Maxwell's equations as mechanical law

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Abstract. Electrodynamics and special relativity are so deeply interconnected that it is difficult decide which of them is the logical consequence of the other, without running the risk of circular reasoning. Following the historical development by introducing electromagnetism phenomenologically first, this issue is not resolved but at best minimized, because even the Lorentz force law itself is fundamentally relativistic. Here we propose a derivation of Maxwell's equations at a beginning graduate level with the goal of circumventing this difficulty. Based on prior work that establishes special relativity independently of any electrodynamic postulates, it is indeed possible to derive Maxwell's equations from relativity by adding suitable assumptions. The contribution of the present work is to choose these assumptions in such a way that they can be regarded as extensions of the same postulates on which special-relativistic point-particle dynamics itself is based in the first place. The biggest leap is to extend the law of energy- and momentum conservation to the electromagnetic field itself. This hinges on the energy-momentum tensor, and the fact that the latter must account not only for the Lorentz force on a charge, but also for the forces exerted by that same charge on its environment.

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1. Introduction

Special relativity is traditionally introduced by relying on various electrodynamic facts leading to a universal speed of light [1], but it can also be presented without invoking any electromagnetic postulates[2]: The existence of a finite limiting speed c, applicable to all objects, is then considered a mechanical axiom without making any references to light propagation. The Lorentz transformation arises unambiguously from this sole modification to Newtonian kinematics.

With this view of special relativity as a part of mechanics[3], it is a valid question how electrodynamics can be deduced from it. The present article proposes an answer to this question that invokes only concepts that students at the advanced undergraduate or beginning graduate level can be expected to master.

1.1. Related ideas for how to derive Maxwell's equations

Maxwell himself sought a (classical) mechanical basis for his field theory [4], but a path to electrodynamics via special relativity is not possible without adding at least some postulates. One might look for these in general relativity, where close parallels to electromagnetism can be uncovered[5, 6]; but our concern is only with flat spacetime where Einstein's field equations can play no role in the derivations. In fact, an important message of the present work is that a deductive path to Maxwell's equations starting from special relativity requires only very modest technical preparation but offers unique physical insight. To put our choice of approach into context, we begin with a brief review of existing methodologies.

A powerful axiomatic foundation of electrodynamics starts from classical field theory by postulating an action and a stationarity principle [7, 8, 9, 10, 11]. However, it has been pointed out [12] that the choice of Lagrangian is ultimately only justified *a posteriori* when the equations of motion it generates are experimentally confirmed. Landau and Lifshitz[10] serve as a good example for how one must invoke yet-to-be-discussed experimental results in such treatments.

Perhaps the most glaring problem with *nonrelativistic* presentations of electromagnetism is that the Lorentz-force law

$$\boldsymbol{F} = q\boldsymbol{E} + qv \times \boldsymbol{B} \tag{1}$$

directly contradicts the postulate of Galilean relativity, which posits that velocities v are relative but forces $\boldsymbol{F}$ are not. It is well-known that (1) can be made consistent only within special relativity[13]. This stumbling block in phenomenological introductions to electricity and magnetism[14] establishes special relativistic dynamics as the antecedent of electrodynamics. Specifically, it has been argued[15, 16] that (1) is an essentially relativistic-mechanical, not electrodynamic, statement if one is looking to write down the simplest possible force law. In the present work, we will adopt this point of view and explore its consequences to deduce Maxwell's equations, in the presence of sources, from relativistic mechanics without the use of Lagrangians or potentials, and without invoking any of the specific electromagnetic phenomena that are associated with the Maxwell equations themselves.

It is also possible to derive Maxwell-like equations from a generalization of (1) by ignoring the inconsistency of Galilean relativity with (1) and using purely nonrelativistic point mechanics: in Ref. [17], this is the starting point in a quantum-mechanical treatment (that can in principle be transcribed to classical mechanics), where nonrelativistic commutation relations for position and (*non*-canonical) momentum operators are assumed. The idea of postulating those commutation relations goes back to unpublished work by Feynman which suffered from the same inconsistency, and was subsequently refined[13] to ensure Lorentz-covariance and derive Maxwell's equations classically, but only in their free-space (*source-free*) form. That author remarks that the meaning of the assumed commutation relations is not clear. An insightful reconstruction of Feynman's idea is found in [18], but a subsequent comment on that work[19] concludes that it does not represent a derivation of Maxwell's equations. The present author's interpretation of this controversy is that a path to Maxwell's equations that begins from ad-hoc assumptions can receive its justification only *a posteriori*. This requires the student to take a leap of faith, creating a dynamic in which the instructor must essentially argue that the ends justify the means.

It may be argued that such a leap of faith is in fact a valid teaching device especially when it establishes paradigms that have wide-ranging applicability in physics. Perhaps the most powerful paradigm of this kind is the idea that *symmetries*[9] govern all fundamental laws. In free space, all the symmetries of the Poincaré group (which includes translations) hold, and Maxwell's source-free equations actually show up by a standard procedure as the equations governing the irreducible representations of that group[20], when considering partner functions corresponding to mass 0 and spin ± 1, i.e., in three-dimensional space. Symmetries are instrumental in deriving the Lorentz transformation in mechanics[2].

However, here we again encounter a pedagogical limitation: Group theory alone cannot tell us *which* of the many irreducible representation of the Poincaré group is realized in nature, and in particular why electromagnetism is the only existing *classical* force between spatially separated objects in free space that is consistent with special relativity – not counting gravity and quantum interactions. Having taught electrodynamics at the advanced undergraduate level for several years, the author finds that in order for the student to gain a foothold in the beautiful but somewhat forbidding edifice comprised of group theory and variational principles, a more constructive, rather than prescriptive, perspective on electrodynamics as a classical field theory was desirable. In particular, this entails a methodology involving fewer leaps of faith and more connections to what students already know from intermediate or advanced classical mechanics.

An intermediate or beginning advanced electrodynamics course should ideally not be taught in isolation from the rest of the curriculum. At this stage in a student's exposure to electromagnetism on the one hand and classical mechanics on the other, it is approximately equally efficient to refresh students' memory of the empirical foundations that give meaning to Maxwell's equations, as it is to review relativistic point-particle mechanics. The latter ties in well with a higher-level mechanics course that traditional curricula schedule around the same time. Approaching the theoretical foundations behind electrodynamics from the mechanics perspective thererefore offers the potential to not only tell the story in a consistent way, but also to leverage synergy with concurrently taught material.

From the technical point of view, perhaps the most challenging concept in the exposition propsed below is the representation of point particles by a delta-function density. There is more synergy to be gained from this challenge, because the Dirac delta function already plays a central role in modern intermediate-level texts not only on electromagnetism[21], but also on quantum physics[22].

Not knowing the equations of motion of the electromagnetic field, we cannot take for granted that electromagnetic interactions conserve energy and momentum locally. The latter will be introduced as a *postulate*. The alternative would be to make assumptions about the nature of the equations of motion at the outset (e.g. that they are first-order in time)[15], so that the conservation laws emerge after the fact. All of these paths are mathematically valid if the outcome is electrodynamics as we know it. However, the differences lie in how to develop the narrative in the most compelling way. Coming from mechanics, conservation of energy and momentum is a compelling principle because we understand its practical utility.

The appeal of the approach presented here is that every step in the deduction can be motivated *a priori*, instead of being justified only by the eventual outcome. This idea also informs a commonly used hybrid approach that follows the spirit of deriving electrodynamics from relativity[23], but begins by empirically familiarizing the reader with Coulomb's or Gauss' Law of electrostatics. This strategy can also help in the presentation of the variational method[7]. A related approach is an inference centered on the continuity equation for the charge and current densities, again assuming electrostatic and magnetostatic laws[24], or even the retarded Green's function[25]. In contrast, we would like to see the phenomena of Coulomb's Law and of wave propagation *derived* from the equations of motion, not the other way around.

