A New Ribbon Basis for Rank-Selected Homology of Geometric Lattices & Representation Stability

(or "What Sheila & Tricia were busy doing last Fall at ICERM")

Patricia Hersh
University of Oregon

- joint work with Sheila Sundaram
University of Minnesota

Plan for Today

I. Background

II. Main Results

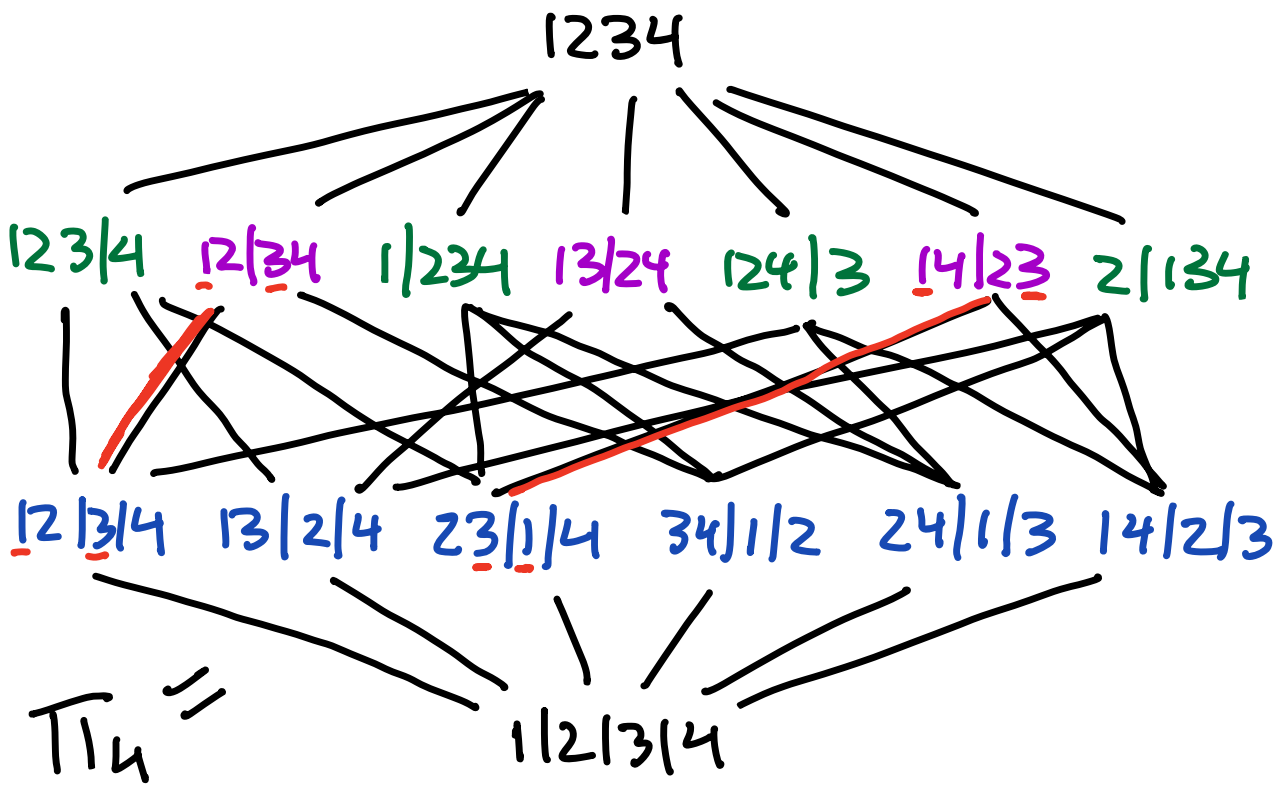
- construct basis for rank-selected homology of any geometric lattice P (called "ribbon basis") (positively answers 1992 question of Anders Björner)

- (1) establish representation stability of $\beta_S(\Pi_n)$ for partition lattice Π_n
  (2) prove sharp repn stability bound of $4 \max S - |S| + 1$ for $\beta_S(\Pi_n)$ (proving conjecture of H.-Reiner)

- establish stability & prove sharp stability bound for Boolean lattice B_n

III. Examples

Partition Lattice Π_n and its S_n-Representations



- Π_n poset of set partitions w/ refinement order
- S_n acts on Π_n by permuting values

  e.g. $(13)[12|3|45] = 32|1|45 \in \Pi_5$
- Action preserves $<$, so permutes chains $u_1 < u_2 < \dots < u_k$, i.e., faces of $\Delta(\Pi_n)$

