(4k+5)(k+1)k\,(2k+1)!!.$$

Theorem 1.6. *For all $0 \leq r \leq k$, we have*

$$E_{2k-r,k} = \sum_{p=1}^{r} \left[\!\!\left[\begin{matrix} 2k-p-1 \\ k-p \end{matrix}\right]\!\!\right] \sum_{i=0}^{r-p} \frac{(-1)^{i+p+1}(2k-p-i)^{k-p-1}}{i!(r-p-i)!}.$$

Example 1.7. When $r = 1$ and $r = 2$, Theorem 1.6 reproduces Theorem 1.5. When $r = 3$, Theorem 1.6 tells us that

$$\begin{aligned} \frac{E_{2k-3,k}}{(2k-3)!!} \quad = \quad & \frac{1}{2}(2k-1)^{k-2} - (2k-2)^{k-2} + \frac{1}{2}(2k-3)^{k-2} \\ & + \frac{2}{3}(k-2)\left((2k-3)^{k-3} - (2k-2)^{k-3}\right) \\ & + \frac{1}{9}(4k-7)(k-2)(k-3)(2k-3)^{k-5}. \end{aligned}$$

Remark 1.8. Let M be a simple quasi series-parallel matroid of rank k on the set $[2k-r]$, and let $\{M_i\}$ be its connected components. Then M_i is a simple series-parallel matroid of rank k_i on a set of cardinality $2k_i - r_i$, and we have $\sum_i k_i = k$ and $\sum_i r_i = r$. Thus $S_{2k-r,k}$ may be computed in terms of $E_{2j-s,j}$ for $j \leq k$ and $s \leq r$. The precise formula can be derived from the first equation in Proposition 1.4.

We prove Theorem 1.6 using the generating functions. Ferroni and Larson provide an expression for the generating function $C(x,y)$ in terms of the compositional inverse of the function

$$\frac{1}{y}\log(1+xy)+\log(1+x)-x,$$

where y is regarded as a parameter (Section 4). We explicitly compute the coefficients of this compositional inverse, which gives us a formula for the numbers $C_{n,k}$ (Corollary 4.4). We then combine this with the third identity in Proposition 1.4 to prove Theorem 1.6.

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2 Two Stirling lemmas

We begin with two lemmas about Stirling numbers that we will need later in the paper. Let $\left\{{n \atop k}\right\}$ be the **Stirling number of the second kind**, which counts partitions of $[n]$ into k nonempty parts.

Lemma 2.1. *We have*

$$\sum_{p=0}^{\ell}(-1)^{\ell+p}\binom{m+p}{\ell+p}\left[\!\!\left[{\ell+p \atop p}\right]\!\!\right]=\left\{{m+1 \atop m-\ell+1}\right\}.$$

Proof. Let us denote the left-hand side of the equation by $T_{m,\ell}$. We have

$$\left\{{m+1 \atop m-\ell+1}\right\}-\left\{{m \atop m-\ell}\right\}=(m-\ell+1)\left\{{m \atop m-\ell+1}\right\},$$

and we will show that $T_{m,\ell}$ satisfies the same recursion. Indeed, we have

$$\begin{aligned}
T_{m,\ell} - T_{m-1,\ell} &= \sum_{p=1}^{\ell} (-1)^{p+\ell} \left(\binom{m+p}{\ell+p} - \binom{m-1+p}{\ell+p} \right) \left[\!\left[\begin{matrix} \ell+p \\ p \end{matrix} \right]\!\right] \\
&= \sum_{p=0}^{\ell} (-1)^{p+\ell} \binom{m+p-1}{\ell+p-1} \left[\!\left[\begin{matrix} \ell+p \\ p \end{matrix} \right]\!\right] \\
&= \sum_{p=0}^{\ell} (-1)^{p+\ell} \binom{m+p-1}{\ell+p-1} (\ell+p-1) \left(\left[\!\left[\begin{matrix} \ell+p-2 \\ p-1 \end{matrix} \right]\!\right] + \left[\!\left[\begin{matrix} \ell+p-1 \\ p \end{matrix} \right]\!\right] \right) \\
&= (m-\ell+1) \sum_{p=0}^{\ell} (-1)^{p+\ell} \binom{m+p-1}{\ell+p-2} \left(\left[\!\left[\begin{matrix} \ell+p-2 \\ p-1 \end{matrix} \right]\!\right] + \left[\!\left[\begin{matrix} \ell+p-1 \\ p \end{matrix} \right]\!\right] \right) \\
&= (m-\ell+1) \sum_{q=0}^{\ell-1} (-1)^{q+\ell} \left(\binom{m+q-1}{\ell+q-2} - \binom{m+q}{\ell+q-1} \right) \left[\!\left[\begin{matrix} \ell+q-1 \\ q \end{matrix} \right]\!\right] \\
&= (m-\ell+1) \sum_{q=0}^{\ell-1} (-1)^{q+\ell-1} \binom{m+q-1}{\ell+q-1} \left[\!\left[\begin{matrix} \ell+q-1 \\ q \end{matrix} \right]\!\right] \\
&= (m-\ell+1) T_{m-1,\ell-1}.
\end{aligned}$$