An arguably even quicker, heuristic way to arrive at Maxwell's equations has been given in Ref. [26], but it requires the reader to accept a larger set of postulates as physically reasonable. In particular, the fact that electromagnetic waves exist and propagate with speed c is taken for granted. As with the use of Green's functions or potentials, this pre-supposes a specific electromagnetic phenomenon (waves) that we instead wish to arrive at deductively. Similar arguments are employed in Ref. [16] to obtain the free-space version of Maxwell's equations, and then to insert terms representing the currents and charges. The latter reference provides a detailed account of prior related work.

1.2. Overview of the new approach

In order to arrive at Maxwell's equations, one invariably needs to expand the role played by the charge q in (1) to include its additional effect as a *source* of the electromagnetic field. This has been called the extended action-reaction principle[15]: If the "action" on the charge is represented by (1), then the appearance of the same charge as a source of the field is the "reaction." This dual role of charge is already contained in a familiar tool of electrodynamics: the Maxwell stress tensor, which is often used as an alternative to (1) for obtaining electromagnetic forces. The stress tensor is the spatial block of the electromagnetic energy-momentum tensor[10], and in order to calculate forces from it, one must consider the *total field* of all the interacting charges. This total field by definition contains "action" and "reaction" contributions of any given charge.

By relying on the energy-momentum tensor, we are led to introduce the field concept for particles, by introducing densities, even before $\boldsymbol{E}$ and $\boldsymbol{B}$ become relevant. Because densities appear in Maxwell's equations, this step is inevitable. Reference [15] takes it at the end of the argument by defining the four-current density *in terms* of the electromagnetic field, and then postulating that it is suitably related to the quantity q in (1) *such that* this same q acts as a field source. By contrast, our presentation explains this dual role of q by asking what properties are required to make the concepts of energy- and momentum density meaningful for an electromagnetic field. The main property is that they must satisfy a conservation law, and we will find that a necessary and sufficient condition for being able to construct this law is given by Maxwell's equations.

The outline of the paper is as follows: Section 2 introduces the tensor formulation for the field and states the fundamental concepts of particle mechanics in the language of energy-momentum tensors and current densities. It also establishes our notation, which has been chosen so as to present the subsequent calculations without a proliferation of indices. Section 3 deduces the existence and properties of the electromagnetic energy-momentum tensor from our postulate that the electromagnetic field must locally conserve energy and momentum. Building on this construction, Maxwell's equations are obtained in Section 4.

2. Tensor formulation

2.1. Electromagnetic field tensor

In the four-vector form of the Lorentz-force law, the electromagnetic field is represented by an antisymmetric rank-2 tensor F (written double-struck to more clearly distinguish it from the force), constructed from the components of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. We shall call F the (electromagnetic) *field tensor*, although this nomenclature is not uniform across the literature[5].

In all our calculations, c is a parameter, entirely decoupled from electrodynamics, that arises purely as a mechanical invariant in the Lorentz transformation[2]. We will use ***bold face*** *for all three-dimensional vectors* to distinguish them from four-vectors, which will receive an *arrow* instead of bold face.

The Lorentz four-force corresponding to (1) is then writable in terms of the four-velocity $\vec{u}$ as a linear relation[15]

$$\vec{F} = \frac{q}{c} \mathsf{F}\, g\vec{u}, \tag{2}$$

where F is the electromagnetic field tensor which will be discussed in more detail below. In this matrix expression, the metric tensor is denoted by g. In this paper, we employ the signature $(+,+,+,-)$ with the temporal coordinate as the fourth dimension[27]. The metric tensor, is then given in terms of the canonical unit vectors $\vec{e}_\nu$ $(\nu = 1, \ldots 4)$ of $\mathbb{R}^4$ by

$$g = \sum_{i=1}^{3} \vec{e}_i \vec{e}_i^{\,t} - \vec{e}_4 \vec{e}_4^{\,t}. \tag{3}$$

Here $\vec{e}_\nu \vec{e}_\nu^{\,t}$ represents an outer product ($\vec{e}_\nu^{\,t}$ denotes the transpose, a row vector). Einstein summation convention will *not* be used, because all the calculations can be done by relying on the rules of matrix multiplications without any additional formal overhead.

In our derivation, it will be important that F can be expressed in terms of a set of matrices from which all elements of the homogeneous Lorentz group[20] are derived: the generators of rotations,

$$\sigma_i = \sum_{j,k=1}^{3} \varepsilon_{i,j,k} \vec{e}_j \vec{e}_k^{\,t} \quad (i = 1, 2, 3) \tag{4}$$

($\varepsilon_{i,j,k}$ is the rank-3 Lévi-Civita tensor), and the boost generators

$$\omega_i = \vec{e}_4 \vec{e}_i^{\,t} - \vec{e}_i \vec{e}_4^{\,t} \quad (i = 1, 2, 3). \tag{5}$$

To make the connection between these generators and the electromagnetic field tensor F explicit, define a matrix-valued function that takes two vector arguments $\boldsymbol{a} = (a_1, a_2, a_3)^t$ and $\boldsymbol{b} = (b_1, b_2, b_3)^t$, and uses them as coefficients in a linear combination

$$\mathsf{F}(\boldsymbol{a}, \boldsymbol{b}) \equiv \sum_{i=1}^{3} a_i \omega_i + \sum_{i=1}^{3} b_i \sigma_i. \tag{6}$$

This is the most general skew-symmetric 4×4 matrix, depending on the six parameters $\boldsymbol{a}$ and $\boldsymbol{b}$. It is straightforward to verify that the tensor appearing in (2) is in fact $\mathsf{F} = \mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})$.

2.2. Densities and energy-momentum tensor for particles

In point mechanics, particles move along world lines, $\vec{x}(\tau)$, where τ is the proper time. An equivalent way to describe the motion of a particle is to adopt a fluid-dynamic point of view where the observer measures the flux of particles through a given point $\boldsymbol{r}$ as a function of time t. It is possible to do this even for a single particle, by turning the quantities of interest into *densities* using Dirac delta functions, as is discussed, e.g., in Ref. [28]. The generalization to arbitrary charge distributions will follow in subsection 2.3.

By introducing densities, $\vec{x}$ is turned into an independent variable. If $\boldsymbol{r}^{(0)}$ and τ are the spatial and temporal coordinates in the rest frame of a particle, then the mass density $\mu^{(0)}$ in the rest frame is a function only of $\boldsymbol{r}^{(0)}$. This, in turn, is related to the coordinates $\vec{x}$ in the observer's frame, so that $\mu^{(0)} = \mu^{(0)}(\boldsymbol{r}^{(0)}(\vec{x}))$. The mass density in the observer's frame is then

$$\mu(\vec{x}) \equiv \mu(\boldsymbol{r}, t) = \gamma\, \mu^{(0)}(\vec{x}), \tag{7}$$

where the factor, where

$$\gamma \equiv \left(1 - (v/c)^2\right)^{-1} \tag{8}$$

accounts for Lorentz contraction of the volume element. From this, one can also obtain the number density by dividing out the particle's rest mass, m.

As an example, the transformation law (7) for the density can be used to define a *four-force density*

$$\vec{f}(\vec{x}) \equiv \frac{\mu^{(0)}(\vec{x})}{m}\, \vec{F}(\vec{x}). \tag{9}$$

The rest-frame quantity $\mu^{(0)}$ is a Lorentz scalar, so that $\vec{f}$ as defined in (9) is a four-vector. To formulate the entire relativistic equation of motion in terms of densities, one introduces the energy-momentum tensor of the particle[10]. The payoff will be a reformulation of mechanics in which particle world lines no longer appear explicitly.

