Background in Topological Combinatorics, Coxeter Groups & Total Positivity

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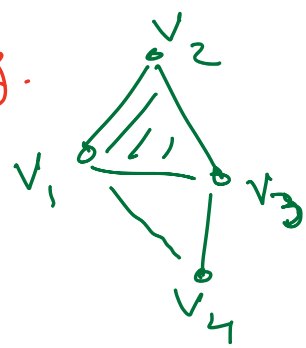
- An abstract simplicial complex is a set V of vertices and a set Δ of subsets of V such that

  (1) $S \in \Delta$ and $T \subseteq S \Rightarrow T \in \Delta$

  (2) $\{v\} \in \Delta \ \forall \ v \in V$

Remark: For each $S \in \Delta$, $\dim(S) := |S| - 1$.

We regard $\emptyset$ as the unique (-1)-dim'l face

e.g. 

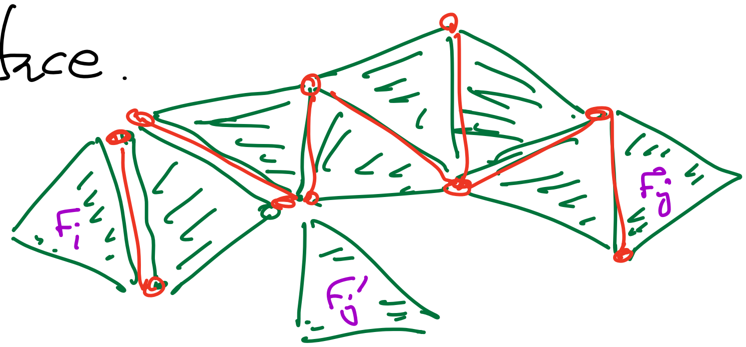
encodes as: $\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \emptyset\}$

- The reduced Euler characteristic of Δ is $\sum_{i \geq -1} (-1)^i \cdot \#\, i\text{-dim'l faces in } \Delta$

e.g. $-1 + 4 - 5 + 1 = -1$

- A simplicial complex is gallery-connected if there is path from any maximal face (i.e. "facet") F_i to any other facet F_j through a series of facets s.t. each consecutive pair share a codimension one face.

e.g.



- F_i to F_j but not F_i to F_j' so not gallery connected

Fact: Shellable (defined next) implies gallery-connected

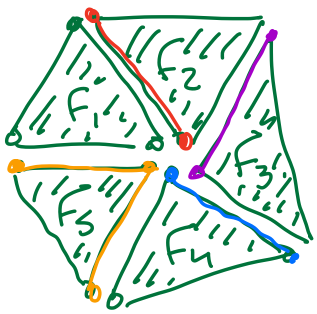
- A simplicial complex is pure of dimension d if all maximal faces are d-dimensional

- A simplicial complex is shellable if there is a total order $F_1, F_2, \ldots, F_k$ on its maximal faces (called "facets") such that

$\overline{F_j} \cap \left(\bigcup_{i<j} \overline{F_i} \right)$ is a pure $\underbrace{(\dim F_j - 1)\text{-dimensional}}_{\text{"codimension one"}}$

subcomplex of $\overline{F_j}$ for each $j \geq 2$.

e.g.



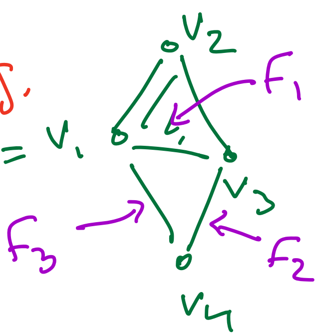
- such orders $F_1, \ldots, F_k$ are called shellings

Key Property: Each $\overline{F_j} \cap \left(\bigcup_{i<j} \overline{F_i} \right)$ is either contractible (due to having "cone point") or equals $\overline{\partial F_j}$. When it is contractible, then attaching $\overline{F_j}$ does not change the homotopy

type of Δ, whereas when it equals ∂F_j then F_j attaches along entire boundary, closing off a sphere.

