Discrete Morse Theory for Poset Order Complexes (& Other Abstract Simplicial Complexes)

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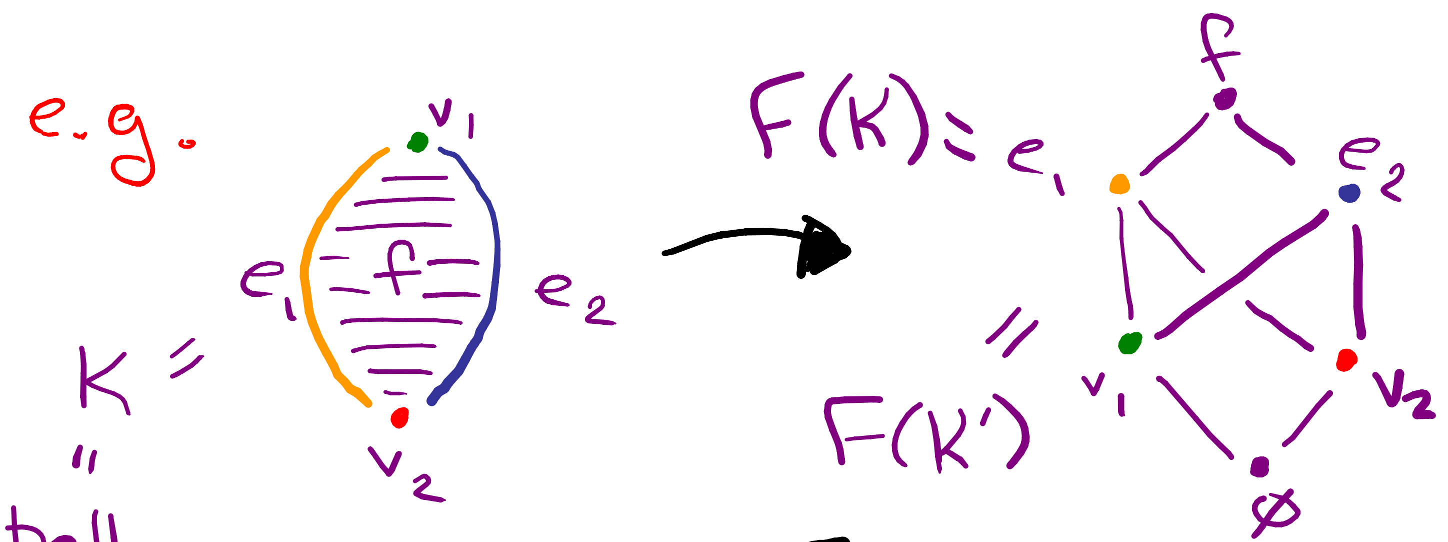
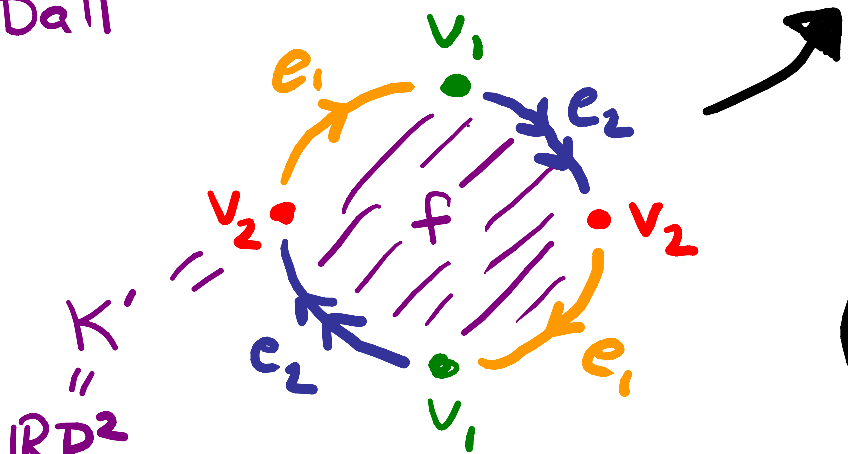
(joint work with Eric Babson)

CW Complexes & their Face Posets





"closure poset"
or "face poset"

