On Counting by Inclusion-Exclusion: Möbius Functions, Shellability & Discrete Morse Theory

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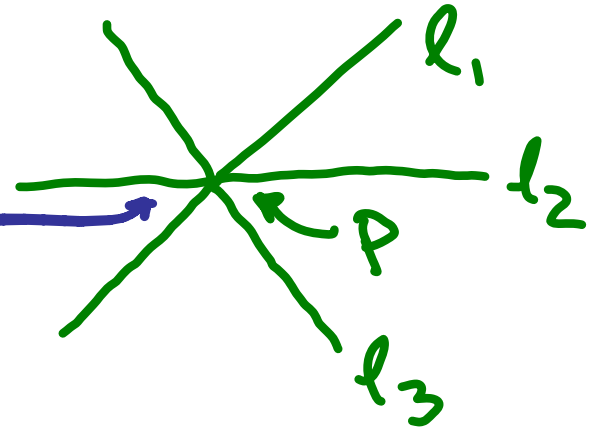
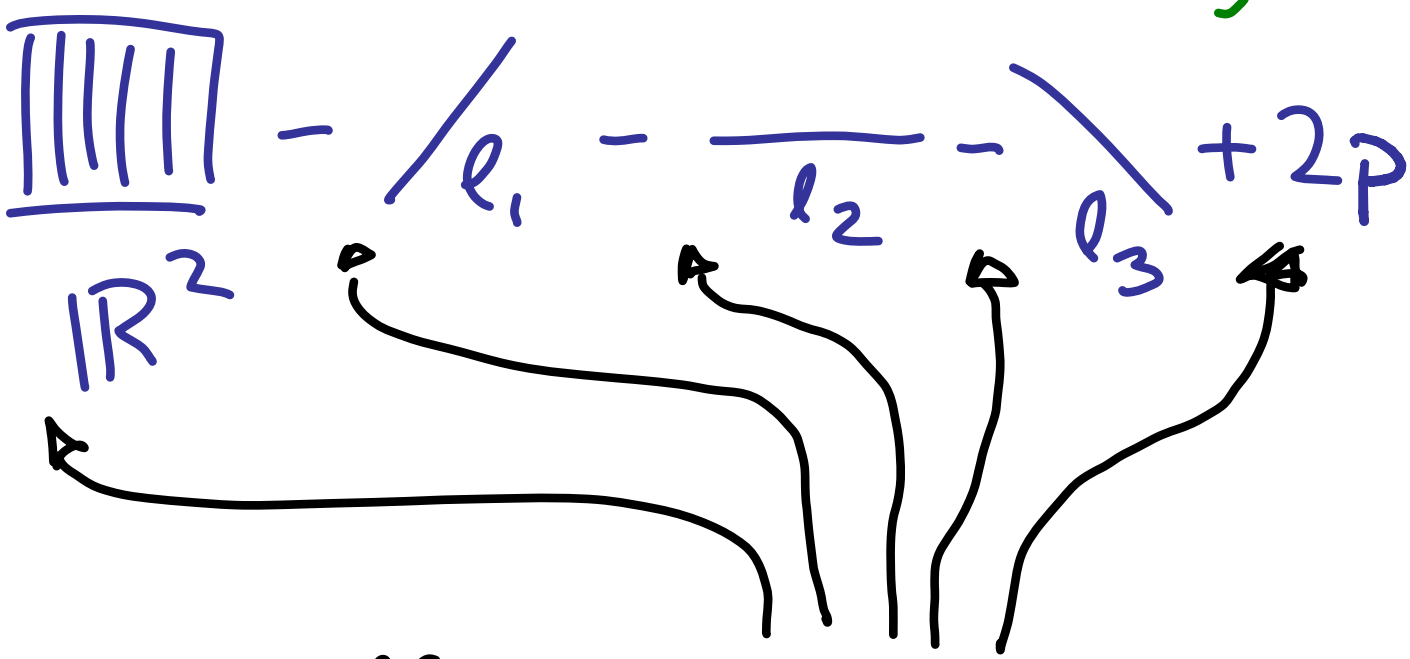
Counting by Inclusion-Exclusion

e.g. "counting" points in the $\mathbb{R}^2$ complement of $\rightsquigarrow$



yields: counted $1-1-1-1+2=0$ times



- Coefficients $1,-1,-1,-1,2$ in such inclusion-exclusion counting formula given by "Möbius function" μ (upcoming)

Calculating these coefficients with Möbius function "μ"

- we'll want:

$$\mu(\mathbb{R}^2, \mathbb{R}^2) = 1 = \text{coef. of } \mathbb{R}^2$$

$$\mu(\mathbb{R}^2, \ell_1) = -1 = \text{coef. of } \ell_1$$

$$\mu(\mathbb{R}^2, \ell_2) = -1 = \text{coef. of } \ell_2$$

$$\mu(\mathbb{R}^2, \ell_3) = -1 = \text{coef. of } \ell_3$$

$$\mu(\mathbb{R}^2, p) = 2 = \text{coef. of } p$$

- how to achieve this:

$$\mu(\mathbb{R}^2, u) = -\sum_{v \text{ contains } u} \mu(\mathbb{R}^2, v)$$

& more generally....

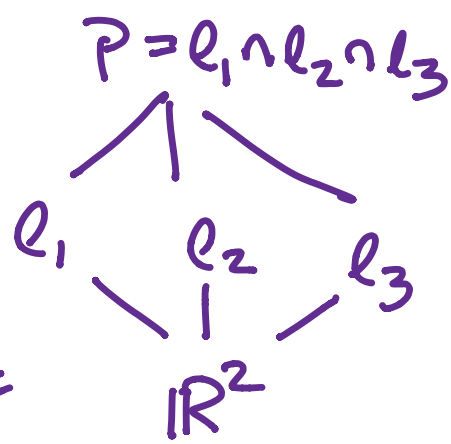
Defn: Given any finite partially ordered set (poset) P, recursively define "Möbius function" $\mu : P \times P \to \mathbb{Z}$ by:

$$\mu_P(u,u) = 1$$

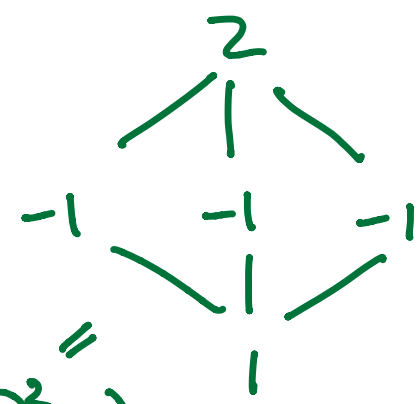
$$\mu_P(u,v) = -\sum_{u \le z < v} \mu_P(u,z)$$

e.g.