Upshot: Δ Shellable $\Rightarrow$ homotopy equivalent to wedge of spheres with # i-spheres = # maximal faces that are i-dim'l attaching along entire bdry. Thus,

$$\underbrace{\tilde{\chi}}_{\substack{\text{reduced} \\ \text{Euler characteristic}}}(\Delta) = \sum_{i \geq 0} (-1)^i \cdot \#\, i\text{-spheres in wedge of spheres}$$

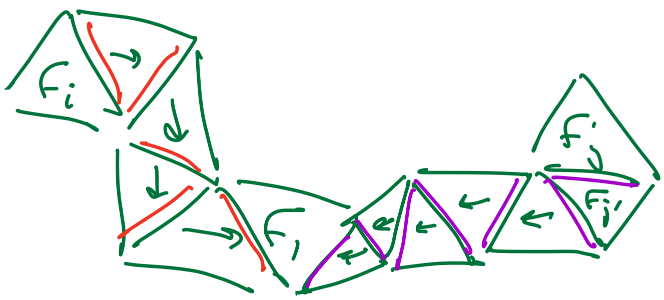
e.g. $\Delta =$ 

is shellable with

$$\tilde{\chi}(\Delta) = (-1)^1 \cdot \underbrace{1}_{\#\,1\text{-spheres}} = -1$$

Qn: Why shellable $\Rightarrow$ gallery-connected?

Ans: Given facets F_i & F_j with $j>1$, there exists codimension one face of F_j shared with earlier facet $F_{j'}$ than F_j in our shelling order. Repeat for $F_{j'}$ if $j'>1$, and keep repeating to get path from F_j to F_1. Likewise get path F_i to F_1.



Next: "Vertex decomposability", a property implying shellability (& gallery-connectedness)

Defn: For vertex $v \in \Delta$, define

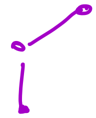
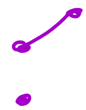
$$lk_\Delta(v) := \{ G \in \Delta \mid G \cup \{v\} \in \Delta, G \cap \{v\} = \emptyset \}$$

"link of v in Δ"

$$del_\Delta(v) := \{ G \in \Delta \mid G \in \Delta, G \cap G = \emptyset \}$$

"vertex deletion of v in Δ"

e.g.

$\Delta =$  $\Longrightarrow$ $\mathrm{del}_\Delta(v) =$  $\mathrm{lk}_\Delta(v) =$ 

- A d-dimensional simplicial complex Δ is **vertex decomposable** if Δ is pure and either
  - (a) Δ is d-simplex, or
  - (b) there exists vertex $v \in \Delta$ such that $\mathrm{link}_\Delta(v)$ & $\mathrm{del}_\Delta(v)$ are vertex decomposable.

Thm (Billera-Provan): Vertex decomposable implies shellable.

Usefulness: if family of simplicial complexes has vertex ordering such that last vertex v in each Δ has $\mathrm{lk}_\Delta v$ & $\mathrm{del}_\Delta v$ both in family (with suitable "base cases"), then all simplicial complexes in family are vertex decomposable.

Key Example: Subword complexes (defined shortly, after reviewing Coxeter groups)
Today

A group W is a **Coxeter group** if it is generated by a set S with relations exactly $s_i^2 = e \ \forall s_i \in S$ and $(s_i s_j)^{m(i,j)} = e$ for some $m(i,j) \in \{2,3,4,\dots,\infty\}$ for each $s_i, s_j \in S$ with $s_i \neq s_j$.

Note: $m(i,j) = \infty$ when no power of $s_i s_j$ equals e

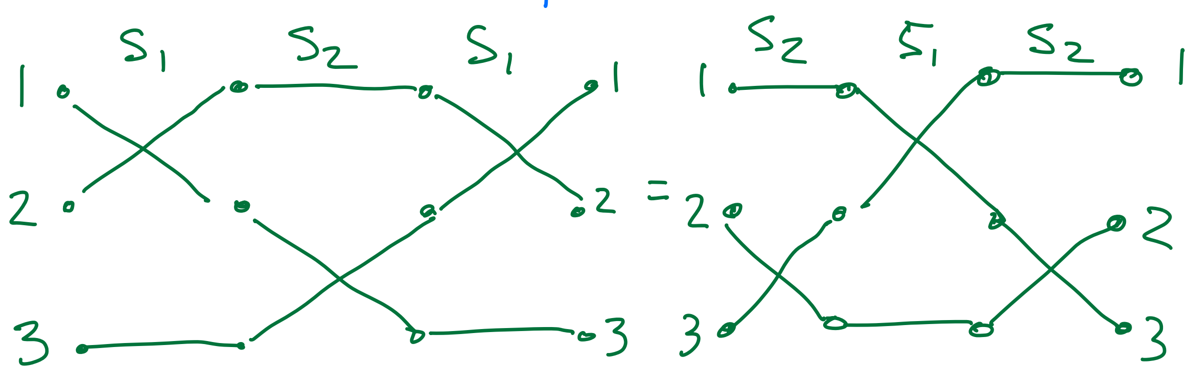
e.g. $W = S_7$ = set of permutations of $1,2,\dots,7$