$(u \le v \iff u \le \bar{v})$

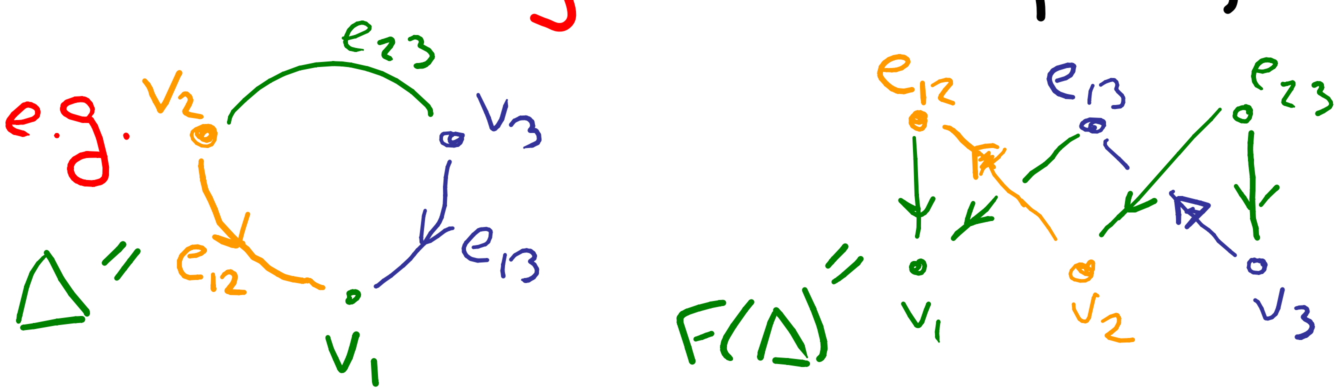
Recall: A **CW complex**: cells $e_\alpha \cong \mathbb{R}^{d(\alpha)}$,
characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{e_\beta \le \bar{e}_\alpha} e_\beta$
& attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

– **regular**: f_α's are homeomorphisms

Face Poset Reformulation of Discrete Morse Theory (M. Chari)

Given any regular CW complex Δ, construct an acyclic matching a.k.a. Morse matching on its face poset, i.e.,

e.g.



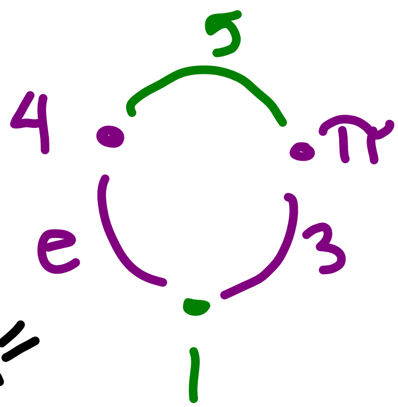
an edge orientation s.t. "up edges" give a matching and directed graph has no cycles.

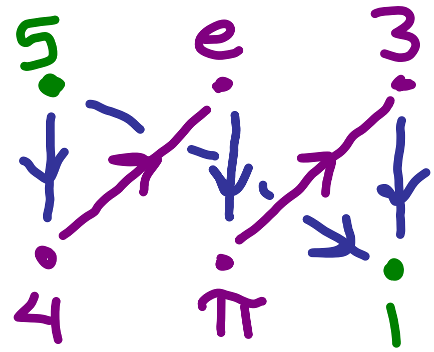
Useful Fact for Proving Acyclicity:

Any directed cycle must alternate "up" & "down" steps

Observations: 1. Discrete Morse fn on Δ induces acyclic matching w/ arrows in direction fn decreases

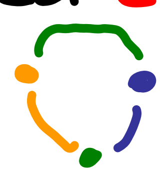
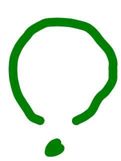
2. Every acyclic matching on face poset is induced by a nonempty set of discrete Morse fn's

