Discrete Morse Theory
for Poset Order Complexes
(及其他抽象紧致性复形)

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(joint work with Eric Babson)
**CW Complexes & Their Face Posets**

**Example:**

- $K$: Ball
- $K'$: Projective Plane ($\mathbb{RP}^2$)

$F(K)$: Faces

$F(K')$: "Closure poset" or "Face poset"

$(u \leq v \iff u \subseteq \overline{v})$

**Recall:** A CW complex: cells $e_d = \mathbb{R}^{d+d}$, characteristic maps $f_d: B^{\dim(e_d)} \rightarrow u \in \partial e_d$

*attaching maps* $f_d|_{\partial B^{\dim(e_d)}}$

- *regular*: $f_d$'s are homeomorphisms
Face Poset Reformulation of Discrete Morse Theory (M. Chari)

Given any regular CW complex $\Delta$, construct an acyclic matching a.k.a. Morse matching on its face poset, i.e.,

an edge orientation s.t. "up edges" give a matching and directed graph has no cycles.

Useful Fact for Proving Acyclicity:

Any directed cycle must alternate "up" & "down" steps
**Observations:**

1. Discrete Morse fn on $\Delta$ induces acyclic matching w/ arrows in direction fn decreases

2. Every acyclic matching on face poset is induced by a nonempty set of discrete Morse fn's

$\Delta^\approx \Delta_M$ a CW complex comprised of the unmatched cells, called critical cells.

E.g., $\circ \approx \bullet$ critical pts of index i
First Examples

1. Boolean algebra of subsets of $\{1,2,\ldots,n\}$, face poset of simplex, matching $S \setminus \Xi \cup \Xi$ with $S \cup \Xi \setminus \Xi$.

Diagram:

- Base pt
- Critical
- O-cell

Matching edge in "reduced homology" version of discrete Morse theory.
2. Any union of acyclic matchings on $F(\Delta_2 \setminus \Delta_1), F(\Delta_3 \setminus \Delta_2), \ldots$, for $\Delta_1 \subseteq \Delta_2 \subseteq \ldots \subseteq \Delta_k = \Delta$ a filtration of subcomplexes is an acyclic matching for $\Delta$

\[ \text{e.g.} \quad \bar{F}_1 \subseteq \bar{F}_1 \cup \bar{F}_2 \subseteq \bar{F}_1 \cup \bar{F}_2 \cup \bar{F}_3 \]

3. Shelling $\Rightarrow$ Discrete Morse fn with homology facets as critical cells

(using a 2nd definition of shelling as total order $F_1, F_2, \ldots, F_k$ s.t. each $F_j \setminus \left( \cup_{i<j} F_i \right)$ has unique minimal face)
Critical cell cancellation via Gradient Path Reversal

- Critical cells $\tau$ and $\sigma$ may be cancelled if there is a unique directed path $\sigma$ to $\tau$ (by reversing path to incorporate endpoints)
- Similar to birth/death of homology classes
Shellability

- Simplicial complex is pure of dim. \(d\) if all maximal faces ("facets") are \(d\)-dimensional.
- Simplicial complex is shellable if there is a total order \(F_1, F_2, \ldots, F_k\), a shelling, on facets s.t. \(\overline{F_j} \cap \bigcup_{i < j} F_i\) is pure, codimension one subcomplex of \(\overline{F_j}\) for each \(j \geq 1\) (hence is \(2\overline{F_j}\) or has a cone point).

- Each facet attachment preserves homotopy type or closes off a new sphere.
Examples from RIPS Complexes

- shellable example

- non-shellable example

- shellable example
**Defn:** The order complex (or nerve) of poset $P$ is the abstract simplicial complex whose $i$-dim' faces are the poset $(i+1)$-chains $v_0 < v_1 < \ldots < v_i$.