$P =$



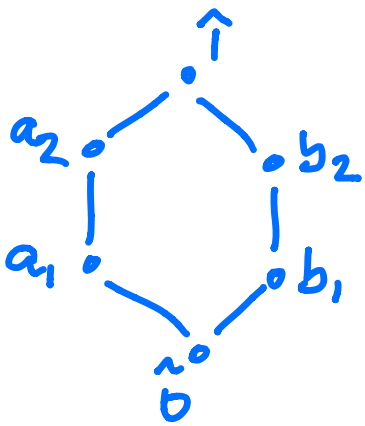
$=$

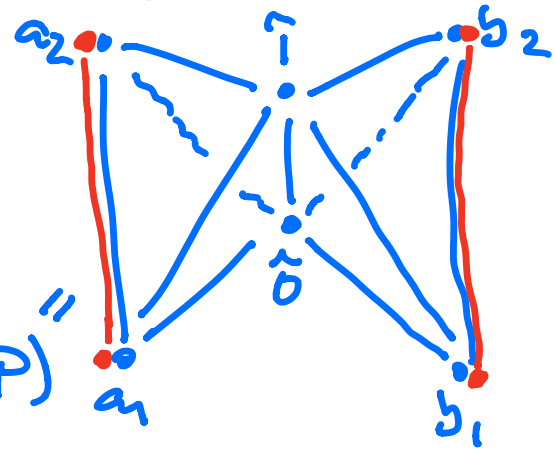


$= \mu_P(\mathbb{R}^2, -)$

Def'n: The order complex of poset P is the abstract simplicial complex, denoted $\Delta(P)$, whose i-dim'l faces are the $(i+1)$-chains $u_0 < u_1 < \cdots < u_i$ in P.

e.g.

$P =$ 

$\Delta(P) =$ 

Key Property (due to Hall & popularized by Rota):

$$\mu_P(x,y) = \tilde{\chi}(\Delta_P(x,y)) = -1 + \#\,0\text{-chains} - \#\,1\text{-chains} + \cdots$$

$$= -1 + \beta_0 - \beta_1 + \underbrace{\beta_2}_{\#\,2\text{-dim'l hole bdry's}} - \cdots$$

$(x,y) = \{z \in P \mid x < z < y\}$

$\Delta_P(x,y) = \Delta(\{z \in P \mid z \in (x,y)\})$

(Reduced) Euler Characteristic

- The **reduced Euler characteristic** of K, denoted $\tilde{\chi}(K) = -1 + \#\text{vertices} - \#\text{edges} + \#\text{triangles} - \ldots$

e.g. 
$\tilde{\chi}(\ ) = -1+4-6+4 = 1$

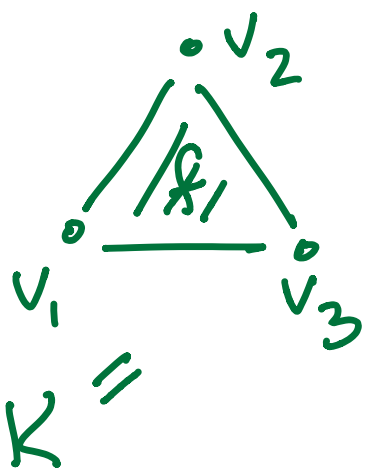

$\tilde{\chi}(\ ) = -1+5-9+6 = 1$

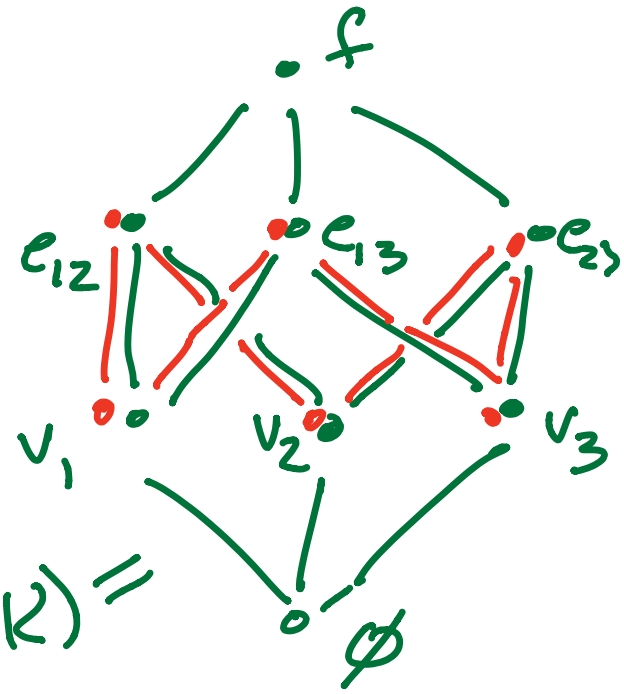
$\Big\}\ \tilde{\chi}(\text{2-sphere})$

Adding faces without changing "Topology" won't change $\tilde{\chi}$!

$$\tilde{\chi} = \underbrace{-1+3-3+1}_{=0} = \underbrace{-1+4-5+2}_{=0} = \underbrace{-1+5-8+4}_{=0}$$

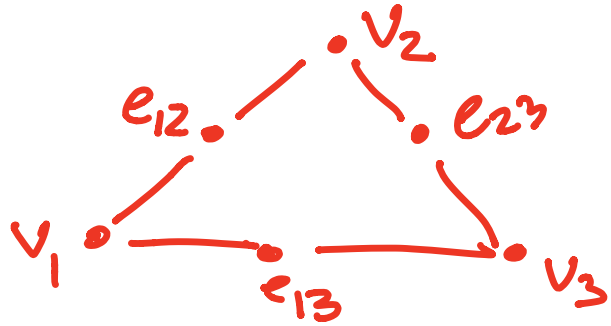
"Topological Proof" of Möbius Function for Poset of Subsets