$\Delta =$ 

$\rightsquigarrow$ $F(\Delta) \cong$ 

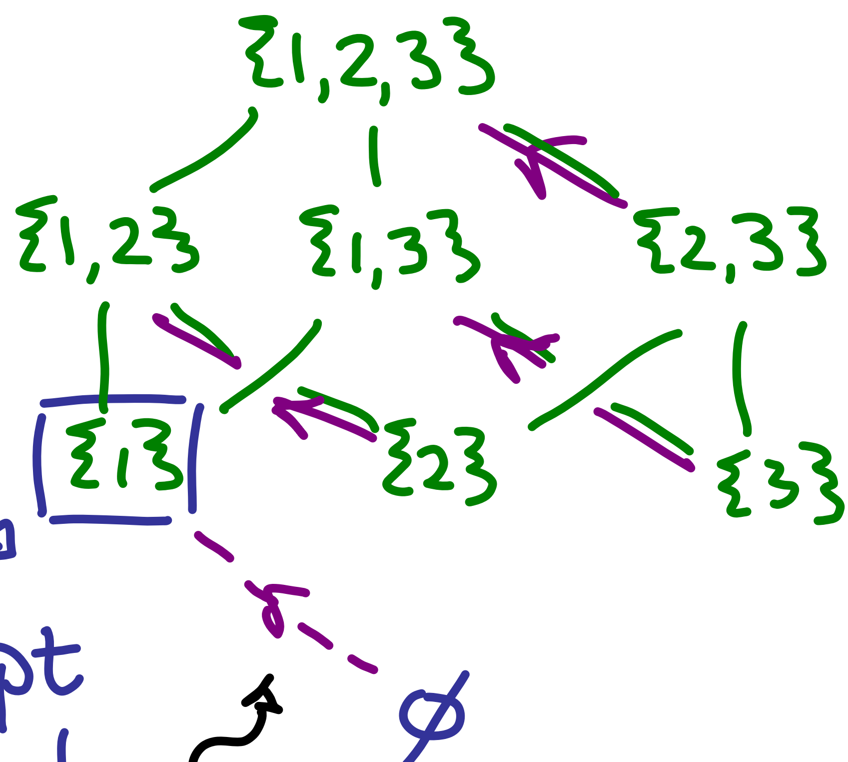
Theorem (Forman): $\Delta \simeq \Delta^M$ a CW complex comprised of the unmatched cells, called critical cells.

critical i-cells $\updownarrow$ critical pts of index i

e.g.  $\simeq$ 

First Examples

1. Boolean algebra of subsets of $\{1,2,\ldots,n\}$, face poset of simplex, matching $S \setminus \{1\}$ with $S \cup \{1\}$ $\forall S$

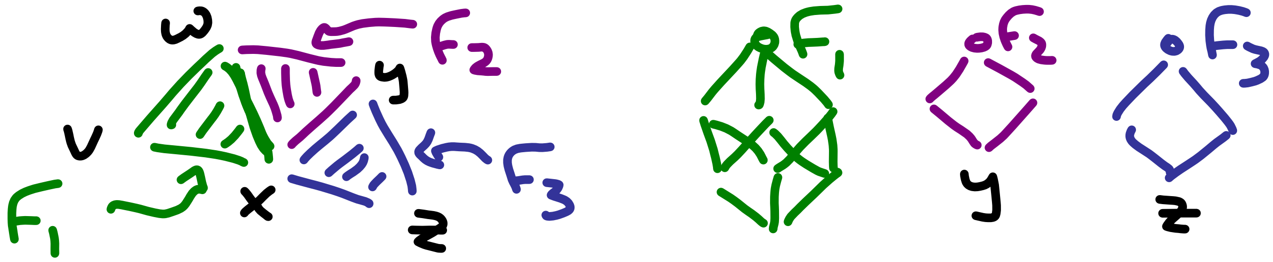


base pt critical 0-cell

matching edge in "reduced homology" version of discrete Morse theory

2. Any union of acyclic matchings on $F(\Delta_2 \setminus \Delta_1), F(\Delta_3 \setminus \Delta_2), \ldots$ for $\Delta_1 \subseteq \Delta_2 \subseteq \ldots \subseteq \Delta_k = \Delta$ a filtration of subcomplexes is an acyclic matching for Δ

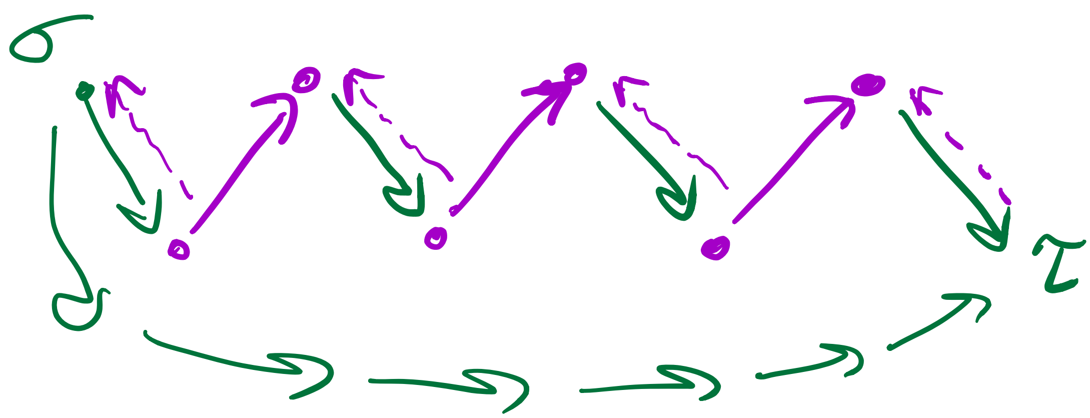
e.g. $\overline{F_1} \subseteq \overline{F_1} \cup \overline{F_2} \subseteq \overline{F_1} \cup \overline{F_2} \cup \overline{F_3}$