Considering the four-velocity $\vec{u}(\vec{x})$ as a function of $\vec{x}$, the energy-momentum tensor of a single particle (indicated by a superscript p) is[10, 5]

$$\mathsf{T}^{(\mathrm{p})}(\vec{x}) \equiv \mu^{(0)}(\vec{x})\, \vec{u}(\vec{x})\, \vec{u}^{\,t}(\vec{x}). \tag{10}$$

With this Lorentz tensor field, one can replace the inertia term in the four-vector form of Newton's Second Law to obtain, in terms of the force density (9):

$$\frac{\partial}{\partial \vec{x}} \mathsf{T}^{(\mathrm{p})}(\vec{x}) = \vec{f}^{\,t}(\vec{x}), \tag{11}$$

where we define the row vector of derivatives following the Jacobian notation:

$$\frac{\partial}{\partial \vec{x}} \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial ct} \right).$$

The reason why $\mathsf{T}^{(\mathrm{p})}$ is called the energy-momentum tensor is that it contains both of these quantities[10]: Recall that for a particle of velocity v, the relativistic momentum is $\boldsymbol{p} = m\gamma v$, and $\varepsilon = m\gamma c^2$ is the energy. The temporal diagonal element of $\mathsf{T}^{(\mathrm{p})}$ is the energy density,

$$\mathsf{T}^{(\mathrm{p})}_{44} = \frac{\mu}{m}\varepsilon. \tag{12}$$

Analogously, the spatial components

$$\mathsf{T}^{(\mathrm{p})}_{i4} = \frac{\mu}{m} p_i c \tag{13}$$

are the densities of the relativistic momentum components $i = 1, 2, 3$. In other words, the temporal column (and row, by symmetry) of $\mathsf{T}^{(\mathrm{p})}$ contains the density of the four-momentum $\vec{p}$. The invariant trace yields the density of rest mass via

$$\mathrm{Tr}\big(g \mathsf{T}^{(\mathrm{p})}\big) \;=\; -\, \mu^{(0)} c^2. \tag{14}$$

The real power of this formulation lies in its ability to describe "swarms" of particles[5] as currents, as we will see next.

2.3. Current density in relation to electromagnetic fields

Equation (11) can be rewritten by inserting the Lorentz force in (2):

$$\frac{\partial}{\partial \vec{x}} \mathsf{T}^{(\mathrm{p})} = -\frac{q}{mc} \mu^{(0)} \vec{u}^{\,t} g \mathsf{F}. \tag{15}$$

Now define the *four-current density*,

$$\vec{j} \equiv \frac{q}{m} \mu^{(0)} \vec{u}. \tag{16}$$

Its temporal component is just the charge density, $j_4/c = \gamma \frac{q}{m} \mu^{(0)} \equiv \rho$, where the factor γ from the definition of $\vec{u}$ is incorporated into ρ for the same reason as in (7). With this, one can also write

$$\vec{j} = \begin{pmatrix} \rho v \\ \rho c \end{pmatrix} \equiv \begin{pmatrix} \boldsymbol{j} \\ \rho c \end{pmatrix} \tag{17}$$

where $\boldsymbol{j} \equiv \rho v$ is the relativistic current density in three dimensions. Then (15) becomes

$$\frac{\partial}{\partial \vec{x}} \mathsf{T}^{(\mathrm{p})} = -\frac{1}{c} \vec{j}^{\,t} g \mathsf{F}. \tag{18}$$

It is worth noting that we do not need to invoke the continuity equation for the charge in our derivation, even though it can be obtained from the above definition. Going through this well-known derivation is of value as a teaching device, but we will skip this step here for brevity. That this law holds can alternatively be deduced directly from Maxwell's equations at the end of our calculations.

To generalize beyond a single particle, consider N particles of charge q_n and mass m_n that are interacting among themselves, and with the environment via electromagnetic forces. *Provided that* F represents the *total* electromagnetic field governing the Lorentz force on all N charges, we can then add N copies of (11):

$$\frac{\partial}{\partial \vec{x}} \sum_{n=1}^{N} \mathsf{T}_n^{(\mathrm{p})} = -\frac{1}{c} \sum_{n=1}^{N} \vec{j}_n^{\,t} g \mathsf{F}, \tag{19}$$

where $\vec{j}_n$ is given by (16) for each charge q_n, irrespective of the fact that the rest frame (indicated by superscript 0) may be different for different particles. But F is common to all terms, so the above collapses back into the form (18) if we introduce a total four-vector of charge-current density

$$\vec{j}(\vec{x}) \equiv \sum_{n=1}^{N} \vec{j}_n(\vec{x}), \tag{20}$$

and a total energy-momentum tensor of all particles combined,

$$\mathsf{T}^{(\mathrm{p})}(\vec{x}) \equiv \sum_{n=1}^{N} \mathsf{T}_n^{(\mathrm{p})}(\vec{x}). \tag{21}$$

Equation (20) can be turned into an integral for $N \to \infty$, so it is now permissible to think of $\vec{j}(\vec{x})$ as a well-behaved function, where point charges are "smeared out" into non-singular distributions.

It remains to be explained what it means that F represents the *total field*, as assumed above (19). To this end, first note that the electromagnetic field obeys the superposition principle, as a direct consequence of the superposition principle for forces. This fundamental axiom of mechanics necessarily translates to the field because (1) is a linear relation between force and fields.

Now focus on a particular charge q_1, and assume for the time being only one other charge q_2 in the environment. The force that q_1 feels according to (1) is proportional to q_1, and it must be accompanied by a force on q_2 that also depends on q_1. Otherwise, we would not be able to construct a conservation law in which changes in energy-momentum of particle q_1 are balanced by changes of these quantities in the environment, for all values of q_1. Because the force exerted by q_1 on q_2 is charge-dependent, it must first of all be electromagnetic. Hence it again takes the form of (1) with q replaced by q_2, and with suitable values of the electric and magnetic fields.

Since "action" and "reaction" forces are distinct quantities in the mechanics of point particles, the field caused by the particle q_1 cannot have any effect on the Lorentz force felt by q_1 itself. Therefore, adding the fields that mediate the forces on q_1 and q_2 according to the superposition principle will create a total field which must still produce the same Lorentz force on each of the particles when used in (1). This condition comes from mechanics and is not a separate postulate of electromagnetism.

To write this in more generality for q_1 as one among N particles, we will use the notation $\mathsf{F}(\mathfrak{E}, c\mathfrak{B})$ for the external field that includes the influence of all particles *other than* q_1. On the other hand, $\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})$ will denote the *total* electromagnetic field that consists of the superposition of the external field and the field caused by q_1. Then the fact that the Lorentz force on q_1 is independent of the field contributed by q_1 itself can be stated in terms of the current density four-vector $\vec{j}_1$ corresponding to q_1:

$$\vec{j}_1^{\,t} g \mathsf{F}(\boldsymbol{E}, c\boldsymbol{B}) = \vec{j}_1^{\,t} g \mathsf{F}(\mathfrak{E}, c\mathfrak{B}). \tag{22}$$

Note that this is *not* the same as $\boldsymbol{E}(\vec{x}) \equiv \mathfrak{E}(\vec{x})$ and $\boldsymbol{B}(\vec{x}) \equiv \mathfrak{B}(\vec{x})$, because we are only equating the components along the given $\vec{j}_1$.

Returning to (19), the argument leading to (22) can be repeated for each particle n, with all the other particles acting as part of the environment for particle n. Note that the external field $\mathsf{F}(\mathfrak{E}, c\mathfrak{B})$ for each particle can also include influences other than the ones caused by the particles under consideration in (19). Whatever the external field on particle n happens to be, (22) then allows that field to be replaced by the total field $\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})$ in (19). This is why all terms in (19) contain the same tensor $\mathsf{F} = \mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})$, corresponding to the total field, caused by all the N particles of interest, and the environment they share.