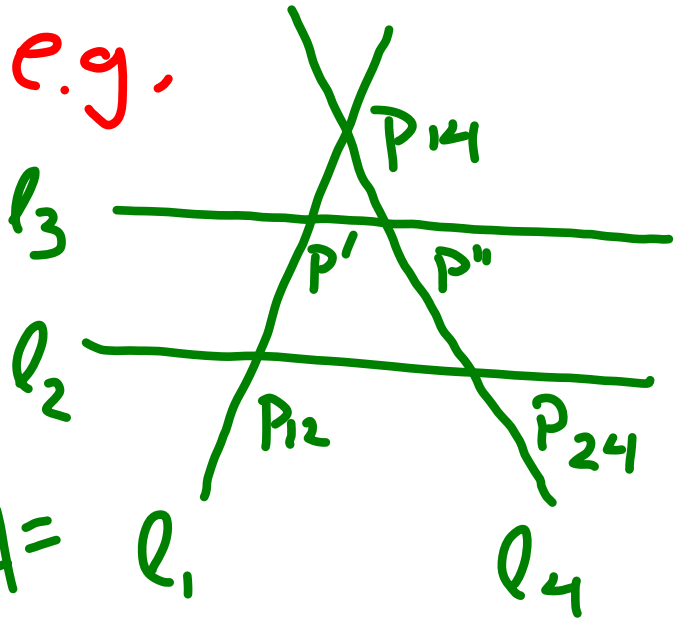
poset of subsets of $\{v_1, v_2, v_3\}$ $= F(K) =$

$$S^1 \cong sd(\partial K) = \Delta(\emptyset, f) =$$

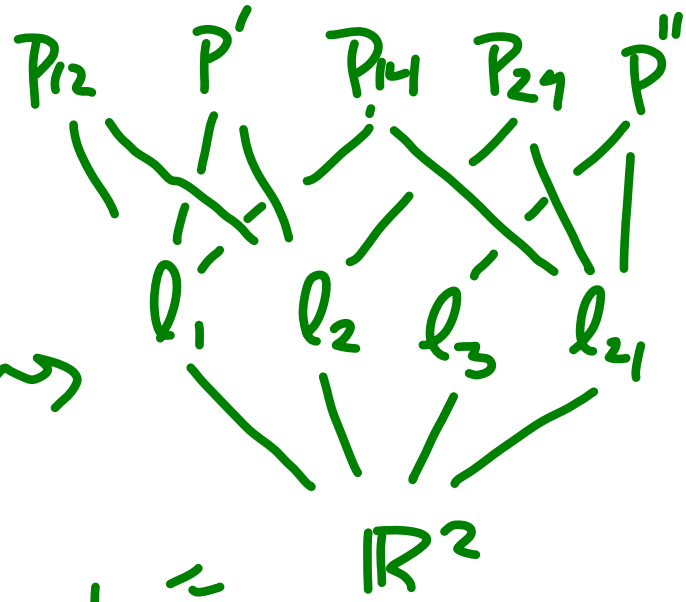


Intersection Posets

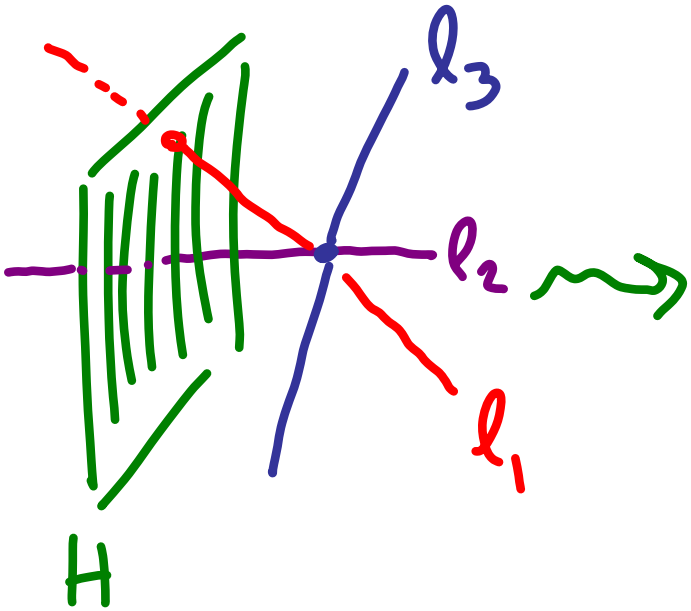
e.g.