S_n-action on chains of P gives rise to S_n-representations on Homology Groups of $\Delta(P)$

- S_n-action on Π_n also preserves poset rank (#steps from $\hat{0}$). S_n-action on $B_n :=$ poset of subsets of $\{1,2,\ldots,n\}$ with $\subseteq$ order preserves $<$ and rank too

- S_n-action on chains of Π_n (resp. B_n) commutes with simplicial boundary map d given by:

$$d(u_0 < \cdots < u_r) = \sum_{i=0}^{r} (-1)^i (u_0 < \ldots < \hat{u_i} < \ldots < u_r)$$

for $\Delta(\Pi_n)$ (resp. $\Delta(B_n)$), hence induces S_n-representation on i^{th} homology group of order complex $\Delta(P)$ for each i.

Def'n: Given graded poset P of rank n with G-action preserving rank & $<$, define for each $S \subseteq \{1,2,\ldots,n-1\}$:

$$S = \{s_1 < s_2 < \cdots < s_k\}$$

$\alpha_S(P) :=$ permutation rep'n of G on chains $c_1 < \cdots < c_k$ in P where $\mathrm{rank}(c_i) = s_i$ for $i = 1,2,\ldots,k$

$$\beta_S(P) := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T(P) = \text{"virtual" } G\text{-rep'n}$$

Fact: If graded P with G-action is "Cohen-Macaulay" (e.g., shellable), then:

(a) $P^S = \{u \in P \mid \mathrm{rk}(u) \in S\}$ has homology concentrated in top degree $\forall\, S \subseteq \{1,\ldots,\mathrm{rk}\,P-1\}$

& (b) $\beta_S(P)$ is an actual G-representation, namely rep'n on top homology of P^S coming from G-action on P^S.

Rank-Selected Homology $\beta_S(\Pi_n)$ of Partition Lattice Π_n

$S = \{1, 2, \ldots, n-2\}$ case:

(where $\Pi_n^{\{1,\ldots,n-2\}} = \Pi_n - \{\hat{0}, \hat{1}\}$ since $\text{rank}(\Pi_n) = n-1$)

Thm (Hanlon + Stanley):

$$\beta_{\{1,\ldots,n-2\}}(\Pi_n) \cong_{S_n} \text{sgn}_n \otimes \left(\psi \uparrow_{C_n}^{S_n}\right)$$

where ψ is linear repn sending generator for C_n to $e^{2\pi i/n}$

Rk: $\dim\left(\beta_{\{1,\ldots,n-2\}}(\Pi_n)\right) = (n-1)! = (-1)^{n-1} \cdot \mu_{\Pi_n}(\hat{0}, \hat{1})$

Thm (Joyal): $\beta_{\{1,\ldots,n-2\}}(\Pi_n) \cong_{S_n} \text{sgn}_n \otimes \underbrace{\text{Lie}_n}_{\text{multilinear part free Lie alg.}}$

(beautiful later proofs by Barcelo & by Wachs)

Qn (Richard Stanley, '82): Understand $\beta_S(\Pi_n)$ for each $S \subseteq \{1,\dots,n-2\}$.

- Lots of past work on $\langle \beta_S(\Pi_n), \mathbb{1} \rangle$, but even that not well understood for $S \neq \{1,2,\dots,n-2\}$

  (encapsulates famous hard questions regarding plethysm of rep'ns)

Today: "representation stability" for $\beta_S(\Pi_n)$ for fixed S (as n grows) & sharp representation stability bound

What rep. stability bound gets you:

- for fixed S, knowing $\beta_S(\Pi_n)$ for $n = 1, 2, 3, \dots, 4\max S - |S| + 1$ determines $\beta_S(\Pi_n)$ for all n.

  ($4\max S - |S| + 1$: stability bound we prove)

  e.g. $S = \{1,3\} \Rightarrow 4\max S - |S| + 1 = 12 - 2 + 1 = 11$

Rk: Rank-selection is poset analogue of partial flag variety (cleanest example: poset of subspaces)