This completes the proof. □

Lemma 2.2. *We have*

$$\left\{ \begin{matrix} n+k \\ m \end{matrix} \right\} = \sum_{j=0}^{k-1} \left\{ \begin{matrix} n+1 \\ m-j \end{matrix} \right\} \sum_{i=0}^{j} \frac{(-1)^i (m-i)^{k-1}}{i!(j-i)!}.$$

Proof. We have

$$\begin{aligned}
m! \left\{ \begin{matrix} n+k \\ m \end{matrix} \right\} &= \big| \{ f : [n+k] \twoheadrightarrow [m] \} \big| \\
&= \sum_{j=1}^{k-1} \binom{m}{j} \big| \{ f : [n+1] \twoheadrightarrow [m-j] \} \big| \cdot \big| \{ f : [k-1] \to [m] \mid [j] \subset \mathrm{im}(f) \} \big| \\
&= \sum_{j=1}^{k-1} \binom{m}{j} (m-j)! \left\{ \begin{matrix} n+1 \\ m-j \end{matrix} \right\} \cdot \big| \{ f : [k-1] \to [m] \mid [j] \subset \mathrm{im}(f) \} \big|,
\end{aligned}$$

and therefore

$$\left\{ \begin{matrix} n+k \\ m \end{matrix} \right\} = \sum_{j=1}^{k-1} \frac{1}{j!} \left\{ \begin{matrix} n+1 \\ m-j \end{matrix} \right\} \cdot \big| \{ f : [k-1] \to [m] \mid [j] \subset \mathrm{im}(f) \} \big|.$$

By the inclusion-exclusion principle,

$$\begin{aligned}
\big|\{f:[k-1]\to[m] \mid [j]\subset \mathrm{im}(f)\}\big| &= \sum_{i=0}^{k-1}(-1)^i\binom{j}{i}\big|\{f:[k-1]\to[m] \mid [i]\not\subset \mathrm{im}(f)\}\big| \\
&= \sum_{i=0}^{k-1}(-1)^i\binom{j}{i}\big|\{f:[k-1]\to[m-i]\}\big| \\
&= \sum_{i=0}^{k-1}(-1)^i\binom{j}{i}(m-i)^{k-1}.
\end{aligned}$$

This completes the proof. ∎

3 Sums of products of reciprocals

Consider the numbers

$$H_{m,k} := \sum_{\substack{j_1+\cdots+j_k=m \\ j_1\geq 1,\ldots,j_k\geq 1}} \frac{1}{(j_1+1)\cdots(j_k+1)}.$$

Lemma 3.1. *We have the recursion*

$$nH_{n-k,k} = kH_{n-k-1,k-1} + (n-1)H_{n-k-1,k}.$$

Proof. We have

$$\begin{aligned}
nH_{n-k,k} &= \frac{n!}{k!}\sum_{\substack{j_1+\cdots+j_k=n-k \\ j_1\geq 1,\ldots,j_k\geq 1}} \frac{1}{(j_1+1)\cdots(j_k+1)} \\
&= \sum_{\substack{j_1+\cdots+j_k=n-k \\ j_1\geq 1,\ldots,j_k\geq 1}} \frac{(j_1+1)+\cdots+(j_k+1)}{(j_1+1)\cdots(j_k+1)}.
\end{aligned}$$