$$S = \{\underset{s_1}{(1,2)}, \underset{s_2}{(2,3)}, \underset{s_3}{(3,4)}, \underset{s_4}{(4,5)}, \underset{s_5}{(5,6)}, \underset{s_6}{(6,7)}\}$$

$m(1,3) = 2$ since $(s_1 s_3)^2 = s_1 s_3 s_1 s_3 = s_1 s_3^2 s_1 = e$

$m(1,2) = 3$ since $s_1 s_2 s_1 = s_2 s_1 s_2 \Rightarrow (s_1 s_2)^3 = e$

$$\begin{tikzcd}
123 \arrow[r, "s_1"] \arrow[d, "s_2"'] & 213 \arrow[r, "s_2"] & 231 \arrow[r, "s_1"] & 321 \\
132 \arrow[r, "s_1"] & 312 \arrow[urr, "s_2"'] & &
\end{tikzcd} \Rightarrow \quad s_1 s_2 s_1 = s_2 s_1 s_2$$



- The set S of generators is called the set of **simple reflections** of W

- The length of $w \in W$ is the smallest r such that $w = s_{i_1} \cdots s_{i_r}$ for some $s_{i_1}, \ldots, s_{i_r} \in S$.

- A reduced expression for $w \in W$ where $\text{length}(w) = r$ is any product $s_{i_1} \cdots s_{i_r}$ of exactly r simple reflections with $s_{i_1} \cdots s_{i_r} = w$

  e.g. $w = 321$ = reverse permutation in S_3
  has reduced expressions $s_1 s_2 s_1$ and
  $s_2 s_1 s_2$ since $\text{length}(w) = 3$

- An expression $s_{i_1} \cdots s_{i_t}$ for w is a nonreduced expression for w if $w = s_{i_1} \cdots s_{i_t}$ but $t > \text{length}(w)$.

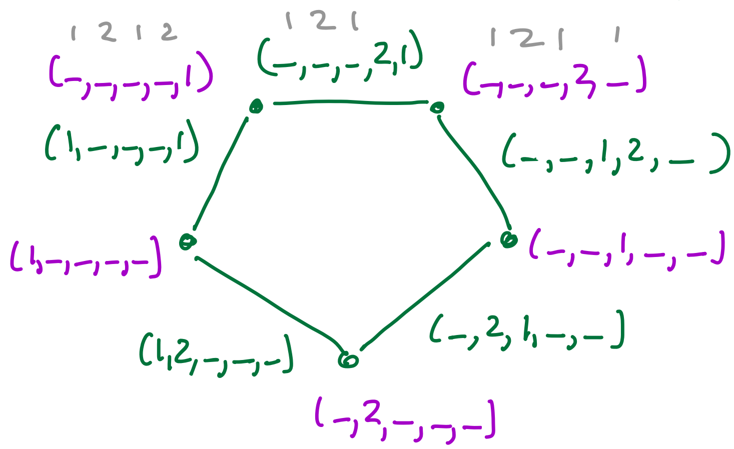
  e.g. $s_1 s_1 s_1$ is nonreduced expression for
  s_1 since $s_1 = s_1^3$ but $\text{length}(s_1) = 1 < 3$

- Any (reduced) expression $s_{i_1} s_{i_2} \cdots s_{i_d}$ can be written more compactly as (reduced) word $(i_1, i_2, \ldots, i_d)$
  e.g. $(1,2,1)$ for $s_1 s_2 s_1$

Def'n (Knutson-Miller): Given Coxeter group (W,S), element $w \in W$, and word Q, the **subword complex** given by Q and w is:

$$\Delta(Q,w) = \{ \underset{\substack{\uparrow \\ \text{subword}}}{Q' \subseteq Q} \mid Q \setminus Q' \text{ contains a reduced word for } w \}$$

e.g. $Q = (1,2,1,2,1)$ with $w = s_1 s_2 s_1 \in S_3$



Thm (Knutson-Miller): $\Delta(Q,w)$ is vertex decomposable, hence shellable (& hence gallery connected)

Idea: Order vertices $v_1, v_2, \ldots, v_r$ by position

e.g. $(1,-,-,-,-) < (-,2,-,-,-) < \ldots < (\underbrace{-,-,-,-,}_{\text{last}}1)$

Show $\mathrm{del}_{\Delta(Q,w)}(v_r)$ & $\mathrm{lk}_{\Delta(Q,w)}(v_r)$ are smaller subword complexes, so in same family of complexes.