3. Shelling $\Rightarrow$ Discrete Morse fn with homology facets as critical cells

(using a 2nd definition of shelling as total order $F_1, F_2, \ldots, F_k$ s.t. each $\overline{F_j} \setminus (\bigcup_{i<j} \overline{F_i})$ has unique minimal face)

Critical Cell Cancellation via Gradient Path Reversal

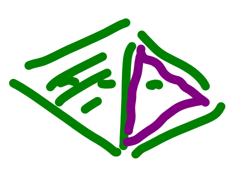


- Critical cells τ & σ may be cancelled if there is unique directed path σ to τ (by reversing path to incorporate endpoints)
- Similar to birth/death of homology classes

Shellability

- simplicial complex is **pure** of dim. d if all maximal faces ("facets") are d-dimensional
- simplicial complex is **shellable** if there is total order $F_1, F_2, \dots, F_k$, a **shelling**, on facets s.t. $\bar{F}_j \cap \left(\bigcup_{i<j} \bar{F}_i\right)$ is pure, codimension one subcomplex of $\bar{F}_j$ for each $j>1$ (hence is $\partial \bar{F}_j$ or has a cone point).

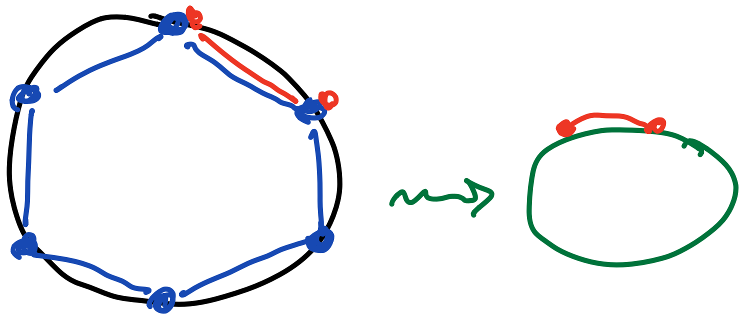
OR



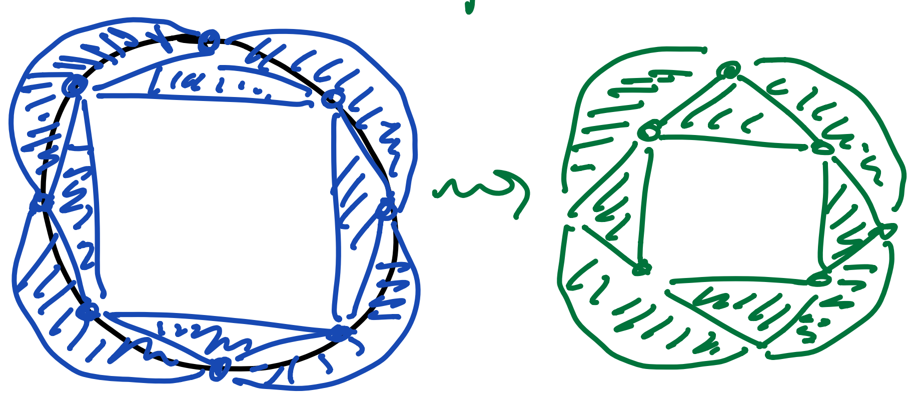
- Each facet attachment preserves homotopy type or closes off a new sphere