By symmetry, we may replace the numerator in the fraction above by $k(j_k+1)$, and we obtain the equation

$$\begin{aligned}
nH_{n-k,k} &= \sum_{\substack{j_1+\cdots+j_k=n-k \\ j_1\geq 1,\ldots,j_k\geq 1}} \frac{k(j_k+1)}{(j_1+1)\cdots(j_k+1)} \\
&= \sum_{\substack{j_1+\cdots+j_k=n-k \\ j_1\geq 1,\ldots,j_k\geq 1}} \frac{k}{(j_1+1)\cdots(j_{k-1}+1)} \\
&= \sum_{j_k\geq 1}\ \sum_{\substack{j_1+\cdots+j_{k-1}=n-k-j_k \\ j_1\geq 1,\ldots,j_{k-1}\geq 1}} \frac{k}{(j_1+1)\cdots(j_{k-1}+1)}.
\end{aligned}$$

Similarly, we have

$$(n-1)H_{n-k-1,k} = \sum_{j_k\geq 1} \sum_{\substack{j_1+\cdots+j_{k-1}=n-k-j_k-1\\ j_1\geq 1,\ldots,j_{k-1}\geq 1}} \frac{k}{(j_1+1)\cdots(j_{k-1}+1)}.$$

Taking the difference, we find that

$$\begin{aligned} nH_{n-k,k} - (n-1)H_{n-k-1,k} &= \sum_{\substack{j_1+\cdots+j_{k-1}=n-k-1\\ j_1\geq 1,\ldots,j_{k-1}\geq 1}} \frac{k}{(j_1+1)\cdots(j_{k-1}+1)} \\ &= kH_{n-k-1,k}. \end{aligned}$$

This completes the proof. □

Lemma 3.2. *We have*

$$H_{n-k,k} = \frac{k!}{n!}\left[\!\left[\begin{matrix} n \\ k \end{matrix}\right]\!\right].$$

Proof. The recursion in Equation (1) matches the one in Lemma 3.1. □

Lemma 3.3. *We have*

$$\sum_{\substack{j_1+\cdots+j_k=m\\ j_1\geq 1,\ldots,j_k\geq 1}} \prod_{i=1}^{k} \frac{1+y^{j_i}}{j_i+1} = \sum_{\ell=0}^{m} y^\ell \sum_{p=0}^{k} \binom{k}{p} H_{\ell,p}H_{m-\ell,k-p}.$$

Proof. We have

$$\begin{aligned} \sum_{\substack{j_1+\cdots+j_k=m\\ j_1\geq 1,\ldots,j_k\geq 1}} \prod_{i=1}^{k} \frac{1+y^{j_i}}{j_i+1} &= \sum_{\substack{j_1+\cdots+j_k=m\\ j_1\geq 1,\ldots,j_k\geq 1}} \sum_{p=0}^{k} \binom{k}{p} \frac{y^{j_1+\cdots+j_p}}{(j_1+1)\cdots(j_k+1)} \\ &= \sum_{p=0}^{k} \binom{k}{p} \sum_{\ell=p}^{m-k+p} y^\ell \sum_{\substack{j_1+\cdots+j_k=m\\ j_1+\cdots+j_p=\ell\\ j_1\geq 1,\ldots,j_k\geq 1}} \frac{1}{(j_1+1)\cdots(j_k+1)} \\ &= \sum_{\ell=0}^{m} y^\ell \sum_{p=0}^{k} \binom{k}{p} H_{\ell,p}H_{m-\ell,k-p}. \end{aligned}$$

This completes the proof. □

Combining Lemmas 3.2 and 3.3 yields the following corollary, which we will use in Section 4.

Corollary 3.4. *We have*

$$\sum_{\substack{j_1+\cdots+j_k=m\\ j_1\geq 1,\ldots,j_k\geq 1}} \prod_{i=1}^{k} \frac{1+y^{j_i}}{j_i+1} = \frac{k!}{(m-k)!} \sum_{\ell=0}^{m} y^\ell \sum_{p=0}^{k} \binom{m-k}{\ell+p} \left[\!\left[\begin{matrix} \ell+p \\ p \end{matrix}\right]\!\right] \left[\!\left[\begin{matrix} m-\ell+k-p \\ k-p \end{matrix}\right]\!\right].$$

4 Inverting a power series

The **partial Bell polynomials** $B_{n,k}(t_1, \ldots, t_{n-k+1})$ are characterized by the identity

$$\exp\left(y \sum_{j=1}^{\infty} t_j \frac{x^j}{j!}\right) = \sum_{0 \le k \le n} B_{n,k}(t_1, \ldots, t_{n-k+1})\, y^k \frac{x^n}{n!}. \tag{2}$$

The following lemma gives an explicit expression for these polynomials.