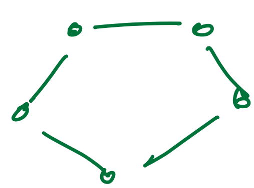
Note: first arose as "Stanley-Reisner ideals" of initial ideals of coordinate rings of matrix Schubert varieties (which helped Knutson-Miller prove these varieties were "Cohen-Macaulay")

Thm (Knutson-Miller) If $\underbrace{\delta(Q)}_{\text{defined next}}=w$, then $\Delta(Q,w)$ is homeomorphic to a sphere.

If $\delta(Q)\neq w$, then $\Delta(Q,w)$ is homeomorphic to a closed ball.

e.g. $Q=(1,2,1,2,1)$ $\quad \delta(Q)=w$

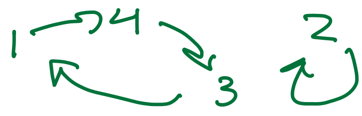
$w=s_1s_2s_2$

$\Delta(Q,w)=$  $\cong S^1 = $ 1-dimensional sphere

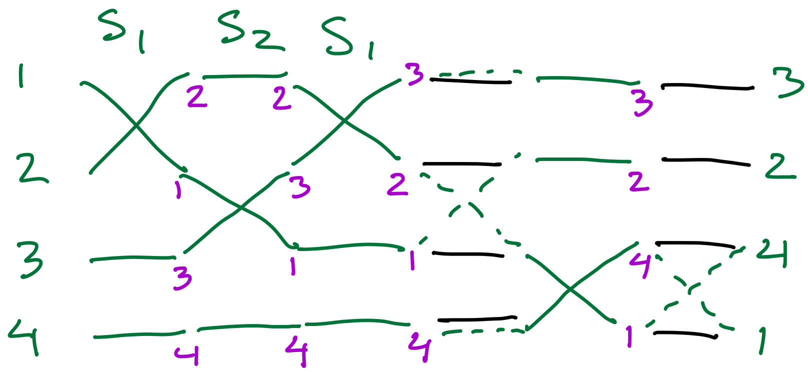
Proof: follows from shellability of subword complexes + property of being "pseudomanifold", namely each codim one face contained in at most two max'l faces

- The Demazure product $\delta(s_{i_1},\dots,s_{i_t})$ is product which omits any steps which decrease length

e.g. $\delta(s_1, s_2, s_1, s_2, s_3, s_3)$

$$= s_1 s_2 s_1 \_ s_3 \_ =$$ 

omitted steps



Formally: • $\delta(s_i) = s_i$

• $\delta(w, s_i) = \begin{cases} w & \text{if } \ell(ws_i) < \ell(w) \\ ws_i & \text{if } \ell(ws_i) > \ell(w) \end{cases}$

• $\delta(s_{i_1},\dots,s_{i_j}) = \delta(\delta(s_{i_1} \dots s_{i_{j-1}}), s_{i_j})$

Useful fact: If $s_{i_1} \dots s_{i_d}$ is reduced expression, then $\delta(s_{i_1},\dots,s_{i_d}) = s_{i_1} \dots s_{i_d}$

Matsumoto Theorem: Any two reduced expressions for same $w \in W$ are connected by a series of commutation moves & braid moves

Any expression for $w \in W$ can be converted to a reduced expression for w by a series of commutation moves, braid moves and "nil-moves" $s_i^2 \to e$.

e.g. $s_1 s_2 s_1 s_2 \underbrace{s_3 s_3}_{e} \xrightarrow[\text{nil}]{} \underbrace{s_1 s_2 s_1}\, s_2$