3. Energy and momentum density of the field

In the foregoing sections, we have introduced the electromagnetic field tensor F, see (6), and the energy-momentum tensor of the particles, $\mathsf{T}^{(\mathrm{p})}$, as defined in (11). The next member in this list of four-tensors will be the energy-momentum tensor of the field, $\mathsf{T}^{(\mathrm{e})}$. The construction of this new tensor gives the present section central significance. It contains the fundamental physical assumptions that the formal derivation in section 4 will rely on.

3.1. Energy-momentum tensor of the electromagnetic field

To disentangle special relativity from electrodynamics, a self-contained justification for the relativistic force law can be found by considering *collisions*[2, 27], where objects can be regarded as freely moving outside an interaction region. Thought experiments along these lines can be found, e.g., in Refs. [29, 30]. Conservation laws then serve as the guiding principle from which the definitions of relativistic momentum, energy and force are obtained. On the other hand, we are now interested precisely in the interaction itself, where the equation of motion for the particles is given by (18). In a direct continuation of the strategy just mentioned, we postulate that the electromagnetic field carries energy and momentum because it is the entity that transmits forces between particles.

This can arguably be called a mechanical, not a specifically electromagnetic postulate. The reason is that the same type of force transmitter already exists in classical mechanics: any connection between two objects for which the forces on the objects satisfy Newton's Third Law is characterized by the fact that it carries energy and momentum, but has no trajectory. Examples are ideal springs, or Newtonian gravity. Newton's Third Law requires them to have no mass, and therefore they also lack a center of mass for which a trajectory could be defined. This lack of a well-defined trajectory is paralleled by the electromagnetic field, and thus the idea of allowing the field to carry energy and momentum has a mechanical precedent.

Newton's Third Law does not carry over to special relativity except when world lines cross[27], so our postulate about the electromagnetic field as a carrier of energy and momentum does go beyond particle mechanics. However, energy and momentum deserve their names only if they satisfy a conservation law, even during the interaction. This statement needs to be made explicit in an equation which holds during the interaction phase, and not just for the asymptotic in- and out states.

For a single particle, energy and momentum are contained in its four-momentum $\vec{p}$, whose density directly appears as the temporal column and row of $\mathsf{T}^{(\mathrm{p})}$, cf. (12) and (13). If the particle is force-free, energy-momentum conservation is equivalent to (18) with $\mathsf{F} = 0$. In the presence of an electromagnetic field, a conservation law of the same form can be constructed for arbitrary number of particles if we are able to find a new rank-2 Lorentz tensor $\mathsf{T}^{(\mathrm{e})}$ such that

$$\frac{\partial}{\partial \vec{x}} \left(\mathsf{T}^{(\mathrm{p})} + \mathsf{T}^{(\mathrm{e})} \right) = \vec{0}^{\,t}. \tag{23}$$

Comparison between (23) and (18) yields the condition

$$\frac{\partial}{\partial \vec{x}} \mathsf{T}^{(\mathrm{e})} = \frac{1}{c} \vec{j}^{\,t} g \mathsf{F}. \tag{24}$$

Here, $\mathsf{T}^{(\mathrm{e})}$ will be called the energy-momentum tensor of the electromagnetic field.

The reason why we do not attempt to write a conservation law in terms of the four-vector $\vec{p}$ directly is that the six independent components of F appearing in the Lorentz force on the right side of (24) cannot be accommodated by a four-vector. Equation (23) is then the only remaining starting point for a conservation law in which particles and forces appear on the same footing.

The tensor $\mathsf{T}^{(\mathrm{e})}$ has to allow us to extract the densities of energy and momentum by the same procedures as with $\mathsf{T}^{(\mathrm{p})}$. If this were not the case, it would not be true for the sum $\mathsf{T}^{(\mathrm{p})} + \mathsf{T}^{(\mathrm{e})}$, and (23) would not be an expression of the energy-momentum conservation law. This must hold covariantly, so $\mathsf{T}^{(\mathrm{e})}$ must transform between inertial frames in the same way as $\mathsf{T}^{(\mathrm{p})}$ does: as a proper Lorentz tensor.

In order to get to Maxwell's equations from (24), the task now is to collect all the physically motivated information about $\mathsf{T}^{(\mathrm{e})}$ that allows the left-hand side of the above equation to be evaluated. Then (24) will become a system of differential equations satisfied by $\boldsymbol{E}$ and $\boldsymbol{B}$.

3.2. Constraints on the energy-momentum tensor

It is important to note that nothing in (23) requires the N-particle system to be isolated, because it is a local statement where the distinction between "external" and "internal" only matters for the definition of the particles we count as part of $\mathsf{T}^{(\mathrm{p})}$ and $\vec{j}$. We will now use $\mathsf{F}(\mathfrak{E}, c\mathfrak{B})$ for the electromagnetic field that has its source in the environment, external to the N-particle system represented by $\vec{j}$. The four-current density $\vec{j}$ can be chosen independently of $\mathsf{F}(\mathfrak{E}, c\mathfrak{B})$, since the mechanical initial conditions for the motion of the particles and their environment are independent.

But by varying $\mathsf{F}(\mathfrak{E}, c\mathfrak{B})$, we also vary the *total field* $\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})$. Since this cannot affect $\vec{j}$, it follows that $\vec{j}$ cannot depend on the function values of $\boldsymbol{E}$ and $\boldsymbol{B}$ at $\vec{x}$, either. Note that this says nothing about field derivatives, which may still be present in $\vec{j}$. Comparing powers of the field in (24), the only way to guarantee that $\vec{j}$ is independent of the values of $\boldsymbol{E}$ and $\boldsymbol{B}$ is by requiring the divergence $(\partial/\partial\vec{x})\,\mathsf{T}^{(\mathrm{e})}$ to be a homogeneous *linear* function of the components of $\boldsymbol{E}$, $\boldsymbol{B}$. This should not be confused with the statement that (24) is a system of linear differential equations – it is not, but will give rise to one in section 4.

Because no point $\vec{x}$ in Minkowski spacetime is special, $\mathsf{T}^{(\mathrm{e})}$ can depend on $\vec{x}$ only through $\boldsymbol{E}$, $\boldsymbol{B}$, and has no explicit $\vec{x}$ dependence. For the same reason, the particle tensors $\mathsf{T}_n^{(\mathrm{p})}$ only depend on the four-velocity of the individual particle. One could allow an arbitrary addition $\mathsf{A}(\boldsymbol{E}, c\boldsymbol{B})$ to $\mathsf{T}^{(\mathrm{e})}$ such that $(\partial/\partial\vec{x})\,\mathsf{A} = \vec{0}$ without violating (24), but this is irrelevant. Our goal is only to find the left-hand side of (24), and we therefore need only the properties of $\mathsf{T}^{(\mathrm{e})}$ that affect the divergence. Additional discussion of the ambiguity in defining energy and momentum of the electromagnetic field is given in the Feynman Lectures[31]. In particular, we have the freedom to add a constant tensor to $\mathsf{T}^{(\mathrm{e})}$ such that $\mathsf{T}^{(\mathrm{e})} = \mathsf{0}$ whenever $\boldsymbol{E} = \boldsymbol{B} = \mathbf{0}$.

All of the above properties can be attained if the tensor $\mathsf{T}^{(\mathrm{e})}$ is taken to be a homogeneous *quadratic* function of the field components. Since $\mathsf{T}^{(\mathrm{e})}$ must be a proper tensor, it can depend on the field components only in a combination that preserves the transformation behavior of the proper tensor $\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})$. A proper tensor that is quadratic in the fields is $\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})g\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})$. In addition, one can promote any Lorentz scalar to a proper tensor by multiplying it with the metric tensor g. This applies to the invariant trace $\mathrm{Tr}(g\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})g\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B}))$, and also to the determinant of $\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})$. But only the former is a quadratic function of the fields.