Lemma 4.1. *We have*

$$B_{n,k}(t_1, \ldots, t_{n-k+1}) = \frac{n!}{k!} \sum_{\substack{j_1 + \cdots + j_k = n \\ j_1 \ge 1, \ldots, j_k \ge 1}} \frac{t_{j_1}}{j_1!} \cdots \frac{t_{j_k}}{j_k!}.$$

Proof. Equation (2) implies that $B_{n,k}(t_1, \ldots, t_{n-k+1})$ is equal to the coefficient of x^n in the power series

$$\frac{n!}{k!}\left(\sum_{j=1}^{\infty} t_j \frac{x^j}{j!}\right)^k.$$

The lemma follows. □

Suppose that

$$F(x) = \sum_{n=1}^{\infty} F_n \frac{x^n}{n!} \qquad \text{and} \qquad G(x) = \sum_{n=1}^{\infty} G_n \frac{x^n}{n!}$$

are power series with coefficients in some commutative $\mathbb{Q}$-algebra R. Suppose further that $F_1 \neq 0$, and let $\hat{F}_n = \frac{F_{n+1}}{(n+1)F_1}$, so that

$$\hat{F}(x) := \sum_{n=1}^{\infty} \hat{F}_n \frac{x^n}{n!} = \frac{F(x) - F_1 x}{x}.$$

The following result is a corollary of the Lagrange inversion theorem [Cha02, Corollary 11.3].

Theorem 4.2. *We have $G(F(x)) = x$ if and only if $G_1 = F_1^{-1}$ and, for all $n > 1$,*

$$\begin{aligned} G_n &= \frac{1}{F_1^n} \sum_{k=1}^{n-1} n(n+1)\cdots(n+k-1) B_{n-1,k}\left(\hat{F}_1, \ldots, \hat{F}_{n-k}\right) \\ &= \frac{1}{F_1^n} \sum_{k=1}^{n-1} (-1)^k \frac{(n+k-1)!}{k!} \sum_{\substack{j_1 + \cdots + j_k = n-1 \\ j_1 \ge 1, \ldots, j_k \ge 1}} \prod_{i=1}^{k} \frac{\hat{F}_{j_i}}{j_i!}. \end{aligned}$$

We now apply Theorem 4.2 to a particular power series with coefficients in the commutative

$\mathbb{Q}$-algebra $\mathbb{Q}[y]$. Let

$$F(x,y) = \sum_{n=1}^{\infty} F_n(y)\frac{x^n}{n!} := \frac{1}{y}\log(1+xy) + \log(1+x) - x.$$

Explicitly, we have $F_1(y) = 1$ and $F_n(y) = (-1)^{n-1}(n-1)!(1+y^{n-1})$ for all $n > 1$. Let

$$G(x,y) = \sum_{n=1}^{\infty} G_n(y)\frac{x^n}{n!} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} G_{n,k}\, y^k \frac{x^n}{n!}$$

be the unique power series with the property that $G(F(x,y),x) = x$.

Proposition 4.3. *We have*

$$G_{n,\ell} = G_{n,n-\ell-1} = \sum_{j=0}^{\ell}(-1)^{j+\ell}\left[\!\!\left[\begin{matrix} j+\ell \\ j \end{matrix} \right]\!\!\right] \left\{ \begin{matrix} n+j \\ j+\ell+1 \end{matrix} \right\}.$$

Proof. Let

$$\hat{F}_n(y) := \frac{F_{n+1}(y)}{(n+1)F_1(y)} = \frac{(-1)^n n!(1+y^n)}{n+1}.$$

By Theorem 4.2, we have

$$\begin{aligned} G_n(y) &= \sum_{k=1}^{n-1}(-1)^k \frac{(n+k-1)!}{k!} \sum_{\substack{j_1+\cdots+j_k=n-1 \\ j_1\geq 1,\ldots,j_k\geq 1}} \prod_{i=1}^{k} \frac{\hat{F}_{j_i}(y)}{j_i!} \\ &= \sum_{k=1}^{n-1}(-1)^{n+k-1} \frac{(n+k-1)!}{k!} \sum_{\substack{j_1+\cdots+j_k=n-1 \\ j_1\geq 1,\ldots,j_k\geq 1}} \prod_{i=1}^{k} \frac{1+y^{j_i}}{j_i+1}. \end{aligned}$$