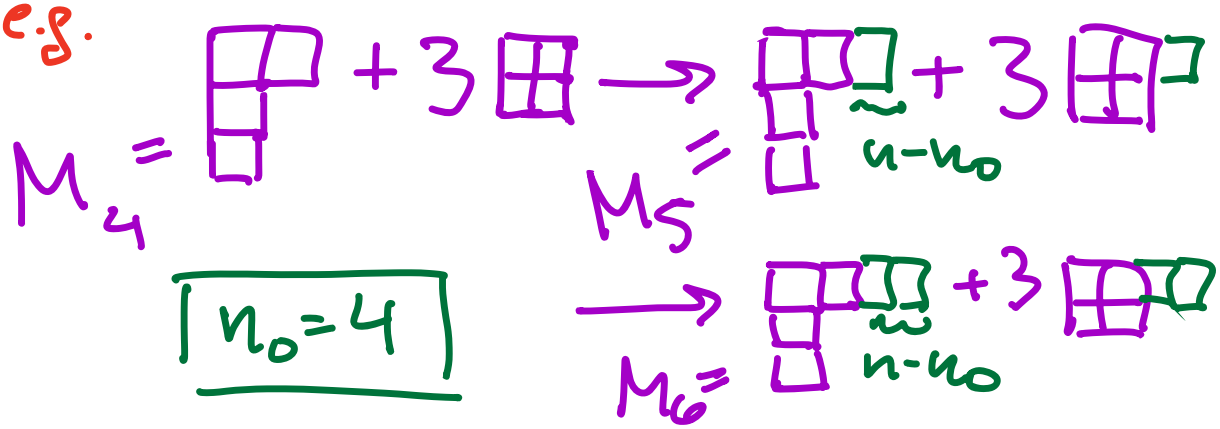
Representation Theoretic Stability

Defn (Church-Farb): A series of S_n-modules $M_1, M_2, M_3, \ldots$ for $n=1,2,3,\ldots$ stabilizes at $n=n_0$ if

$$M_{n_0} = \bigoplus_{\lambda \vdash n_0} c_\lambda S^\lambda \Rightarrow M_n = \bigoplus_{\lambda \vdash n_0} c_\lambda S^{(\lambda_1 + n - n_0, \lambda_2, \lambda_3, \ldots)}$$

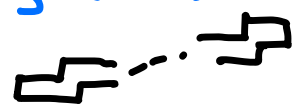
(for all $n \geq n_0$)

e.g.



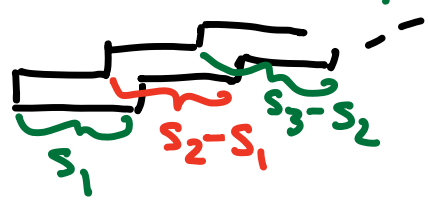
Stability Proof Method: Decompose each M_n into components $S^\lambda \otimes \mathbb{1}_{n-|\lambda|} \uparrow_{S_{|\lambda|} \times S_{n-|\lambda|}}^{S_n}$ and give upper bound on $|\lambda| + \lambda_1$.

S_n-Module Structure for $\beta_S(B_n)$

Thm (Solomon): $\beta_S(B_n) \cong_{S_n} S$ 

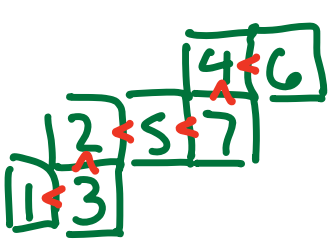
for $S = \{s_1, \dots, s_k\}$
with $1 \le s_1 < \dots < s_k \le n-1$

Specht module of ribbon shape

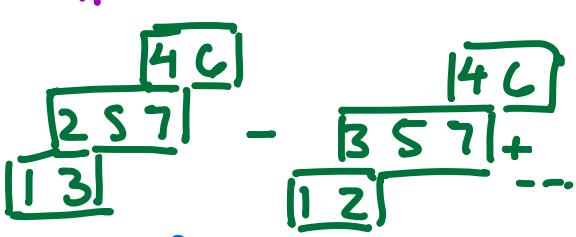


Plan: Homology basis for $\beta_S(P)$ for any geometric lattice P. Shares features of "polytabloid" basis for Specht modules. Use features to calculate multiplicities of irrep's via Young symmetrizers.

e.g. (Boolean lattice case)

$F =$ 

$\rightsquigarrow$ v_F = polytabloid basis element

v_F = 

F = standard Young tableau

element of our ribbon basis

$\rightsquigarrow$ $(\{1,3\} < \{1,3,2,5,7\} < \dots)$
$-(\{1,2\} < \{1,2,3,5,7\} < \dots) + \dots$

II. Main Results

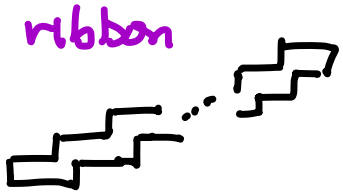
1. Fix finite $S \subseteq \{1,2,\ldots\} = \mathbb{Z}_{>0}$. Letting n grow, we prove:

   - $\beta_S(B_n)$ is representation stable.
   - it has sharp uniform representation stability bound $2\max S - |S| + 1$

2. We construct new "ribbon basis" for $\beta_S(P)$ for any geometric lattice P

   -answers question of Anders Björner

   -introduces matroid/graphical analogue of Specht modules (for ribbon shapes)



3. Fixing $S \subseteq \mathbb{Z}_{>0}$ and letting n grow, we prove:

- representation stability for $\beta_S(\Pi_n)$
- $\beta_S(\Pi_n)$ satisfies sharp (uniform) representation stability bd of $4 \max S - |S| + 1$ (as conjectured by H.-Reiner)

Note: special cases of $S = \{1, 2, \ldots, i\}$ & $S = \{i\}$ previously done in H-Reiner

Motivations: computer pfs, understand multiplicities...

Techniques for (3):

- symmetric fns & Whitney homology (for Boolean lattices)
- Young symmetrizers applied to "ribbon basis" in rather nonstandard way (for partition lattices)

III. Examples

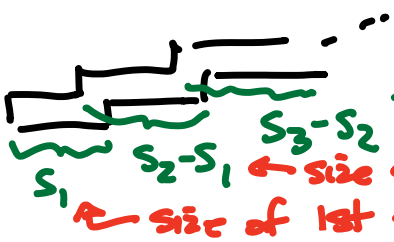
Homology Basis Construction for $\beta_S(\Pi_n)$

- totally order atoms $a_1, a_2, \ldots, a_n$ (elements covering $\hat{0}$)

  e.g. $12 <_n 13 <_n 23 <_n 14 <_n 24 <_n 34 <_n \cdots$ in Π_n

  where $12 := 12|3|4|\cdots|n$ $\quad ij := ij|1|2|\cdots|n$
  $13 := 13|2|4|5|\cdots|n$

- consider "standard" fillings of ribbon shape



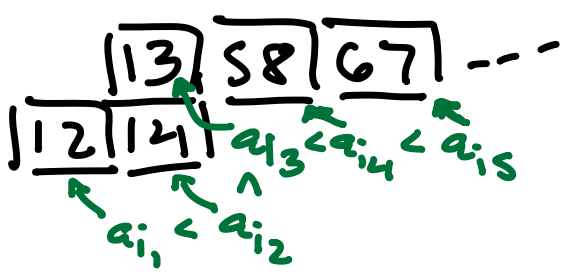
for $S = \{s_1 < s_2 < \cdots\}$

$s_2 - s_1$ ← size of 2nd poset rank jump in P^S

s_1 ← size of 1st poset rank jump in P^S

with atoms (increase L to R in rows & decrease up columns)

e.g.



such that

(1) $a_{i_1} <\cdot\ a_{i_1} \vee a_{i_2} <\cdot\ a_{i_1} \vee a_{i_2} \vee a_{i_3} <\cdot \cdots$

in geometric lattice P

(2) each a_{i_j} small as possible w.r.t. Γ

yielding poset chain $a_{i_1} \lessdot a_{i_1} \vee a_{i_2} \lessdot \cdots$

e.g. $12 \vee 14 = 12 \vee 24$ but $14 <_\Gamma 24$

so $\cancel{\boxed{12}\boxed{24}}$ $\quad \boxed{12}\boxed{14}$ ✓

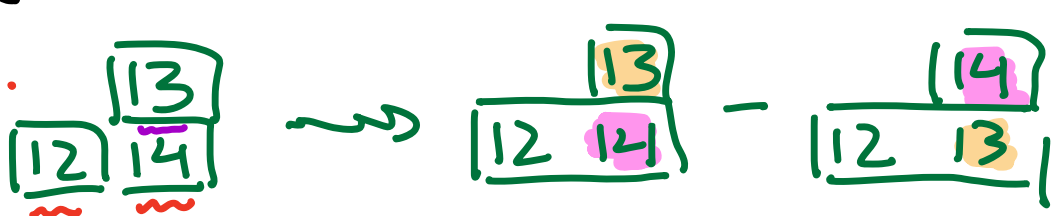
- allowable std. fillings map bijectively to the "homology facets" in shelling for $\Delta(P^S)$

← facets "closing off" spheres



- Send each such ribbon filling F to "polytabloid" v_F

e.g.