Another tensor that could appear in $\mathsf{T}^{(\mathrm{e})}$ is the *dual field tensor* (Hodge dual relative to g), defined as

$$\mathsf{G}(\boldsymbol{E}, c\boldsymbol{B}) \equiv \mathsf{F}(-c\boldsymbol{B}, \boldsymbol{E}). \tag{25}$$

Here, we formally just replaced $\boldsymbol{E} \to -c\boldsymbol{B}$ and $c\boldsymbol{B} \to \boldsymbol{E}$ in F. If $\boldsymbol{E} \cdot \boldsymbol{B} \neq 0$, then $g\mathsf{F}(-c\boldsymbol{B}, \boldsymbol{E})$ is proportional to the inverse of $\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})$. One finds straightforwardly that

$$\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})\, g\, \mathsf{G}(\boldsymbol{E}, c\boldsymbol{B}) = -\left(c\boldsymbol{E} \cdot \boldsymbol{B}\right)\, g. \tag{26}$$

The reason this tensor may appear in $\mathsf{T}^{(\mathrm{e})}$ is that the quadratic combination $\mathsf{G}g\mathsf{G}$ is also a proper tensor, because it is directly related to the previous two proper tensors, $\mathsf{F}g\mathsf{F}$ and $g\mathrm{Tr}(g\mathsf{F}g\mathsf{F})$:

$$\frac{1}{2}g\mathrm{Tr}(g\mathsf{F}g\mathsf{F}) = -\frac{1}{2}g\mathrm{Tr}(g\mathsf{G}g\mathsf{G}) = \mathsf{F}g\mathsf{F} - \mathsf{G}g\mathsf{G}. \tag{27}$$

In fact, this shows that to build the energy-momentum tensor, the traces on the left-hand side are redundant if we instead use only $\mathsf{F}g\mathsf{F}$ and $\mathsf{G}g\mathsf{G}$. To prove (27), one can verify that the right-hand side of (27) is

$$\mathsf{F}g\mathsf{F} - \mathsf{G}g\mathsf{G} = g\left(E^2 - c^2B^2\right), \tag{28}$$

and that the invariant traces are

$$\mathrm{Tr}(g\mathsf{F}g\mathsf{F}) = -\mathrm{Tr}(g\mathsf{G}g\mathsf{G}) = 2\left(E^2 - c^2B^2\right). \tag{29}$$

Because of (27), the following *ansatz* captures all distinct terms that can contribute to the electromagnetic energy-momentum tensor:

$$\mathsf{T}^{(\mathrm{e})}(\boldsymbol{E}, \boldsymbol{B}) = -\frac{\epsilon}{2}\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B})g\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B}) - \frac{\alpha}{2}\mathsf{G}(\boldsymbol{E}, c\boldsymbol{B})g\mathsf{G}(\boldsymbol{E}, c\boldsymbol{B}). \tag{30}$$

Here, ϵ and α are undetermined constants, and the overall minus sign has been introduced for later convenience. To explain why the quadratic combination in (26) has not been included as a possible term in $\mathsf{T}^{(\mathrm{e})}$, note that it is an *improper* tensor. This means that it changes sign under inversion, as can be seen by observing from (1) that inversion changes only the sign of $\boldsymbol{E}$ and not of $\boldsymbol{B}$ (it is an axial vector), so that the right-hand side of (26) flips sign. Since the energy-momentum tensor of the particles does not show such a sign change, an inversion would not preserve (23) if $\mathsf{T}^{(\mathrm{e})}$ were to include $\mathsf{F}g\mathsf{G}$.

3.3. Scalar invariant from energy and momentum

From (13) it follows that (30) already implies a momentum density $\boldsymbol{\pi}$ of the electromagnetic field which transforms like a vector under spatial rotations, because it is proportional to $\boldsymbol{E} \times \boldsymbol{B}$, no matter what the constants α and ϵ are. A similar calculation shows that the energy density ε turns out to be a scalar under rotations. In summary, one finds

$$\begin{pmatrix} \boldsymbol{\pi} \\ \varepsilon \end{pmatrix} \equiv \mathsf{T}^{(\mathrm{e})}\vec{e}_4 = \begin{pmatrix} (\epsilon + \alpha)c\boldsymbol{E} \times \boldsymbol{B} \\ \epsilon E^2 + \alpha c^2 B^2 \end{pmatrix}. \tag{31}$$

But in particular for the energy density ε, the definition is incomplete because the relative weights of E^2 and B^2 are undetermined.

To discover an independent constraint that fixes the ratio between ϵ and α, recall that the energy and momentum of a particle are actually not independent of each other, because they must satisfy $\varepsilon^2 - p^2c^2 = m^2c^4$. This is a Lorentz scalar because it is given in terms of the particle's four-momentum $\vec{p}$ as an inner product, $-\vec{p}^{\,t}g\vec{p}$. However, (31) is a density, and does not transform as a Lorentz four-vector. But the Lorentz-invariant constraint we seek must be writable in terms of $\mathsf{T}^{(\mathrm{e})}$, because there is no analogue to $\vec{p}$ for the electromagnetic field.

We do not wish to impose any condition on the densities of energy and momentum in (31) that cannot *also* be imposed on the corresponding densities for a particle. For the latter, there is indeed one special case in which the densities of energy and momentum in $\mathsf{T}^{(\mathrm{p})}\vec{e}_4$ can be combined almost trivially into a Lorentz scalar density proportional to $\vec{p}^{\,t}g\vec{p}$: if the particle has mass $m = 0$. Equations (12) and (13) only contain the number density μ/m, which is mass-independent, so the energy- and momentum densities of a massless particle can and will in general be nonzero‡. But crucially, we then find that they combine into $\left(\mathsf{T}^{(\mathrm{p})}\vec{e}_4\right)^t g \left(\mathsf{T}^{(\mathrm{p})}\vec{e}_4\right) = 0$, because $\varepsilon = pc$. Zero is certainly a scalar density. Having found a case for which particles possess a scalar density of this form, we now require that the same combination,

$$\lambda^2 \equiv \left(\mathsf{T}^{(\mathrm{e})}\vec{e}_4\right)^t g \left(\mathsf{T}^{(\mathrm{e})}\vec{e}_4\right) = \varepsilon^2 - \pi^2 c^2,$$

must be a scalar density for the electromagnetic field. We do not demand $\lambda = 0$, only the *existence* of the invariant.

Equation (31) then yields the explicit form of the desired invariant,

$$\lambda^2 = (\epsilon + \alpha)^2 c^2 \left(\boldsymbol{E} \times \boldsymbol{B}\right)^2 - \left(\epsilon E^2 + \alpha c^2 B^2\right)^2. \tag{32}$$

This does in fact become a Lorentz scalar *if and only if* $\alpha = \epsilon$. To show this, first observe that for $\alpha = \epsilon$, (32) is proportional to

$$4c^2 \left(\boldsymbol{E} \times \boldsymbol{B}\right)^2 - \left(E^2 + c^2B^2\right)^2 = -4c^2 \left(\boldsymbol{E} \cdot \boldsymbol{B}\right)^2 - \left(E^2 - c^2B^2\right)^2.$$

‡ For mathematical simplicity, think of a swarm of particles as in (21) with $N \to \infty$, all having the same mass, energy and momentum but with a smooth density instead of a singular delta function.