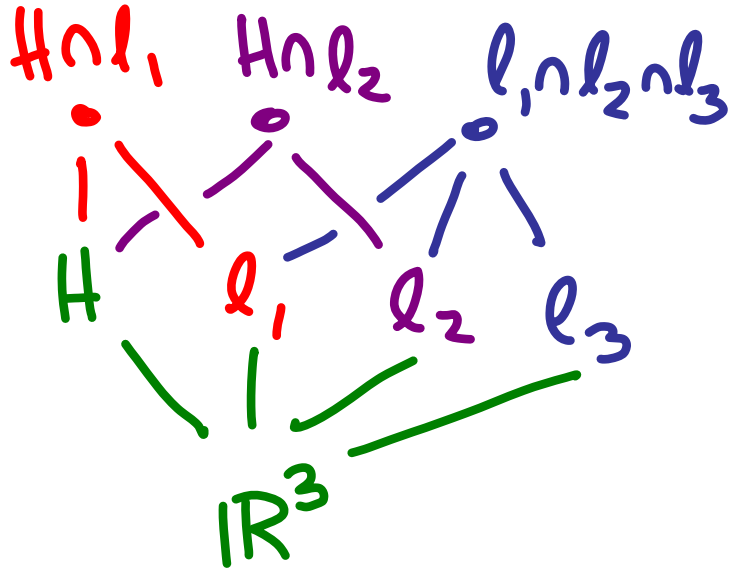
$\rightsquigarrow$



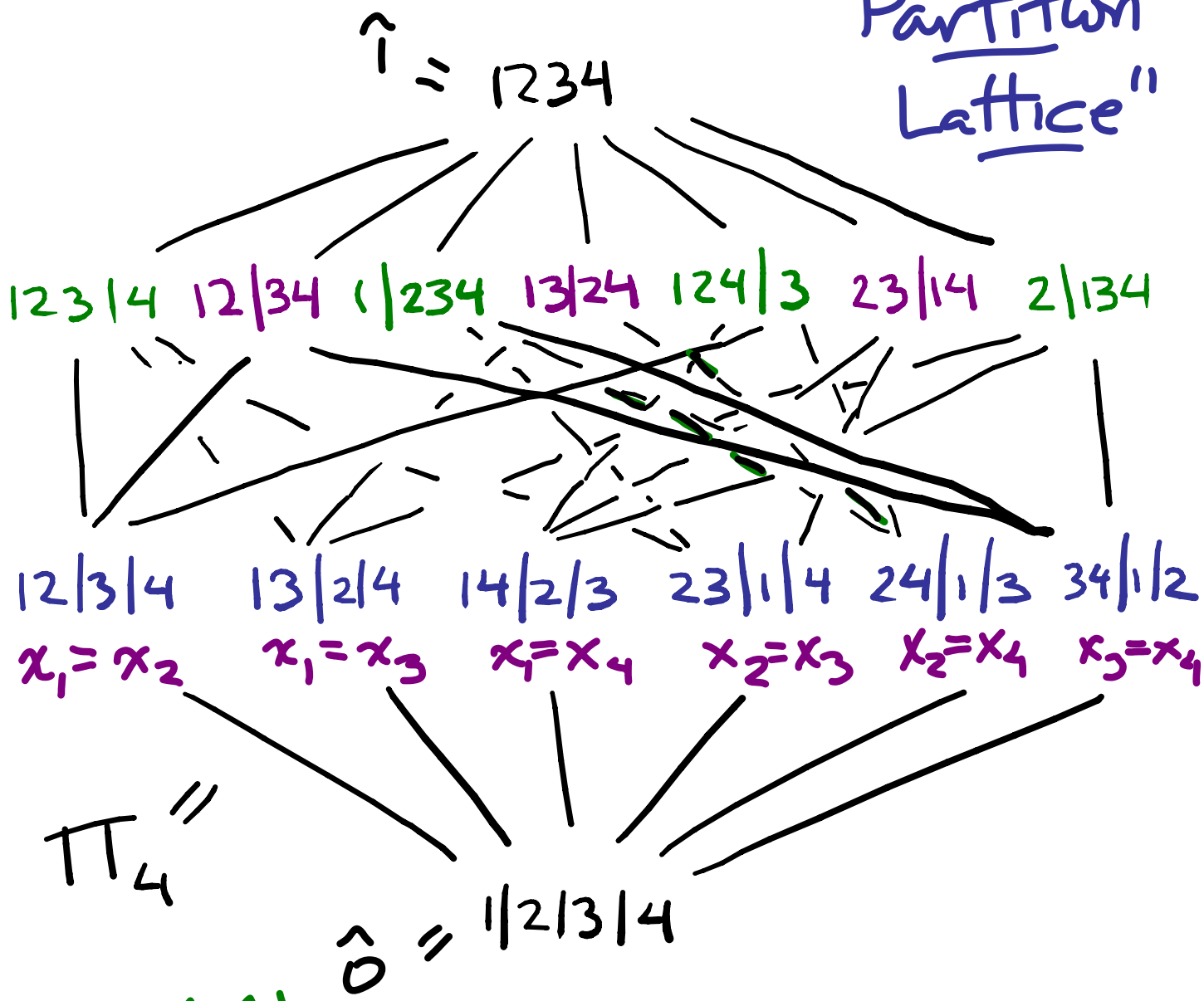
$L_A =$ (intersection poset)



$\rightsquigarrow$



Intersection Poset L_A for $A = \{x_i = x_j \mid 1 \le i < j \le n\}$ the "Partition Lattice"



$\Pi_4 =$

$\mu_{\Pi_4}(\hat{0}, \hat{1}) = -6$

Some Applications of Möbius Functions & "Shellability"

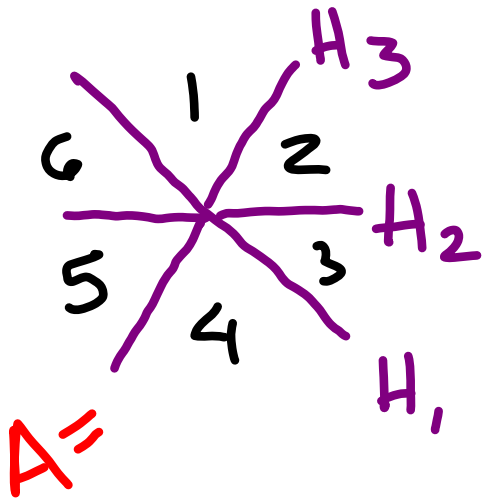
1. Shellability of intersection posets of hyperplane arrangements due to shellability of "geometric lattices"

(Anders Björner) & "geometric semilattices"

(Michelle Wachs, James Walker), yielding Möbius fns of "intersection posets" of hyperplane arrangements

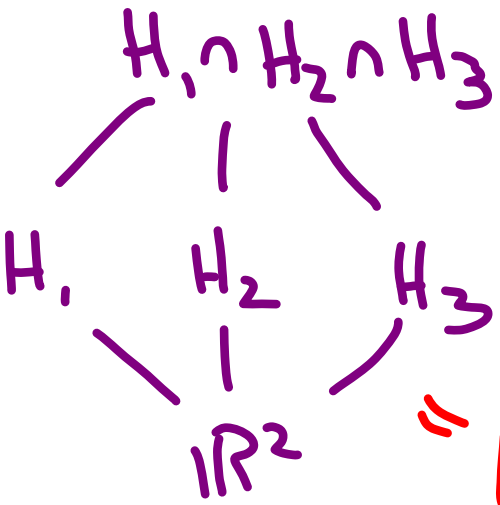
useful e.g. for ...

2. Tom Zaslavsky: region counting formulas for the complement of $\mathbb{R}$-hyperplane arr't A



$$\#\text{regions} = \sum_{u \in L_A} |\mu(\hat{0}, u)|$$

$$\#\text{bdd regions} = \left|\sum_{u \in L_A} \mu(\hat{0}, u)\right|$$



$= L_A =$ "intersection poset"