$\xrightarrow[\text{braid}]{} s_2 s_1 \underbrace{s_2\ s_2}$

$\xrightarrow[\text{nil}]{} s_2 s_1$

Demazure Product Variation: Any sequence $s_{i_1}, \dots, s_{i_r}$ of simple reflections can be converted to one with the same Demazure product that is reduced expression by series of braid moves, commutation moves and "modified nil-moves" $s_i s_i \to s_i$

e.g. $\delta(s_1,s_2,s_1,s_2,\underbrace{s_3,s_3}_{s_3}) = \delta(\underbrace{s_1,s_2,s_1}_{},s_2,s_3)$

$= \delta(s_2,s_1,\underbrace{s_2,s_2}_{s_2},s_3)$

$= \delta(\underbrace{s_2,s_1,s_2,s_3}_{\text{reduced}})$

$= s_2s_1s_2s_3$

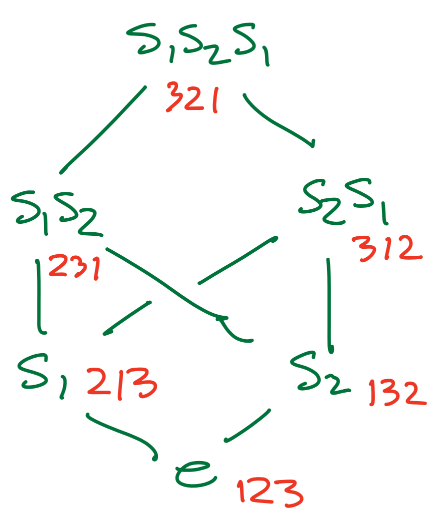
Bruhat Order on Coxeter Group W: Given any $u,v\in W$, we say $u\leq_{Br} v$ if and only if every reduced expression for v has a subexpression that is reduced expression for u

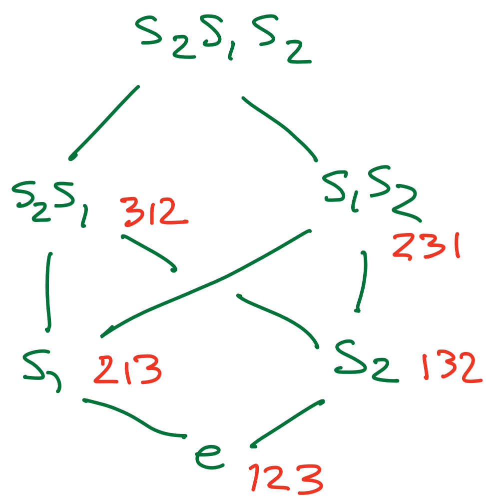
e.g.

$W=S_3$

$v=s_1s_2s_1$

$=s_2s_1s_2$

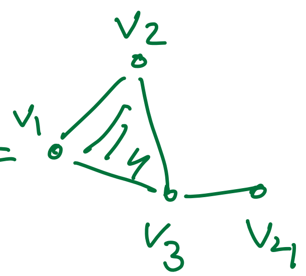


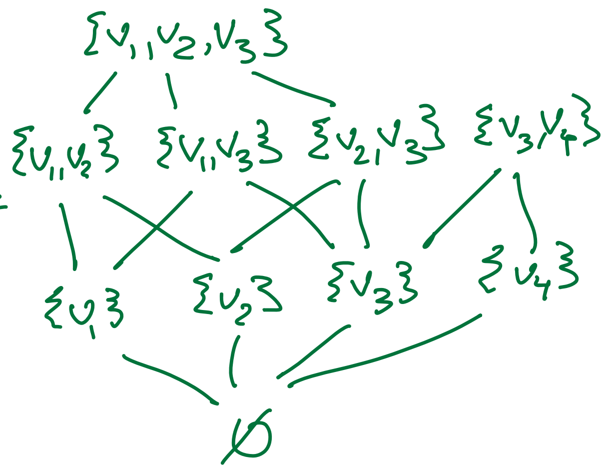


$u\leq_{Br} s_1s_2s_1$ for all $u\in S_3$ $\quad s_1\leq s_1s_2 \quad s_1s_2\not\leq s_2s_1$

The face poset $F(\Delta)$ of simplicial complex Δ is the partial order on faces with $f_1 \leq f_2 \iff f_1 \subseteq f_2$