*Example:* 

- $\Delta(F(K)) = sd(K) \cong K$
- Shellable $\Rightarrow$ link of each face homotopy equivalent to wedge of spheres
Question (H.): Is there a good way to "complete the square":

lexicographic shelling => ??

↓  ↓  ↓
shelling => discrete Morse function

to understand posets that fail to be shellable (e.g. not wedge of spheres)?

Proposed Answer: "lexicographic discrete Morse functions"
Lexicographic Discrete Morse Functions: A General Construction
(partly joint work with E. Babson)

**Step 1**: Any edge labeling on poset $P$ induces lexicographic order $F_1, F_2, \ldots, F_m$ on maximal faces (facets) of $\Delta(P)$.

*Example:* $P = \{1, 2, 3\}$ with labels $\{5, 4, 7\}$ gives $F_1 = 135$, $F_2 = 147$, $F_3 = 297$.

(Usually not EL-labeling!)
Step 2: Morse matching on each $\overline{F}_j \setminus (u, \overline{F}_i)$ s.t.

1. Each $\overline{F}_j \setminus (u, \overline{F}_i)$ has 0 or 1 unmatched (critical) cells
2. Union of these matchings is Morse matching for $\Delta(P)$

Theorem (Babson-H.) Any edge labeling on any finite poset gives rise to a lexicographic discrete Morse fn $\mathfrak{s}$.t. critical cells $\leftrightarrow$ facets whose attachment changes the homotopy type of complex.
Description of Critical Cells

"interval system"  
\[ I \sim \rightarrow J \]

\[ \text{critical cell} \rightarrow \text{lowest element of each (truncated) interval} \]

\[ \{ \text{Faces in } F_j \} \rightarrow \{ \text{Subsets of ranks in } F_j \text{ that "hit" all intervals in } I\text{-system} \} \]

- \[ \{ F_j \setminus (\cup_{i \in j} \bar{F}_i) \} \]
- No critical cell unless truncated interval system J fully covers F_j
Remarks: (1) Lexicographic shellability is a special case with all intervals in $I$ of size one

(2) Saturated chain does not contribute critical cell unless fully covered by $J$-system

match by include/exclude uncovered rank

(3) Critical cell dimension is $|J|-1$, since it consists of $\{i \mid i = \min(j) \text{ for some } j \in J\}$

(4) Upper bd on interval size for all $F_j \Rightarrow$ lower bd on connectivity of $\Delta(\bar{P})$

(5) Match based on uncovered elt or lowest $J$-interval differing from critical cell
**Truncation Algorithm**

Start with interval system $I$ and initialize truncated system $J$ to $\emptyset$.

*E.g.* $I = \{ [1,2], [2,3], [3,4] \} \uparrow J = \emptyset$

Repetedly: (1) move $\min(I)$ to truncated system $J$ after truncating all other elements of $I$ to eliminate overlap w/ $\min(I)$

(2) throw away elements of $I$ no longer minimal

*E.g.*

$I = \left\{ \begin{array}{c}
[1,2] \\
[2,3] \\
[3,4] \\
\end{array} \right\}$

(1) $\rightarrow \left\{ \begin{array}{c}
[3] \\
[3,4] \\
\end{array} \right\}$

(2) $\rightarrow \left\{ \begin{array}{c}
[3] \\
\end{array} \right\}$

$J = \emptyset \rightarrow \left\{ [1,2] \right\} \rightarrow \left\{ [1,2], [3] \right\}$

(4 uncovered, so no critical cell)
Natural Labeling & Lexic. Discrete Morse Fn for Monoidal Posets

e.g. $R[x_1, x_2, x_3, x_4, x_5, x_6] / (x_2 x_6 - x_1^2)$

$R[a, b, a^2, c, d, e, b^2]$

\[
\begin{align*}
\text{1: Interval system given by:} \\
\text{1. "descents" such as } x_0 x_5 = x_3 x_0 \\
\text{2. "syzgygies" such as } x_3 (x_2 x_6 - x_1^2) = 0
\end{align*}
\]