Examples from RIPS Complexes

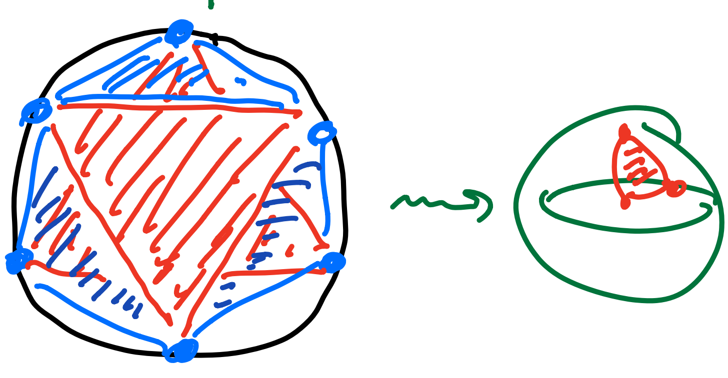
- shellable example



- non-shellable example

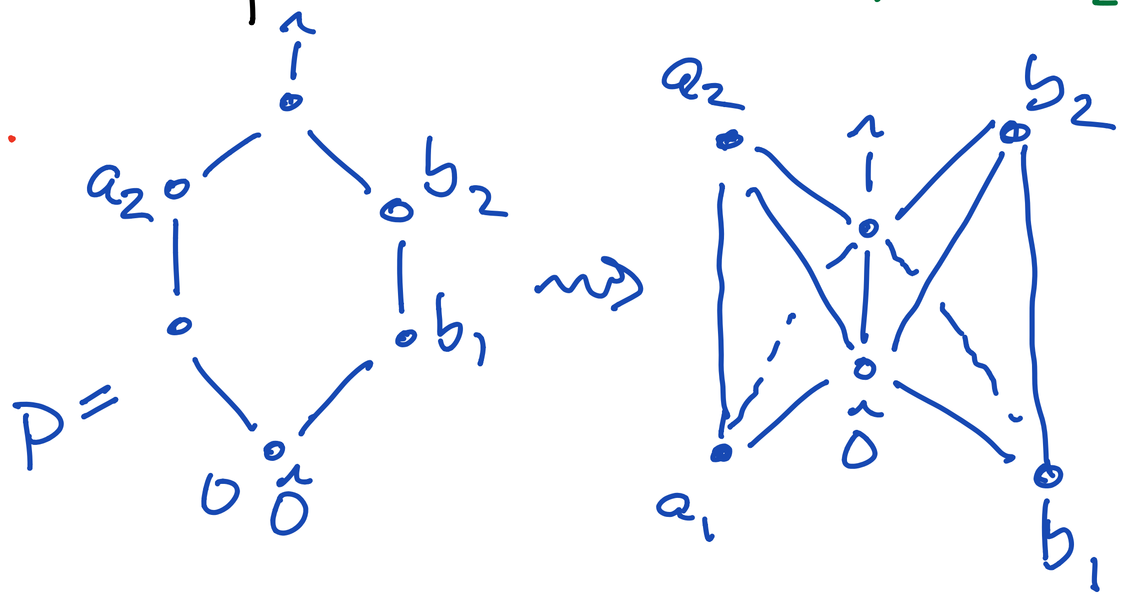


- shellable example



Def'n: The order complex (or nerve) of poset P is the abstract simplicial complex whose i-dim'l faces are the poset $(i+1)$-chains $v_0 < v_1 < \dots < v_i$

e.g.



- $\Delta(F(K)) = sd(K) \cong K$

  (F: face poset; K: simplicial complex)

- shellable $\Rightarrow$ link of each face homotopy equiv to wedge of spheres

Question (H.): Is there a good way to "complete the square":

$$\begin{tikzcd}
\text{lexicographic shelling} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & ?? \arrow[d, Rightarrow] \\
\text{shelling} \arrow[r, Rightarrow] & \text{discrete Morse function}
\end{tikzcd}$$

to understand posets that fail to be shellable (e.g. not wedge of spheres)?

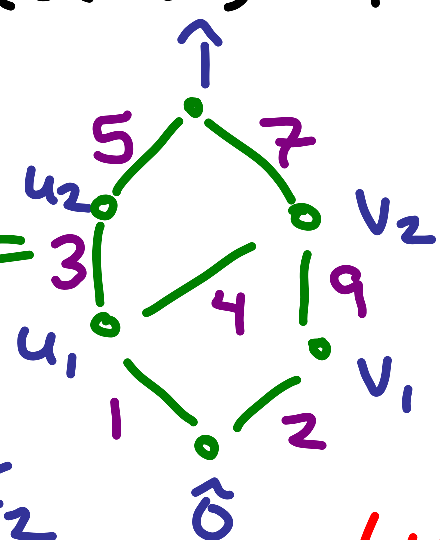
Proposed Answer: "lexicographic discrete Morse functions"