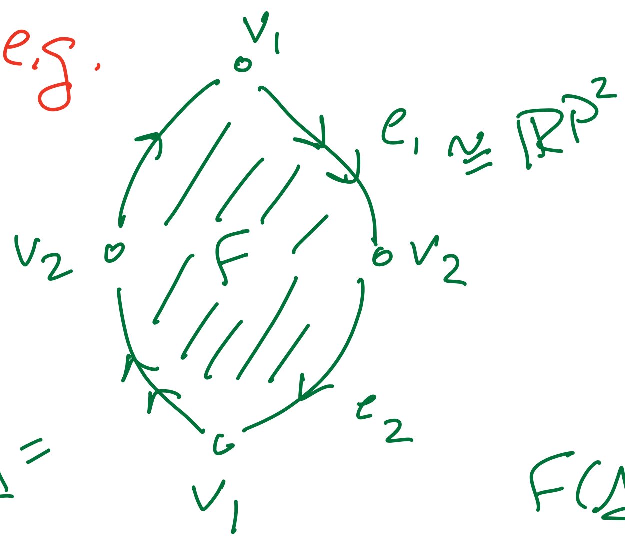
e.g.

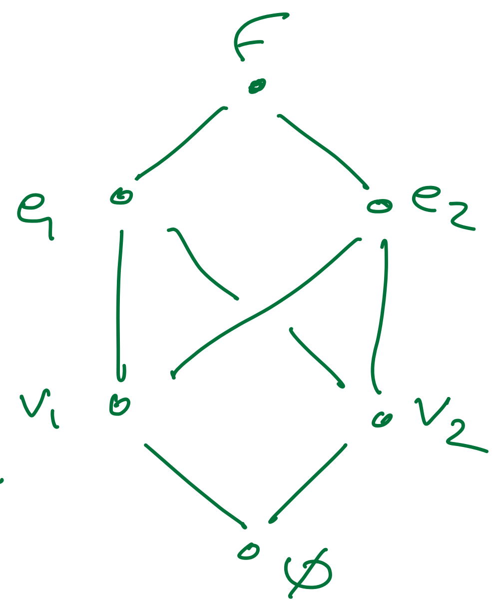
$\Delta =$ 

$F(\Delta) =$ 

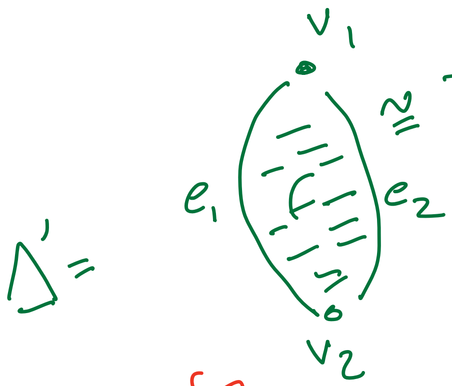
More generally, the face poset of stratified space (e.g. "regular CW complex") is partial order on strata (e.g. cells) with $\sigma \leq \tau \iff \sigma \subseteq \overline{\tau}$

e.g.