The right-hand side is manifestly a proper Lorentz scalar, because it contains only the squares of the scalars in (26) and (29). Forming the difference between (32) for $\alpha = \epsilon$ and for $\alpha \neq \epsilon$ yields

$$2c^2\left(\alpha - \epsilon\right) B^2 \left(\left(\alpha + \epsilon\right) c^2 B^2 + \epsilon E^2\right). \tag{33}$$

This is clearly not Lorentz-invariant: for example, if $B = 0$ in some inertial frame, the expression vanishes. But in a moving frame with relative velocity $\boldsymbol{V}$, there will generally be a magnetic field dictated by the transformation of the field tensor with the corresponding Lorentz boost, $\Lambda(\boldsymbol{V})$:

$$\mathsf{F}(\boldsymbol{E}', c\boldsymbol{B}') = \Lambda(\boldsymbol{V})\,\mathsf{F}(\boldsymbol{E}, \boldsymbol{0})\,\Lambda(\boldsymbol{V}).$$

This will yield a nonzero result in (33), *unless* $\alpha = \epsilon$. Only under this condition will λ^2 be a Lorentz scalar density.

Equation (30) with $\alpha = \epsilon$ now states

$$\mathsf{T}^{(\mathrm{e})} = -\frac{\epsilon}{2}\left(\mathsf{F}g\mathsf{F} + \mathsf{G}g\mathsf{G}\right). \tag{34}$$

As a further consequence of $\alpha = \epsilon$, (29) then implies that the traces of the two terms in (30) precisely cancel each other,

$$\mathrm{Tr}\big(g\mathsf{T}^{(\mathrm{e})}\big) = \left(\epsilon - \alpha\right)\left(c^2 B^2 - E^2\right) = 0. \tag{35}$$

Comparison with (14) suggests that one may thus refer to the electromagnetic field as having a vanishing mass density. Note, however, that our argument for setting $\alpha = \epsilon$ is not based on the notion that the field has a property called "mass density" at all. We merely argued that mechanics allows particles for which the analogous invariant can be constructed from $\mathsf{T}^{(\mathrm{p})}$, and this gave us a constraint that was needed to resolve the ambiguity in the definition of the electromagnetic energy density, involving only the concepts of energy and momentum themselves.

Even though the interpretation of (35) as a vanishing mass density is irrelevant for the following, it is consistent with the case of the massless particle that served as the mechanics-based template for the invariance of (32), and also with the known property that the Lorentz force is *rest-mass preserving*[27]. One can furthermore make $\lambda = 0$ by choosing $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ and $E = cB$, which will turn out to correspond to the special case of electromagnetic plane waves.

The simplicity of (34) may provide additional confidence in its validity, but the true test will be whether it is possible to unambiguously deduce the known laws of electrodynamics from it. The following section will show that this is indeed the case, without invoking any postulates that have not already been used. Then the foregoing discussion will have served its purpose: to use the postulate of local energy-momentum conservation to explain the construction of the electromagnetic energy-momentum tensor from which Maxwell's equations will arise.

4. Deriving Maxwell's equations

4.1. Separating the fields and their derivatives

Equation (24) takes the following form if (34) is inserted:

$$-\frac{\epsilon}{2}\frac{\partial}{\partial\vec{x}}\left(\mathsf{F}g\mathsf{F}+\mathsf{G}g\mathsf{G}\right)=\frac{1}{c}\vec{j}^{\,t}g\mathsf{F}. \tag{36}$$

Due to the symmetry between F and G on the left-hand side, *magnetic monopoles* readily suggest themselves in (36), in the form of a new term on the right:

$$-\frac{\epsilon}{2}\frac{\partial}{\partial\vec{x}}\left(\mathsf{F}g\mathsf{F}+\mathsf{G}g\mathsf{G}\right)=\frac{1}{c}\vec{j}^{\,t}g\mathsf{F}-\frac{1}{c}\vec{j}^{\,t}_{\mathrm{m}}g\mathsf{G}. \tag{37}$$

This is a modification of the Lorentz force in which a new four-current $\vec{j}_{\mathrm{m}}$ is introduced. The symmetry of (37) can already be said to exist[27] in (1). The sign choice for $\vec{j}_{\mathrm{m}}$ is arbitrary and can be considered part of the definition of magnetic charge. Because there exists no experimental evidence for magnetic monopoles in free space[7], this is an optional addition, but it adds virtually no complexity to the calculation. So we shall continue with (37), treating $\vec{j}_{\mathrm{m}}$ as an auxiliary variable that can be set to $\vec{j}_{\mathrm{m}}=\vec{0}$ at the end, if desired.

The field tensors will now be written out in terms of the generators in (4), (5), because these are *constants* relative to the derivative operator in (37). This has two main purposes: it exposes the six functions in $\boldsymbol{E}$ and $\boldsymbol{B}$ on which the derivatives actually operate, and it will later allow us to change the structure under the derivative, by right-multiplying both sides of the equation with other constant matrices.

Replacing F and G by (6) and (25), the divergence is

$$\begin{aligned}\frac{\partial}{\partial\vec{x}}\mathsf{T}^{(\mathrm{e})}= & -\frac{\epsilon}{2}\sum_{i,j=1}^{3}\left[\frac{\partial}{\partial\vec{x}}\left(\omega_i g\omega_j+\sigma_i\sigma_j\right)\left(E_iE_j+c^2B_iB_j\right)\right]- \\ & -\frac{\epsilon}{2}\sum_{i,j=1}^{3}\left[c\frac{\partial}{\partial\vec{x}}\left(\omega_i\sigma_j-\sigma_i\omega_j\right)\left(E_iB_j-B_iE_j\right)\right]\end{aligned}$$

Here, the gradient row vector $\partial/\partial\vec{x}$ is multiplied by one of the constant matrices

$$\mathsf{D}^{(1)}_{i,j}\equiv\omega_i g\omega_j+\sigma_i\sigma_j, \tag{38}$$

$$\mathsf{D}^{(2)}_{i,j}\equiv\omega_i\sigma_j-\sigma_i\omega_j \tag{39}$$

(for i, $j=1,2,3$). For any two functions $f_1(\vec{x})$ and $f_2(\vec{x})$, and any constant 4×4 matrix D, the *Leibnitz rule* applies as follows§:

$$\frac{\partial}{\partial\vec{x}}\mathsf{D}\,f_1f_2=\left(\frac{\partial}{\partial\vec{x}}\mathsf{D}\,f_1\right)f_2+f_1\left(\frac{\partial}{\partial\vec{x}}\mathsf{D}\,f_2\right). \tag{40}$$

Using this on the products of electric and magnetic field components in $\mathsf{T}^{(\mathrm{e})}$,

$$\frac{\partial}{\partial\vec{x}}\mathsf{T}^{(\mathrm{e})}=-\frac{\epsilon}{2}\sum_{i,j=1}^{3}\left[E_i\frac{\partial}{\partial\vec{x}}\left(\mathsf{D}^{(1)}_{i,j}E_j+c\mathsf{D}^{(2)}_{i,j}B_j\right)+cB_i\frac{\partial}{\partial\vec{x}}\left(c\mathsf{D}^{(1)}_{i,j}B_j-\mathsf{D}^{(2)}_{i,j}E_j\right)\right]-$$

§ The analogous rules for the tensors in (37) would look much uglier.

$$-\frac{\epsilon}{2}\sum_{i,j=1}^{3}\left[E_j\frac{\partial}{\partial\vec{x}}\left(\mathsf{D}^{(1)}_{i,j}E_i-c\mathsf{D}^{(2)}_{i,j}B_i\right)+cB_j\frac{\partial}{\partial\vec{x}}\left(c\mathsf{D}^{(1)}_{i,j}B_i+\mathsf{D}^{(2)}_{i,j}E_i\right)\right]$$