e.g. $\#\text{regions} = 1+3+2$

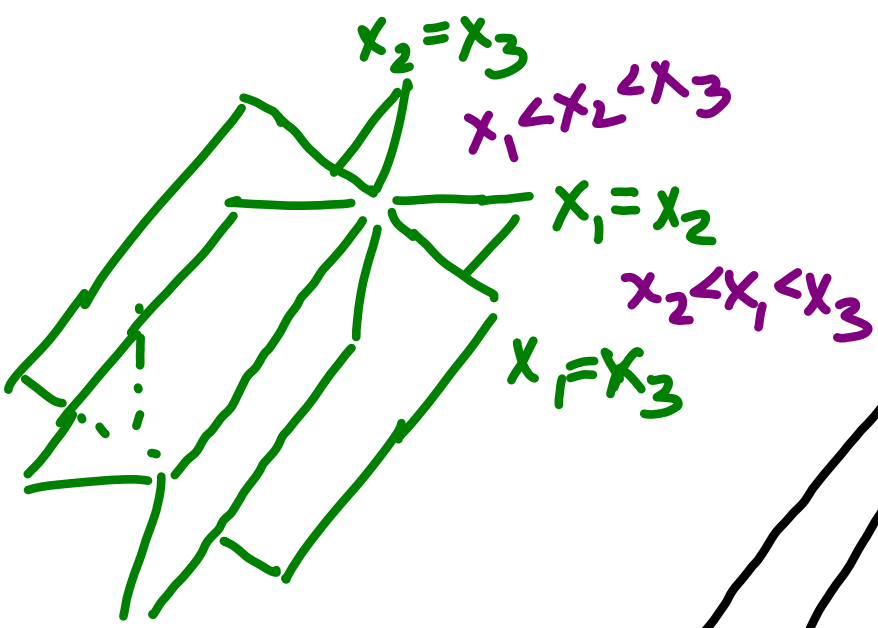
$\#\text{bdd regions} = 1-3+2$

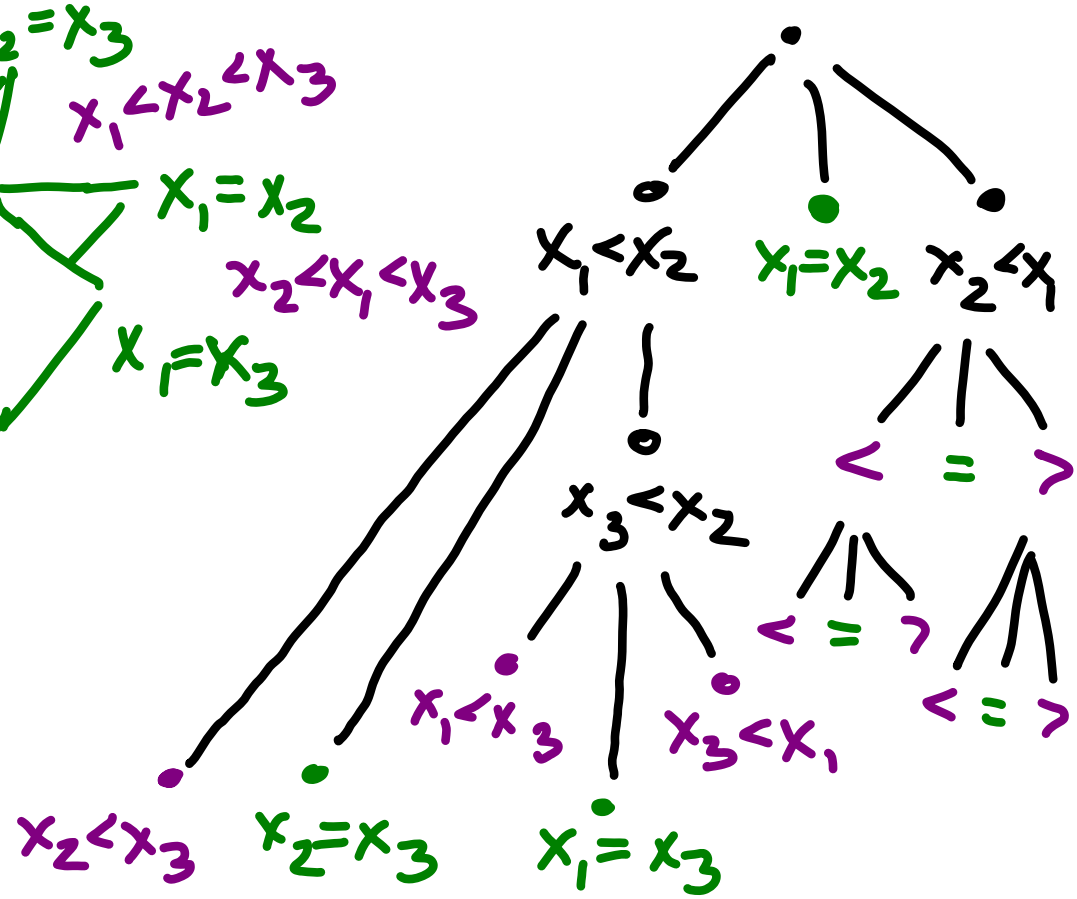
$\mu(\mathbb{R}^2, \mathbb{R}^2) = 1$

$\mu(\mathbb{R}^2, H_i) = -1$ for $i = 1, 2, 3$

$\mu(\mathbb{R}^2, H_1 \cap H_2 \cap H_3) = 2$

3. Björner-Lovász-Yao: lower bound via Möbius fns for deciding if there are k equal coordinates in $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ by pairwise coord. comparisons, i.e. deciding whether $\vec{a}$ lies on "k-equal arr't" of subspaces $x_{i_1} = \ldots = x_{i_k}$





- lower bd on # leaves (and hence on $\log_3(\text{depth})$) was given in terms of betti #'s (i.e. # holes in each dimension) in topological space $\mathbb{R}^n$ – k-equal subspace arrangement

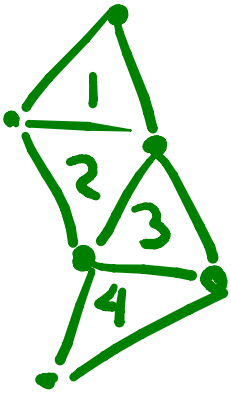
  subspaces like $x_1 = x_2 = x_4$ for $k=3$

- Mark Goresky & Robert MacPherson showed how to compute these betti #'s from poset order complexes

- Björner & Wachs found shellings for these poset order complexes, namely intersection posets for "k-equal arrangement"

Techniques Yielding Möbius Functions (& Poset Topology)

- (Lexicographic) shellability



- EL-labelings (Anders Björner)

- CL-labelings (Anders Björner & Michelle Wachs)

$\Rightarrow \Delta(P) \simeq$ 

(telling us $\tilde{\chi}$ hence μ)

- Lexicographic discrete Morse functions (Babson-H.)

(for other topol. types)

Technique: Shellability

- simplicial complex is pure of dimension d if all maximal faces ("facets") have dim. d
- simplicial complex is shellable if there is facet order $F_1, F_2, \ldots$ called a shelling s.t. $\forall\, j \geq 2$

$$\overline{F_j} \cap \Big(\bigcup_{i<j} \overline{F_i}\Big) \text{ is pure of}$$

dimension one less than F_j

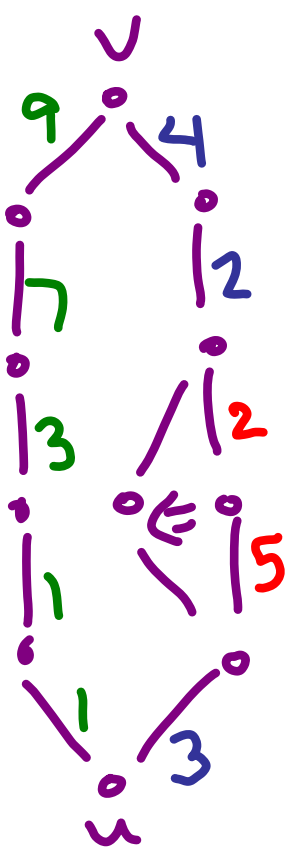
e.g. 