Lexicographic Discrete Morse Functions: A General Construction

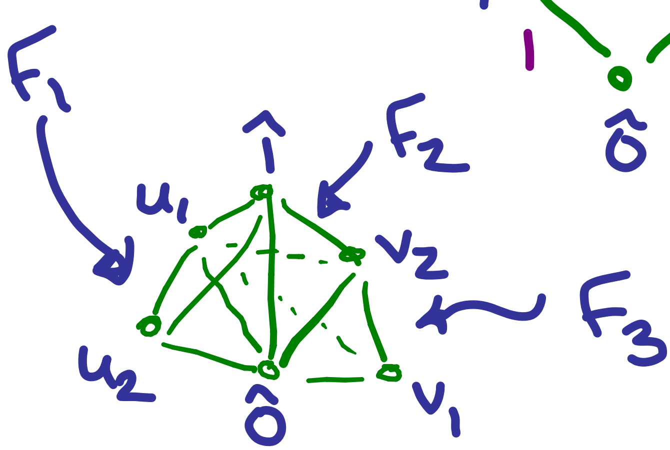
(partly joint work with E. Babson)

Step 1: Any edge labeling on poset P induces lexicographic order $F_1, F_2, \ldots, F_m$ on maximal faces (facets) of $\Delta(P)$

e.g.

$P =$ 

$F_1 = 135$
$F_2 = 147$
$F_3 = 297$

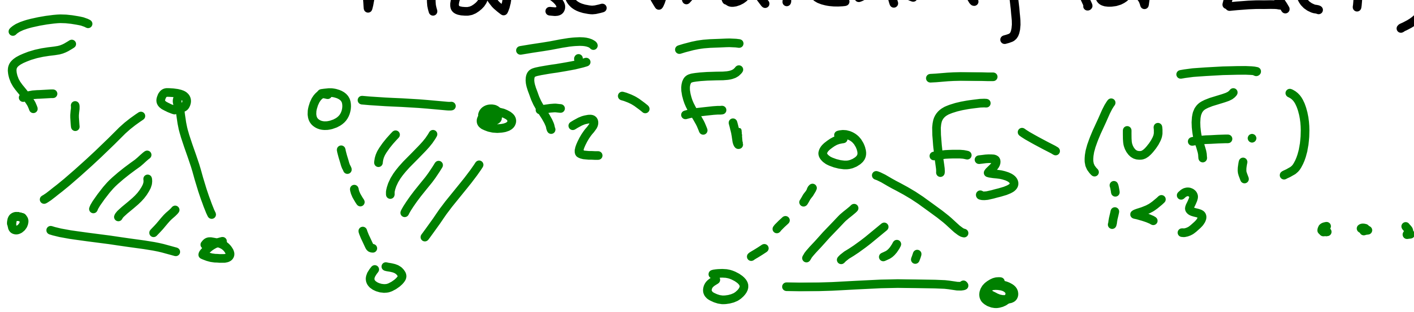


(Usually <u>not</u> EL-labeling!)

Step 2: Morse matching on each $\overline{F}_j \setminus (\cup_{i<j} \overline{F}_i)$ s.t.

(1) Each $\overline{F}_j \setminus (\cup_{i<j} \overline{F}_i)$ has 0 or 1 unmatched (critical) cells