Note that this polynomial is clearly palindromic of degree $n-1$, which implies that $G_{n,\ell} = G_{n,n-\ell-1}$. By Corollary 3.4, $G_n(y)$ is equal to

$$\sum_{\ell=0}^{n-1} y^\ell \sum_{k=1}^{n-1}(-1)^{n+k-1} \sum_{p=0}^{\ell} \left[\!\!\left[\begin{matrix} \ell+p \\ p \end{matrix} \right]\!\!\right] \left[\!\!\left[\begin{matrix} n-1-\ell+k-p \\ k-p \end{matrix} \right]\!\!\right] \binom{n+k-1}{\ell+p}.$$

Taking the coefficient of y^ℓ and reindexing with $j = k-p$, we get

$$G_{n,\ell} = \sum_{j=0}^{n-\ell-1} (-1)^{n+j-\ell-1} \left[\!\!\left[\begin{matrix} n-1-\ell+j \\ j \end{matrix} \right]\!\!\right] \sum_{p=0}^{\ell} (-1)^{\ell+p} \left[\!\!\left[\begin{matrix} \ell+p \\ p \end{matrix} \right]\!\!\right] \binom{n-1+j+p}{\ell+p}.$$

Note that the symmetry $G_{n,\ell} = G_{n,n-1-\ell}$ can be seen by exchanging j and p in the summation above. By Lemma 2.1 with $m = n-1+j$, we have

$$G_{n,\ell} = \sum_{j=0}^{n-\ell-1} (-1)^{n+j-\ell-1} \left[\!\left[{n-1-\ell+j \atop j} \right]\!\right] \left\{ {n+j \atop n+j-\ell} \right\}.$$

Replacing ℓ with $n-1-\ell$ allows us to rewrite our expression as

$$G_{n,\ell} = G_{n,n-1-\ell} = \sum_{j=0}^{\ell} (-1)^{j+\ell} \left[\!\left[{j+\ell \atop j} \right]\!\right] \left\{ {n+j \atop j+\ell+1} \right\}.$$

This completes the proof. □

Proposition 4.3, along with a theorem of Ferroni and Larson, provides a formula for $C_{n,\ell}$.

Corollary 4.4. *For all $n \geq 2$, we have*

$$C_{n,\ell} = \sum_{k=0}^{\ell-1} (-1)^{k+\ell-1} \left[\!\left[{k+\ell-1 \atop k} \right]\!\right] \left\{ {n-1+k \atop k+\ell} \right\}.$$

Proof. Using the work of Drake [Dra08, Example 1.5.1], Ferroni and Larson [FL24, Proposition 2.3] show that

$$C(x,y) = (1+y)x + y \int G(x,y)\, dx,$$

where the improper integral is taken to have no constant term. This means that, for all $n \geq 2$, $C_{n,\ell} = G_{n-1,\ell-1}$. The Corollary then follows from Proposition 4.3. □

Remark 4.5. In Proposition 4.3, we gave an algebraic proof of the identity $G_{n,\ell} = G_{n,n-1-\ell}$. We can reinterpret this identity as saying that $C_{n+1,\ell+1} = C_{n+1,n-\ell}$, which follows from the fact that matroid duality is a bijection from the set of series-parallel matroids on $[n+1]$ of rank $\ell+1$ to the set of series-parallel matroids on $[n+1]$ of rank $n-\ell$.

5 Proof of Theorem 1.6

This section is devoted to using Corollary 4.4 to prove Theorem 1.6.