- map v_F to alternating sum of maximal chains in P^S giving ribbon basis element indexed by F.

e.g. $(12 \vee 14 < 12 \vee 14 \vee 13) - (12 \vee 13 < 12 \vee 13 \vee 14)$

More Detailed Example of Ribbon Basis Element

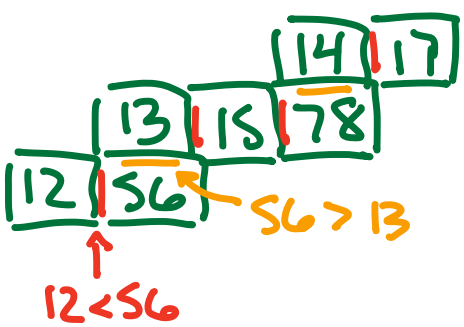
e.g. $n = 8$, $S = \{2,5\}$ $\rightsquigarrow$ ribbon row lengths $2,5\text{-}2,7\text{-}5$

Total order on atoms of Π_8:

$$12 < 13 < 23 < 14 < 24 < 34 < 15 < 25 < \cdots$$

$\updownarrow$

$12|3|4|\cdots|8$

$F =$ 

$=$ "standard" Young tableau whose entries are NBC$^+$ basis

$\overline{f}_{\text{chain}}(\{F\}) = e_{12} \vee e_{56} < e_{12} \vee e_{56} \vee e_{13} \vee e_{15} \vee e_{\cdots}$

(F: tabloid for F)

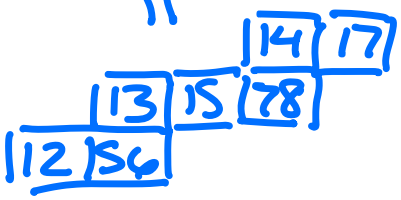
$12|56|3|4|7|8 \qquad 12356|78|4$

max chain in Π_n^S for $S = \{2,5\}$

$\overline{f}_{\text{chain}}$: map sending tabloids to max chains in P^S

Basis Element indexed by F

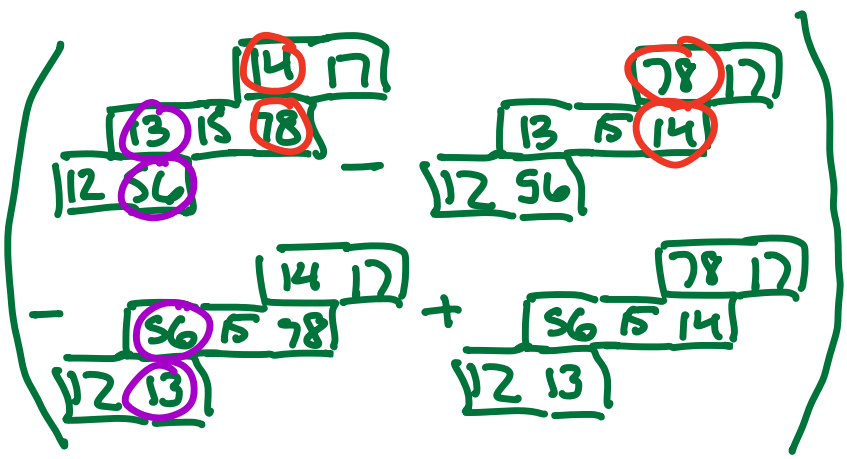
$=$



$\bar{f}_{chain}(v_F)$

$=$

$\bar{f}_{chain}$



$= (12 \vee 56 < (12 \vee 56) \vee (13 \vee 15 \vee 78))$

$- (12 \vee 56 < (12 \vee 56) \vee (13 \vee 15 \vee 14))$

$- (12 \vee 13 < (12 \vee 13) \vee (56 \vee 15 \vee 78))$

$+ (12 \vee 13 < (12 \vee 13) \vee (56 \vee 15 \vee 14))$

Example: Stability Bound for $B_S(\Pi_n)$

- Prove $|\lambda| + \lambda_1 > 4 \max S - |S| + 1 \Rightarrow$

related to B_S: $\langle \widehat{\omega H}_S(\Pi_n), S^\lambda \rangle = 0$

to deduce stability bound $4 \max S - |S| + 1$

e.g. $S = \{1,3\}$, $\lambda = (6)$

$\max S = 3$ $\quad |S| = 2$

$T =$ 

$|\lambda| = 6$

$\lambda_1 = 6$

shape $= (6^1)$

$4 \max S - |S| + 1 = 4 \cdot 3 - 2 + 1 = 11 < 12 = 6 + 6 = |\lambda| + \lambda_1$