(2) Union of these matchings is Morse matching for $\Delta(P)$

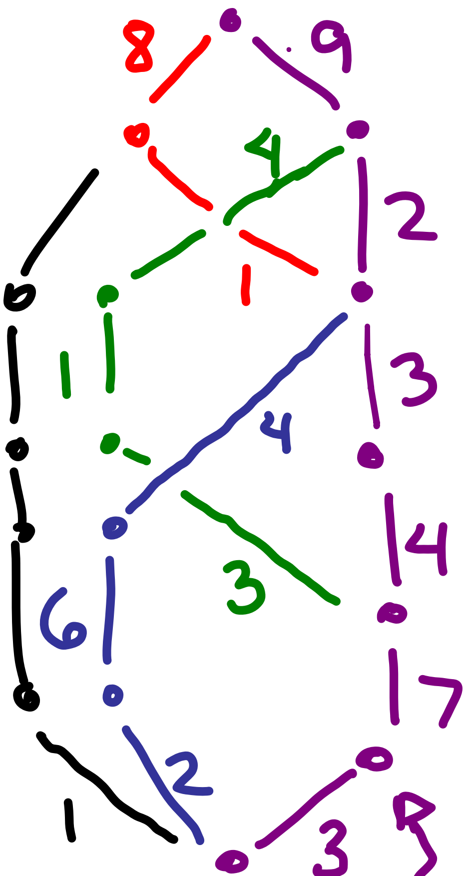


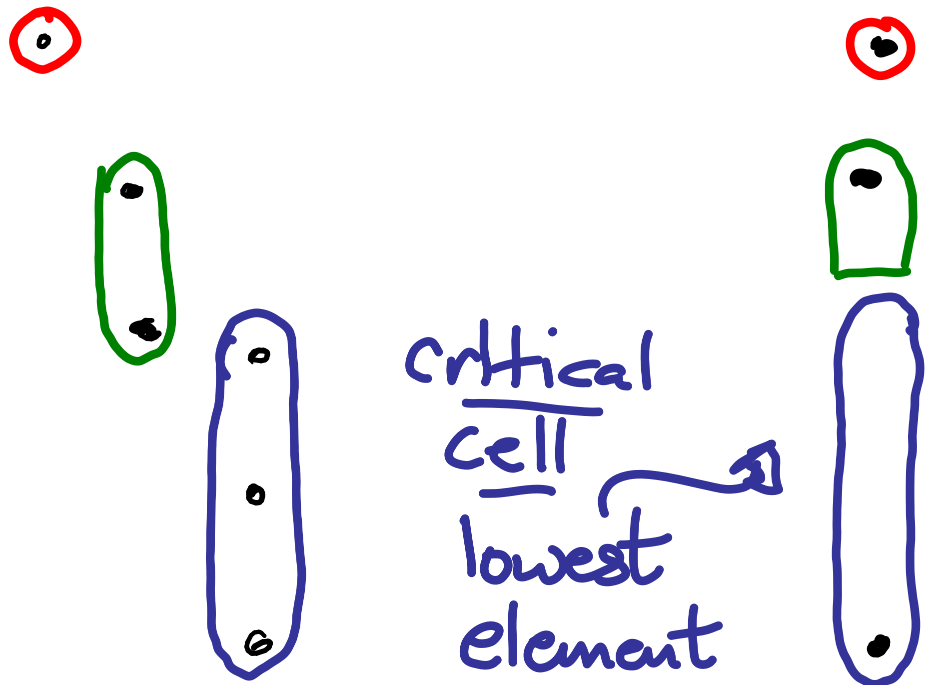
Theorem (Babson-H.) Any edge labeling on any finite poset gives rise to a lexicographic discrete Morse fn s.t. critical cells $\leftrightarrow$ facets whose attachment changes the homotopy type of complex.

Description of Critical Cells

"interval system"

$I \rightsquigarrow J$





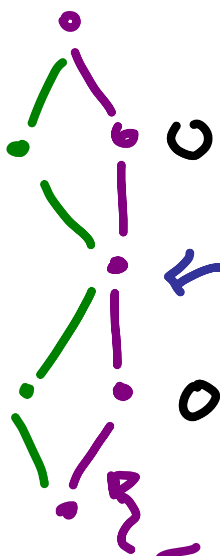
critical cell

lowest element of each (truncated) interval

F_j

- $\left\{ \text{Faces in } \overline{F}_j \setminus \left(\bigcup_{i<j} \overline{F}_i \right) \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{subsets of ranks in} \\ F_j \text{ that "hit" all} \\ \text{intervals in } I\text{-system} \end{array} \right\}$
- No critical cell unless truncated interval system J fully covers F_j

Remarks: (1) Lexicographic shellability is a special case with all intervals in I of size one



F_j

(2) Saturated chain does not contribute critical cell unless fully covered by J-system

match by include/exclude uncovered rank

(3) Critical cell dimension is $|J|-1$, since it consists of $\{i \mid i = \min(j) \text{ for some } j \in J\}$

(4) Upper bd on interval size for all $F_j \Rightarrow$ lower bd on connectivity of $\Delta(\bar{P})$

(5) Match based on uncovered elt <u>or</u> lowest J-interval differing from critical cell

Truncation Algorithm

Start with interval system I & initialize truncated system J to $\emptyset$

e.g. $I = \{[1,2],[2,3],[3,4]\}$ & $J = \emptyset$

Repeatedly: (1) move $\min(I)$ to truncated system J after truncating all other elements of I to eliminate overlap w/ $\min(I)$