Feature: Each attachment preserves topol. or completes a sphere

Technique 1*: Lexicographic Shellability

(Anders Björner & Michelle Wachs)

A poset P is EL-shellable if it admits labeling λ of its cover relations $x \lessdot y$ w/ integers (called an EL-labeling) s.t. $u < v$ implies:



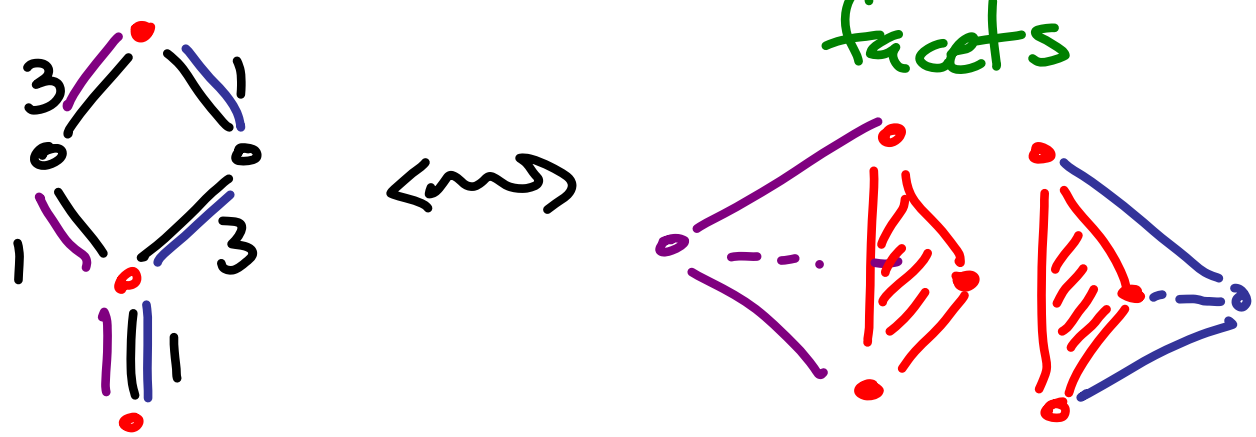
(1) there is unique saturated chain $u \lessdot u_1 \lessdot \dots \lessdot u_k \lessdot v$ s.t. $\lambda(u,u_1) \leq \lambda(u_1,u_2) \leq \dots \leq \lambda(u_k,v)$ and

(2) $(\lambda(u,u_1), \lambda(u_1,u_2), \dots, \lambda(u_k,v))$ is lexicographically smaller than the label sequences on all other saturated chains from u to v.

Thm (Björner): EL-labeling $\Rightarrow$ Shelling

Idea: Lexicographic order on maximal chains (breaking ties arbitrarily) induces shelling order on corresponding facets of $\Delta(P)$.

- "descents in labeling" $\leftrightsquigarrow$ codim. one overlap of facets



- "descending" chains $\leftrightsquigarrow$ facets attaching along entire belly $\leftrightsquigarrow$ spheres

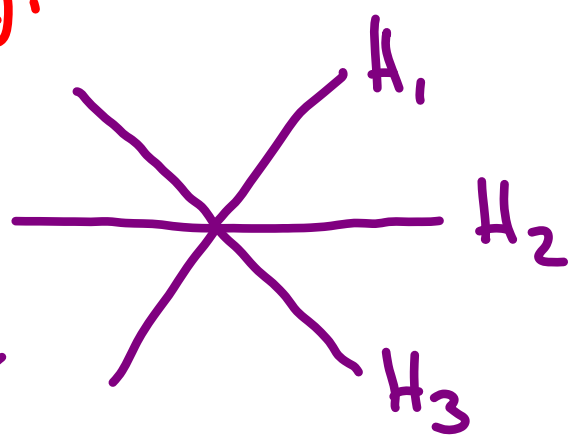
- $\mu_P(u,v) = \pm \#$ descending chains u to v (for P graded)

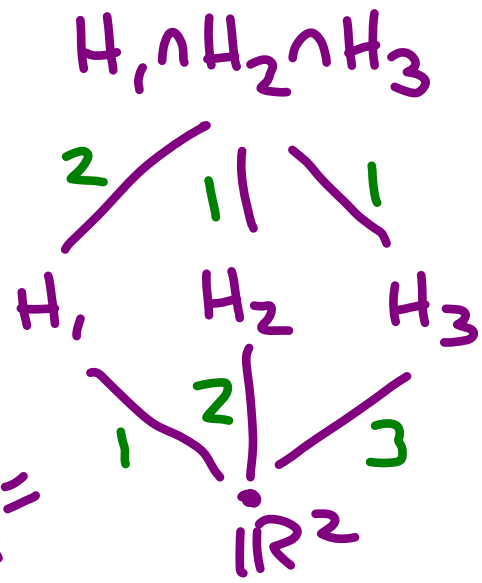
Example: Intersection Posets of Hyperplane Arrangements

- Choose any total order $H_1, H_2, \ldots, H_k$ on hyperplanes (resp. "atoms")
- Label $u \lessdot v$ with

$$\min\{ i \mid H_i \not\leq u \text{ and } H_i \leq v \}$$

e.g.