In the double sum on the last line, we now switch indices $i\leftrightarrow j$. Collecting the field components E_i and B_i, the result then simplifies to

$$\frac{\partial}{\partial\vec{x}}\mathsf{T}^{(\mathrm{e})}=-\frac{\epsilon}{2}\sum_{i,j=1}^{3}E_i\frac{\partial}{\partial\vec{x}}\left[\left(\mathsf{D}^{(1)}_{i,j}+\mathsf{D}^{(1)}_{j,i}\right)E_j+c\left(\mathsf{D}^{(2)}_{i,j}-\mathsf{D}^{(2)}_{j,i}\right)B_j\right]- \\ -\frac{\epsilon}{2}\sum_{i,j=1}^{3}cB_i\frac{\partial}{\partial\vec{x}}\left[c\left(\mathsf{D}^{(1)}_{i,j}+\mathsf{D}^{(1)}_{j,i}\right)B_j-\left(\mathsf{D}^{(2)}_{i,j}-\mathsf{D}^{(2)}_{j,i}\right)E_j\right]. \tag{41}$$

The two combinations of $\mathsf{D}^{(k)}_{i,j}$ appearing above can now be simplified. Commuting the terms in $\mathsf{D}^{(1)}_{i,j}$ as defined by (38),

$$\mathsf{D}^{(1)}_{i,j}+\mathsf{D}^{(1)}_{j,i}=2\left(\omega_j g\omega_i+\sigma_j\sigma_i\right) \tag{42}$$

In the same way, reversing the order of terms in $\mathsf{D}^{(2)}_{i,j}$ from (39) yields

$$\mathsf{D}^{(2)}_{i,j}-\mathsf{D}^{(2)}_{j,i}=2\left(\sigma_j\omega_i-\omega_j\sigma_i\right). \tag{43}$$

Equations (42) and (43) can be inserted in (41). This is then equated to the right-hand side of the Lorentz force law (37), which expands as follows:

$$\frac{\partial}{\partial\vec{x}}\mathsf{T}^{(\mathrm{e})}=\frac{1}{c}\sum_{i=1}^{3}\left(\vec{j}^{\,t}g\omega_iE_i+c\vec{j}^{\,t}g\sigma_iB_i+c\vec{j}^{\,t}_{\mathrm{m}}g\omega_iB_i-\vec{j}^{\,t}_{\mathrm{m}}g\sigma_iE_i\right). \tag{44}$$

With this, (37) now reads

$$\vec{0}^{\,t}=\sum_{i=1}^{3}E_i\left\{\frac{1}{c}\vec{j}^{\,t}g\omega_i-\frac{1}{c}\vec{j}^{\,t}_{\mathrm{m}}g\sigma_i+\epsilon\frac{\partial}{\partial\vec{x}}\sum_{j=1}^{3}\left[\left(\omega_jg\omega_i+\sigma_j\sigma_i\right)E_j+c\left(\sigma_j\omega_i-\omega_j\sigma_i\right)B_j\right]\right\}+ \\ +c\sum_{i=1}^{3}B_i\left\{\frac{1}{c}\vec{j}^{\,t}g\sigma_i+\frac{1}{c}\vec{j}^{\,t}_{\mathrm{m}}g\omega_i+\epsilon\frac{\partial}{\partial\vec{x}}\sum_{j=1}^{3}\left[c\left(\omega_jg\omega_i+\sigma_j\sigma_i\right)B_j-\left(\sigma_j\omega_i-\omega_j\sigma_i\right)E_j\right]\right\}. \tag{45}$$

None of the quantities in the curly braces depend on the values of E_i, B_i – only on their derivatives.

4.2. Deriving Maxwell's equations

Now recall that $\boldsymbol{E}$ and $\boldsymbol{B}$ are a superposition with the external field $\mathfrak{E}$, $\mathfrak{B}$, which can take on arbitrary values because the sources lie in the environment, not in $\vec{j}$ or $\vec{j}_{\mathrm{m}}$. On the other hand, the *derivatives* of E_i, B_i are related to $\vec{j}$ and $\vec{j}_{\mathrm{m}}$, in addition to the external field, since it is these derivatives that ultimately contain the information about the force *on* the environment.

As an elementary example‖, consider a parallel-plate capacitor with a charge q moving through it. The capacitor creates a uniform electric field $\mathfrak{E}$ (i.e., vanishing

‖ Examples using electromagnetic phenomena are illustrative but not necessary for the logical coherence of our argument derivatives). Repeating the experiment with q made to cross the same point at the same velocity every time, but with the uniform capacitor field adjusted to a different strength and direction, the total field $\boldsymbol{E}$ would have a different value at the instant the charge is at the chosen point, but it *would* have the same derivatives everywhere, because the latter are determined by the charge and its motion. The magnetic field $\boldsymbol{B}$ could be manipulated in the same way by surrounding q with a solenoid.

Stated more generally: at a *given* $\vec{x}$, we have the freedom to vary the external fields $\mathfrak{E}(\vec{x})$ and $\mathfrak{B}(\vec{x})$ from one repetition of the experiment to the next in such a way that their x_i*-derivatives* stay locally unchanged. Assume that all other initial conditions are chosen just right every time we try a new external field, so as to produce the same $\vec{j}(\vec{x})$ and $\vec{j}_{\mathrm{m}}(\vec{x})$ at the event $\vec{x}$ of interest. Then the derivatives of $\boldsymbol{E}(\vec{x})$ and $\boldsymbol{B}(\vec{x})$ will also stay unchanged, but the *function values* of the components $E_i(\vec{x})$ and $B_i(\vec{x})$ will be different.

As noted, the curly braces in (45) contain only quantities that stay *constant* while E_i and B_i are changed as described. Therefore, in order to guarantee that the sum (45) remains zero, the braces multiplying E_i and B_i must independently be zero, for each $i = 1, 2, 3$. This procedure makes no assumptions about the *functional form* of $\boldsymbol{E}(\vec{x})$ and $\boldsymbol{B}(\vec{x})$, because $\vec{x}$ is held fixed. As a consequence, the following two necessary conditions must hold:

$$\epsilon\frac{\partial}{\partial\vec{x}}\sum_{j=1}^{3}\left[\left(\omega_j g\omega_i+\sigma_j\sigma_i\right)E_j+c\left(\sigma_j\omega_i-\omega_j\sigma_i\right)B_j\right]=-\frac{1}{c}\vec{j}^{\,t}g\omega_i+\frac{1}{c}\vec{j}_{\mathrm{m}}^{\,t}g\sigma_i \tag{46}$$

$$\epsilon\frac{\partial}{\partial\vec{x}}\sum_{j=1}^{3}\left[c\left(\omega_j g\omega_i+\sigma_j\sigma_i\right)B_j-\left(\sigma_j\omega_i-\omega_j\sigma_i\right)E_j\right]=-\frac{1}{c}\vec{j}^{\,t}\sigma_i-\frac{1}{c}\vec{j}_{\mathrm{m}}^{\,t}g\omega_i \tag{47}$$

This step constitutes the *complete derivation of Maxwell's equations*, in the presence of electric and magnetic sources, from the above assumptions about the energy-momentum tensor. To verify this, one only needs to group the system of coupled first-order differential equations (46) and (47) in a more recognizable form. This will also give the parameters ϵ and c their familiar meaning.