(2) throw away elements of I no longer minimal

e.g.

$$I = \left\{\begin{matrix}[1,2]\\ [2,3]\\ [3,4]\end{matrix}\right\} \xrightarrow{(1)} \left\{\begin{matrix}[3]\\ [3,4]\end{matrix}\right\} \xrightarrow{(2)} \left\{[3]\right\} \xrightarrow{(1)\,\&\,(2)} \left\{\quad\right\}$$

$$J = \emptyset \longrightarrow \left\{[1,2]\right\} \longrightarrow \left\{\begin{matrix}[1,2]\\ [3]\end{matrix}\right\}$$

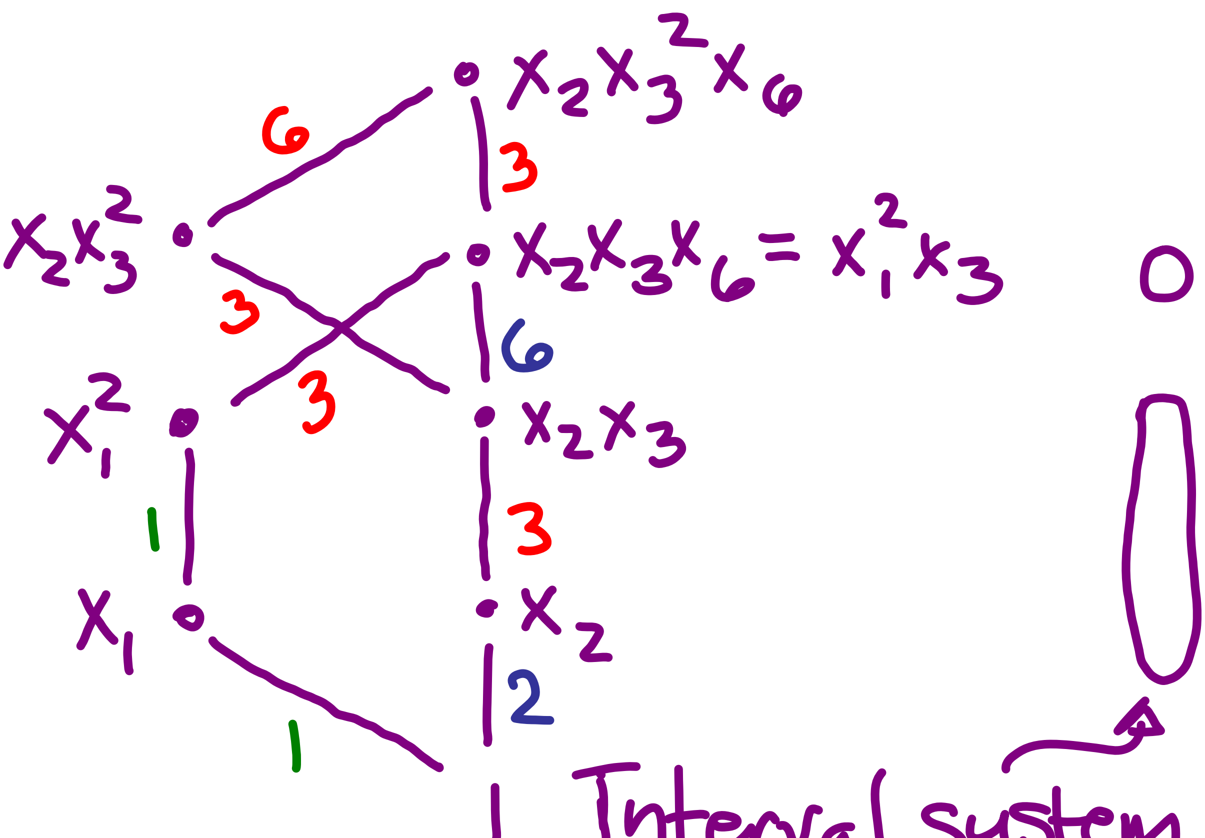
(4 uncovered, so no critical cell)

Natural Labeling & Lexic. Discrete Morse Fn for Monoid Posets

e.g. $k[x_1, x_2, x_3, x_4, x_5, x_6] \big/ (\underline{x_2 x_6} - x_1^2)$

$\cong$

$k[ab, a^2, c, d, e, b^2]$



Interval system given by:

1. "descents" such as $x_6x_3 = x_3x_6$
2. "syzygies" such as $x_3(\underline{x_2x_6} - x_1^2) = 0$