($4 \max S$: $\max S$; $-|S|$: $|S|$)

$$b_T a_T = (1\!\!1)\left(\sum_{\sigma \in S_6} \sigma\right)$$

"Young symmetrizer"

$(1\!\!1)$: column antisymmetrizer; $\sum_{\sigma \in S_6} \sigma$: row symmetrizer

$$\bar{f}_{\text{chain}}(v_F) = \bar{f}_{\text{chain}}\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end\!\!\begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array} - \begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline\end\!\!\begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array}\right)$$

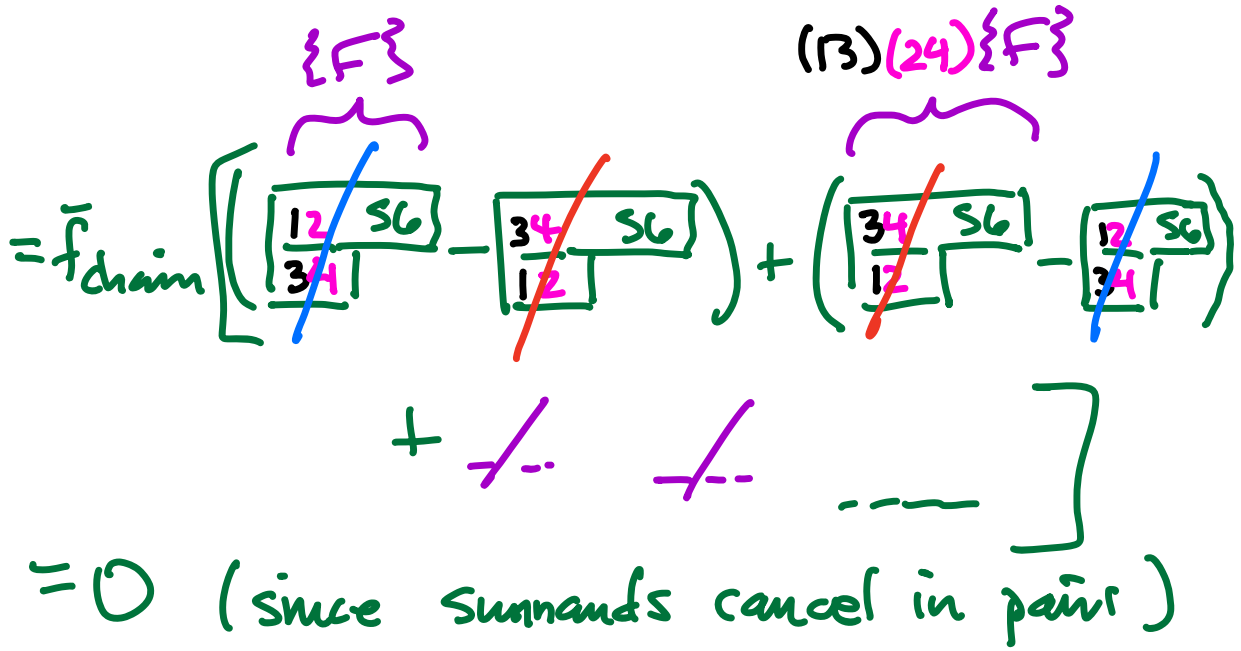
$$= \boxed{(1|2|34|5|6 < 12|34|56) - (12|3|4|5|6 < 12|34|56)}$$

$\bar{f}_{\text{chain}}$: ribbon basis elt

$b_T a_T(\sim)$: apply to ribbon basis elt

$$b_T a_T(\sim) = \bar{f}_{\text{chain}}\left(\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end\!\!\begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array} - \begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline\end\!\!\begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array}\right) + (13)(24)(v_F) \pm \cdots\right)$$

v_F; $(13)(24) \in S_6$



$= \bar{f}_{chain}[\ldots] = 0$ (since summands cancel in pair)

One may show:

$b_T a_T \bar{f}_{chain}(v_F) = 0$ $\forall$ T of shape  & all $\bar{f}_{chain}(v_F)$ in ribbon basis

so $b_T a_T(\widehat{WH}_{\{1,3\}}(\Pi_n)) = 0$ (since $b_T a_T$ sends basis to 0)

so $\langle \widehat{WH}_{\{1,3\}}(\Pi_n), S^{\square\square\square} \rangle = 0$ (argument works for $|\lambda| + \lambda_1 > 4 \max S - |S| + 1$)

implying stability for $n \geq 4 \max S - |S| + 1$.

Thank you!