$A =$ 

$L_A =$ 

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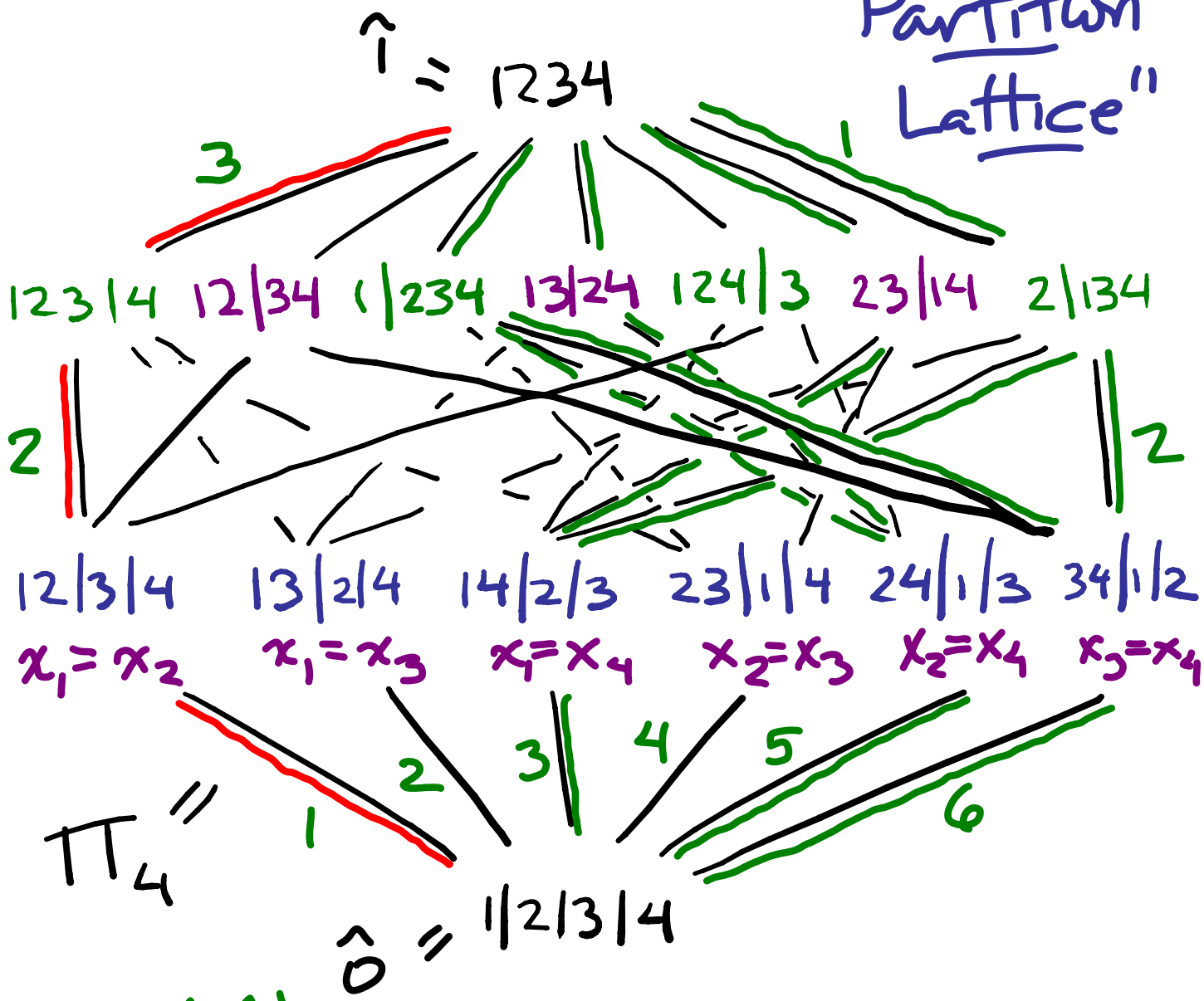


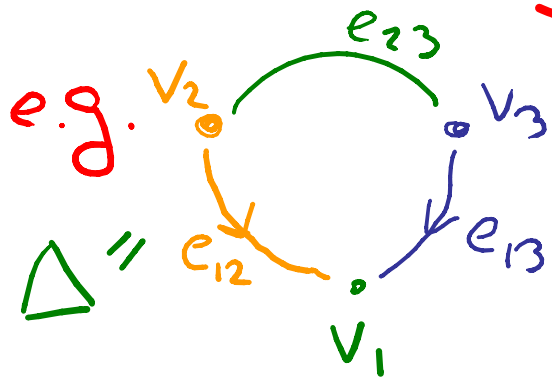
$\mu_{\Pi_4}(\hat{0}, \hat{1}) = -6$

Discrete Morse Theory

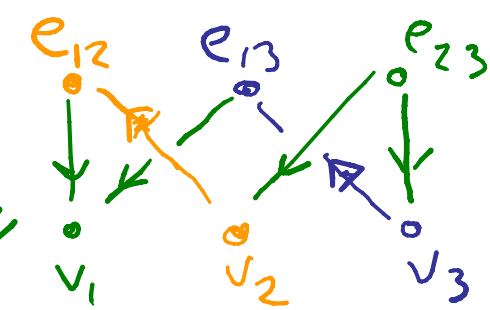
(due to Forman & reformulated by Chari)

Given any regular CW complex Δ, construct an **acyclic matching** a.k.a. **Morse matching** on its face poset, i.e.,

e.g. $\Delta =$



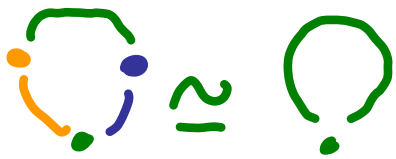
$F(\Delta) =$



an edge orientation s.t. "up edges" give a matching and directed graph has no cycles.

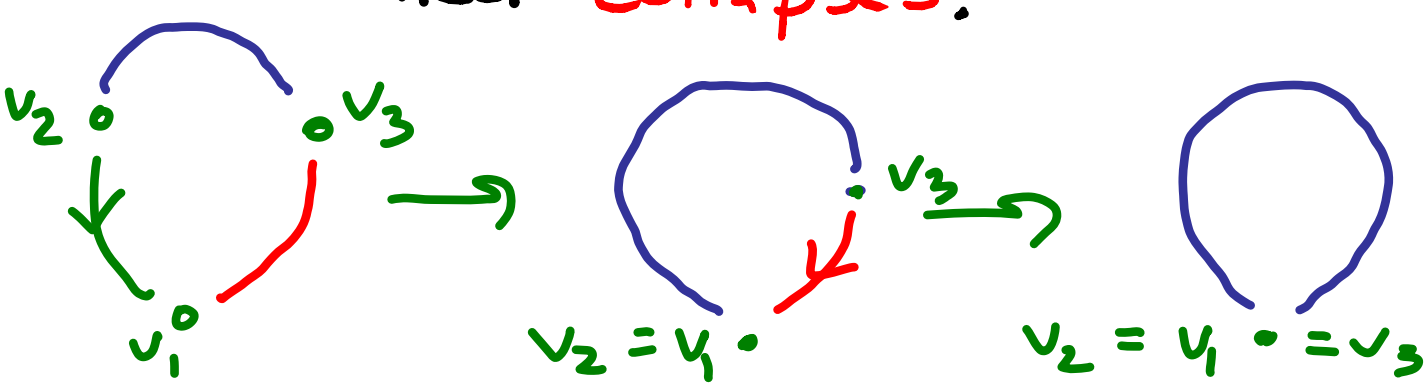
(A **matching** is a collection of graph edges s.t. no vertex is in more than one edge)

Theorem (Forman): $\Delta \simeq \Delta^M$ a CW complex comprised of the unmatched cells, called critical cells.

e.g. 