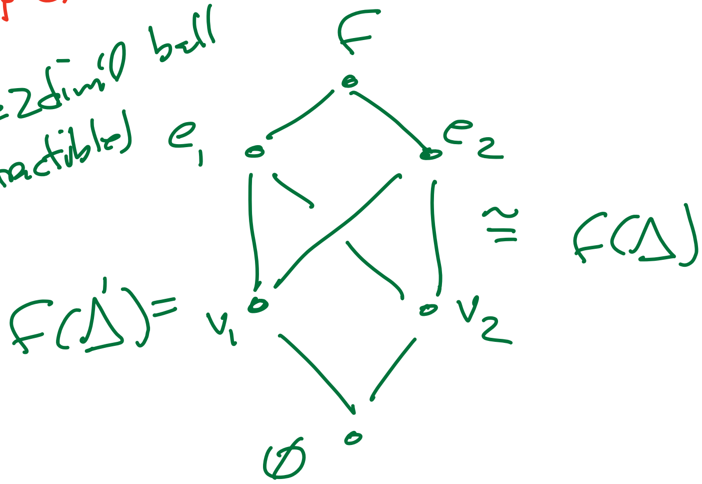
$\Delta =$ 

$F(\Delta) =$ 

- Above example is CW complex but not regular CW complex



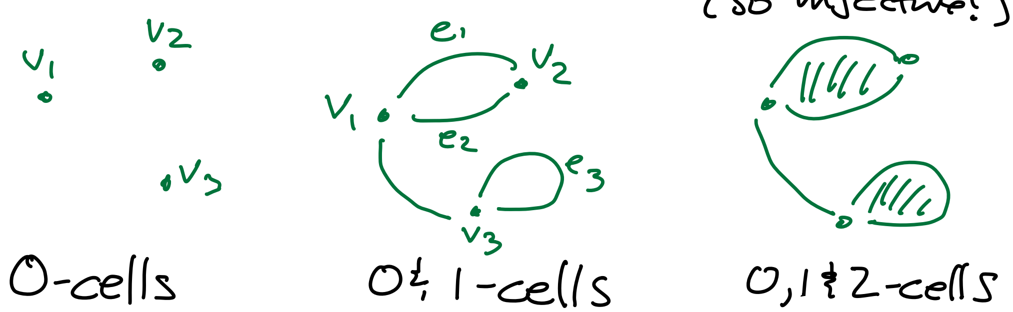
$\cong B^2 = 2$ dim'l ball (contractible)



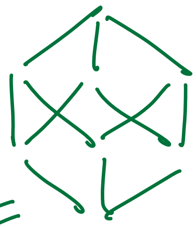
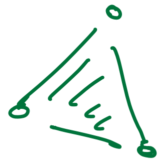
example of "regular" CW complex

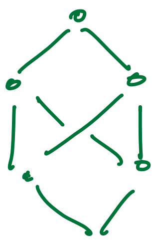
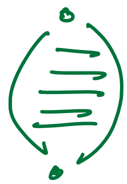
- Roughly speaking, K is regular CW complex if built by attaching "m-cells" (pieces homeomorphic to $(0,1)^m$) for $m=0$, then $m=1$, then $m=2$, etc. with each m-cell glued to union of lower cells via continuous map called "attaching map" that is homeom. (so injective!)

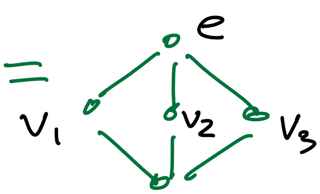
e.g.



A face poset is a **CW poset** if it is the face poset of a regular CW complex,

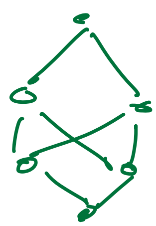
e.g. $P =$  face poset for 

$P =$  face poset for 

$P =$ 

not CW poset (would need edge with 3 vertices)

<u>Note</u>: CW poset can <u>also</u> be face poset for other stratified space

e.g.  for 

Total Positivity

- A real matrix is *totally positive* if all its minors are positive. It is *totally nonnegative* if all minors are nonnegative.

  e.g. $\begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ is totally nonnegative since

  all entries (1×1 minors) are nonnegative, 2×2 minors (e.g. $\begin{vmatrix} 3 & 7 \\ 1 & 4 \end{vmatrix} = 5$ and $\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$) are nonnegative and 3×3 minor is 1, so nonneg.

- Any product of totally nonnegative matrices is totally nonneg. Each minor in $A\cdot B$ is a sum of products of minors of A & B

  e.g. $\begin{pmatrix} 1 & 3 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & 5 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 7 & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3+7 & 3\cdot5 \\ & 1 & 5 \\ & & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 12 & 35 \\ & 1 & 5 \\ & & 1 \end{pmatrix}$$

Today: Spaces of totally nonneg. matrices, stratified based on which minors strictly positive

Maps to Space of Totally Nonnegative Matrices

Given any reduced word $(i_1, i_2, \ldots, i_d)$ for $\pi \in S_n$,
define map $f_{(i_1,i_2,\ldots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow \underbrace{(M_{n\times n})_{\geq 0}}_{\text{totally nonnegative } n\times n \text{ matrices}}$

$$(t_1, \ldots, t_d) \longmapsto x_{i_1}(t_1) \cdots x_{i_d}(t_d)$$

where
$$x_i(t) = \begin{array}{cc} & \text{column } i+1 \\ \text{row } i & \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & t & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \end{array} = I_n + t\,E_{i,i+1}$$

e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = \underbrace{\begin{pmatrix} 1 & t_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{x_1(t_1)} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}}_{x_2(t_2)} \underbrace{\begin{pmatrix} 1 & t_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{x_1(t_3)}$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_2t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Maps arise in work e.g. of Lusztig on dual canonical bases.
We'll discuss a stratification of the fibers, heavily relying upon subword complexes & Demazure product to describe the structure.