Lemma 5.1. *For all $n \geq 2$, we have*

$$C_{n,\ell} = \sum_{m=\ell}^{n} \left\{ {n \atop m} \right\} E_{m,\ell}.$$

Proof. This can be derived from the third identity in Proposition 1.4, or one can prove it directly using the same combinatorial reasoning employed in the proof of Proposition 1.4. That is, a series-parallel matroid on $[n]$ is given by a partition of $[n]$ into m parallel classes for some m, along with a simple series-parallel matroid on the set of parallel classes. The lemma follows. □

Let

$$\tilde{E}_{n,\ell} := \sum_{p=1}^{2\ell-n} \left[\!\!\left[\begin{matrix} 2\ell-p-1 \\ \ell-p \end{matrix} \right]\!\!\right] \sum_{i=0}^{2\ell-n-p} \frac{(-1)^{i+p+1}(2\ell-p-i)^{\ell-p-1}}{i!(2\ell-n-p-i)!},$$

so that

$$\tilde{E}_{2k-r,k} = \sum_{p=1}^{r} \left[\!\!\left[\begin{matrix} 2k-p-1 \\ k-p \end{matrix} \right]\!\!\right] \sum_{i=0}^{r-p} \frac{(-1)^{i+p+1}(2k-p-i)^{k-p-1}}{i!(r-p-i)!}$$

is the expression appearing on the right-hand side of the equation in the statement of the theorem. We next prove the analogue of Lemma 5.1 for $\tilde{E}$.

Lemma 5.2. *For all $n \geq 2$, we have*

$$C_{n,\ell} = \sum_{m=\ell}^{n} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \tilde{E}_{m,\ell}.$$

Proof. By Corollary 4.4 and using Lemma 2.2, we have

$$C_{n,\ell} = \sum_{k=0}^{\ell-1} (-1)^{k+\ell-1} \left[\!\!\left[\begin{matrix} k+\ell-1 \\ k \end{matrix} \right]\!\!\right] \sum_{j=0}^{k-1} \left\{ \begin{matrix} n \\ k+\ell-j \end{matrix} \right\} \sum_{i=0}^{j} \frac{(-1)^i (k+\ell-i)^{k-1}}{i!(j-i)!},$$

Setting $m = k+\ell-j$, we get

$$C_{n,\ell} = \sum_{m=1}^{n} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \sum_{k=0}^{\ell-1} (-1)^{k+\ell-1} \left[\!\!\left[\begin{matrix} k+\ell-1 \\ k \end{matrix} \right]\!\!\right] \sum_{i=0}^{k+\ell-m} (-1)^i \frac{(k+\ell-i)^{k-1}}{i!(k+\ell-m-i)!},$$

thus it will suffice to show that

$$\sum_{k=0}^{\ell-1} (-1)^{k+\ell-1} \left[\!\!\left[\begin{matrix} k+\ell-1 \\ k \end{matrix} \right]\!\!\right] \sum_{i=0}^{k+\ell-m} \frac{(-1)^i (k+\ell-i)^{k-1}}{i!(k+\ell-m-i)!}$$

is equal to

$$\sum_{p=1}^{2\ell-m} \left[\!\!\left[\begin{matrix} 2\ell-p-1 \\ \ell-p \end{matrix} \right]\!\!\right] \sum_{i=0}^{2\ell-m-p} \frac{(-1)^{i+p+1}(2\ell-p-i)^{\ell-p-1}}{i!(2\ell-m-p-i)!}.$$

This is readily seen by setting $k = \ell - p$. □

Proof of Theorem 1.6. We need to prove that $E_{n,\ell} = \tilde{E}_{n,\ell}$ for all $n \geq \ell \geq 1$. We fix $\ell \geq 1$ and proceed by induction on n. If $n = \ell = 1$, we can verify the equality directly. Otherwise we have $n \geq 2$, so Equation (5.1) and Lemma 5.2 tell us that

$$\left\{ \begin{matrix} n \\ \ell \end{matrix} \right\} E_{\ell,\ell} + \left\{ \begin{matrix} n \\ \ell+1 \end{matrix} \right\} E_{\ell+1,\ell} + \cdots + \left\{ \begin{matrix} n \\ n \end{matrix} \right\} E_{n,\ell} = C_{n,\ell} = \left\{ \begin{matrix} n \\ \ell \end{matrix} \right\} \tilde{E}_{\ell,\ell} + \left\{ \begin{matrix} n \\ \ell+1 \end{matrix} \right\} \tilde{E}_{\ell+1,\ell} + \cdots + \left\{ \begin{matrix} n \\ n \end{matrix} \right\} \tilde{E}_{n,\ell}.$$

By our inductive hypothesis, we can conclude that $E_{n,\ell} = \tilde{E}_{n,\ell}$. □

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