There are 24 equations, because i takes on three values and each line above has four components – but only 8 of these equations are independent. To extract the latter, one can use the following relations[20] between the generators of rotations and boosts:

$$-\frac{1}{3}\sum_{i=1}^{3}\left(\omega_i^2+\sigma_i^2\right)=1, \tag{48}$$

(note the absence of g in ω_i^2), and

$$\sum_{i=1}^{3}\omega_i\sigma_i=\sum_{i=1}^{3}\sigma_i\omega_i=0, \tag{49}$$

which acts as a matrix orthogonality relation. To apply (49), right-multiply (46) by ω_i, and (47) by σ_i. Summing over i then removes the source term containing $\vec{j}_{\mathrm{m}}$. Then add the two equations to exploit (48):

$$\frac{3}{c\epsilon}\vec{j}^{\,t}g = \frac{\partial}{\partial\vec{x}}\sum_{i,j=1}^{3}\left[\left(\omega_j g\omega_i + \sigma_j\sigma_i\right)\omega_i - \left(\sigma_j\omega_i - \omega_j\sigma_i\right)\sigma_i\right]E_j +$$

$$+\, c\frac{\partial}{\partial\vec{x}}\sum_{i,j=1}^{3}\left[\left(\omega_j g\omega_i + \sigma_j\sigma_k\right)\sigma_i + \left(\sigma_j\omega_i - \omega_j\sigma_i\right)\omega_i\right]B_j$$

$$= -3\frac{\partial}{\partial\vec{x}}\sum_{j=1}^{3}\left(E_j\omega_j g + cB_j\sigma_j\right), \tag{50}$$

On the right-hand side, we used the following two matrix identities which can be directly read off from (48) and (49):

$$\sum_{i=1}^{3}\left[\left(\sigma_j\omega_i - \omega_j\sigma_i\right)\sigma_i - \left(\omega_j g\omega_i + \sigma_j\sigma_i\right)\omega_i\right] = 3\omega_j g \tag{51}$$

$$\sum_{i=1}^{3}\left[\left(\omega_j g\omega_i + \sigma_j\sigma_i\right)\sigma_i + \left(\sigma_j\omega_i - \omega_j\sigma_i\right)\omega_i\right] = -3\sigma_j. \tag{52}$$

The sum in (50) is the electromagnetic field tensor F, as defined in (6). After right-multiplying by $g/3$ and using $\sigma_j g = \sigma_j$, we arrive at

$$\frac{1}{c\epsilon}\vec{j}^{\,t} = -\frac{\partial}{\partial\vec{x}}\mathsf{F}(\boldsymbol{E}, c\boldsymbol{B}) \tag{53}$$

This is the first half of Maxwell's equations in the presence of sources. For example, right-multiplying (53) by $\vec{e}_4$ and recalling (17), one finds Gauss' Law:

$$\frac{1}{\epsilon}\rho = \frac{\partial}{\partial\vec{x}}\sum_{j=1}^{3}E_j\vec{e}_j = \nabla\cdot\boldsymbol{E}. \tag{54}$$

Comparison with experiment finally identifies ϵ as the *dielectric constant of free space.*

To obtain the second half of Maxwell's equations, right-multiply (46) by σ_i, and (47) by ω_i. Then sum over i and subtract the second from the first equation. Equation (48) then yields

$$-\frac{3}{c\epsilon}\vec{j}^{\,t}_{\mathrm{m}}g = \frac{\partial}{\partial\vec{x}}\sum_{i,j=1}^{3}\left[\left(\omega_j g\omega_i + \sigma_j\sigma_i\right)\sigma_i + \left(\sigma_j\omega_i - \omega_j\sigma_i\right)\omega_i\right]E_j +$$

$$+\, c\frac{\partial}{\partial\vec{x}}\sum_{i,j=1}^{3}\left[\left(\sigma_j\omega_i - \omega_j\sigma_i\right)\sigma_i - \left(\omega_j g\omega_i + \sigma_j\sigma_i\right)\omega_i\right]B_j$$

$$= -3\frac{\partial}{\partial\vec{x}}\sum_{j=1}^{3}\left(E_j\sigma_j - cB_j\omega_j g\right), \tag{55}$$

again using equations (42), (43), (51) and (52). Multiplying by $-g/3$ from the right and recognizing the dual field tensor G from (25), we arrive at

$$\frac{1}{c\epsilon}\vec{j}^{\,t}_{\mathrm{m}} = \frac{\partial}{\partial\vec{x}}\mathsf{G}(\boldsymbol{E}, c\boldsymbol{B}). \tag{56}$$

Equations (53) and (56) are Maxwell's equations. They were obtained from our physical understanding of $\boldsymbol{E}$ and $\boldsymbol{B}$ as being composed of two parts, corresponding to the external field $\mathfrak{E}$, $\mathfrak{B}$, and the effects of $\vec{j}$ on its environment, encoded in the derivatives of $\boldsymbol{E}$, $\boldsymbol{B}$. We then used the fact that $\mathfrak{E}$, $\mathfrak{B}$ are independent of the electromagnetic field generated by the charge under consideration, to allow a variation of $\boldsymbol{E}$ and $\boldsymbol{B}$ without varying any of their derivatives. Our procedure is mathematically unrelated to the Euler-Lagrange equation because it involved no action integral.

We can now return to any standard text that derives (37) *from* Maxwell's equations[10] to confirm that (53) and (56) are not only necessary, but also sufficient to satisfy the original condition in (37)¶. All the usual implications of electrodynamics follow, including most significantly the value of the parameter c in terms of the experimentally measured constants ϵ and $1/(c^2\epsilon)$, the *magnetic permeability of free space*. Maxwell's wave equation then reveals the meaning of c as the speed of electromagnetic waves in vacuum. Crucially, this was not taken for granted *a priori* in our mechanics-centered approach. Therefore, we have arrived at the speed of light as the limiting speed of relativistic mechanics in a deductive way.

For $\vec{j}_{\mathrm{m}} \neq \vec{0}$, more physical insight can be gained by returning to (1) with the addition of the magnetic monopoles[7]. There, one encounters the well-known issue that the transformation behavior of the Lorentz force requires magnetic charge to be a pseudoscalar. As in all other frameworks, a departure from $\nabla \cdot \boldsymbol{B} = 0$ therefore makes some aspect of the theory look more aesthetic, but simultaneously spoils the simplicity of another aspect.

5. Conclusion

In our path to Maxwell's equations, no reference is made to the empirical results of electromagnetism that give the four famous equations their individual names. This abstraction from phenomenology also underlies the action principle, but our approach requires much less methodological overhead to derive electrodynamics. We endeavored to construct the central equation (36) in a bottom-up manner that does not require the reader to to take large leaps of faith before seeing the successful outcome of the deduction.

There are three distinct steps to this program that can also be taken independently of each other: first, a derivation of the Lorentz force from relativistic mechanics given in Ref. [15]; second, an inference of the energy-momentum tensor (up to irrelevant terms) from the assumption that the electromagnetic field can carry energy and momentum; and finally, a derivation of Maxwell's equations using the fact that the external field on a charge distribution can vary independently of the field created by that distribution, while both must contribute to the energy-momentum tensor.

The third part is the central piece of this work and can stand alone if one just desires

¶ With some additional work, one can of course also check directly that (46) and (47) are strictly equivalent to (53) and (56).

a derivation of Maxwell's equations that takes the Lorentz force and energy-momentum tensor for granted. However, all three parts are important in order to check that we have not circularly relied on the relations of electrodynamics that were to be deduced. In each step, it can be argued that the main content is a direct extension of mechanics. This explains the title of the present paper.

From a pedagogical perspective, the present work can provide an avenue to reacquaint students at the advanced undergraduate or beginning graduate level with Maxwell's equations from a novel point of view. The central importance of the energy-momentum tensor in our discussion provides a strong motivation to rearrange the teaching of some other electrodynamic topics that would follow from Maxwell's equations. In particular, one could readily delve into the Maxwell stress tensor as an alternative way to look at electromagnetic forces.

Future work may help confer additional *a priory* physical motivation to the electromagnetic Lagrangian[10], which for the field itself is proportional to (29). The course of action would be to gather experience and student feedback from a two-week unit developing the material along the lines proposed here, followed by standard topics rearranged as suggested above, but *not* including the action principle. How the students react to the emergence of energy-momentum conservation in our presentation will then inform the development of the action principle in later parts of the course.

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