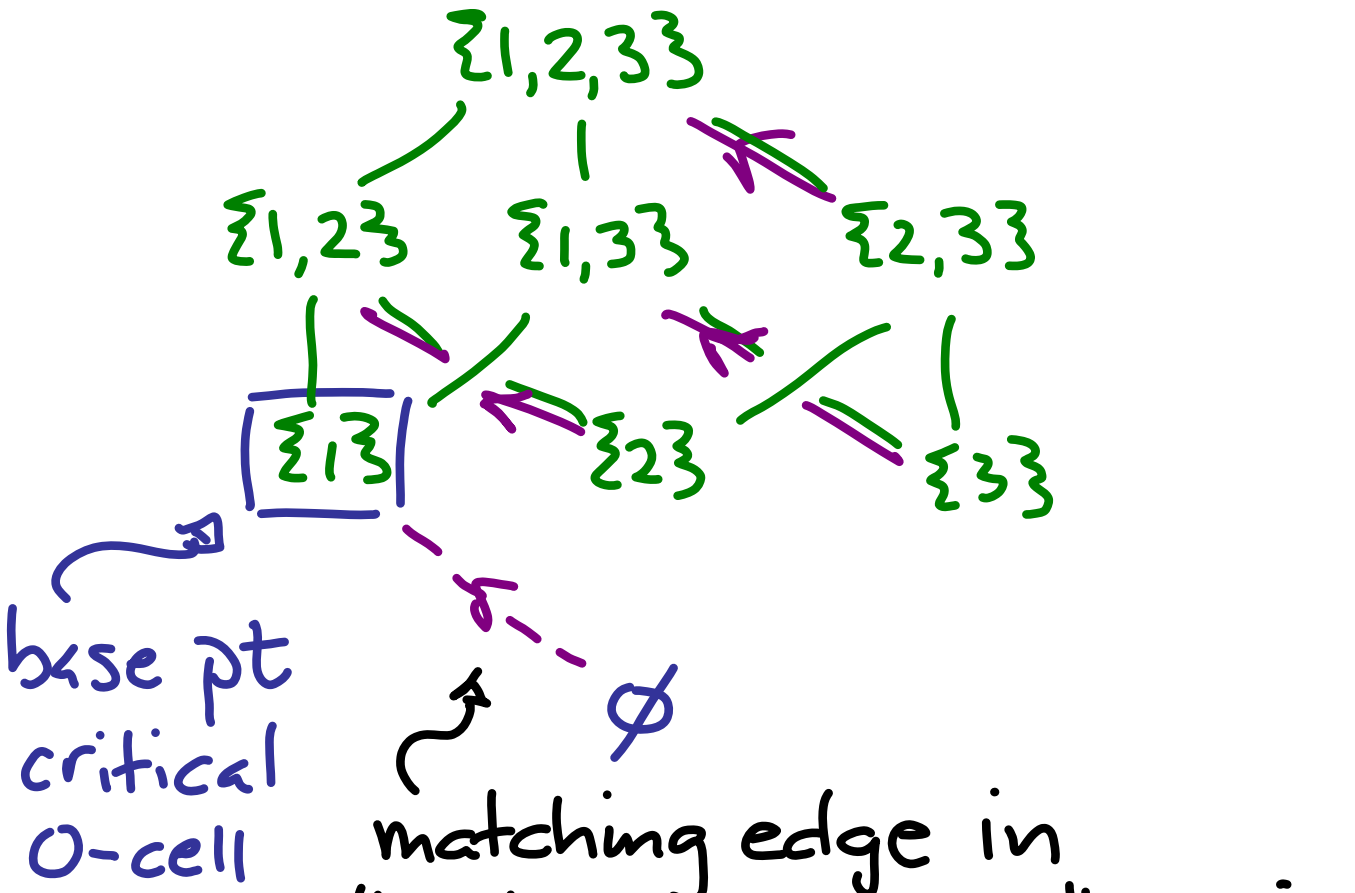
same topological structure (same homology groups + more!)

Idea: Find pairs of faces where one can be "pulled across" other eliminating both without changing topology, via moves called "collapses".



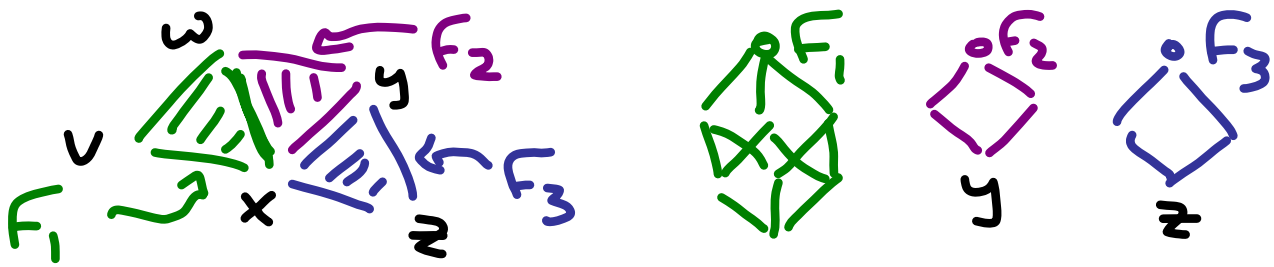
First Examples

1. Boolean algebra of subsets of $\{1,2,\ldots,n\}$, face poset of simplex, matching $S \setminus \{1\}$ with $S \cup \{1\}$ $\forall S$



2. Any union of acyclic matchings on $F(\Delta_2 \setminus \Delta_1), F(\Delta_3 \setminus \Delta_2), \ldots$ for $\Delta_1 \subseteq \Delta_2 \subseteq \ldots \subseteq \Delta_k = \Delta$ a filtration of subcomplexes is an acyclic matching for Δ

e.g. $\bar{F}_1 \subseteq \bar{F}_1 \cup \bar{F}_2 \subseteq \bar{F}_1 \cup \bar{F}_2 \cup \bar{F}_3$



3. Shelling $\Rightarrow$ Discrete Morse fn whose critical cells are the maximal faces attaching along their entire boundary

Explanation for $\Delta \simeq \Delta^M$: Matching edges specify (internal) elementary collapses preserving homotopy type

Some Consequences of $\Delta \simeq \Delta^M$:

1. If $F(\Delta)$ has complete acyclic matching (w/ $\emptyset \in F(\Delta)$) then Δ is collapsible.

Recall: Some contractible complexes are not collapsible.

e.g. dunce cap

2. $\tilde{\chi}(\Delta) = \tilde{\chi}(\Delta^M)$

$$= -1 + \#\,0\text{-cells} - \#\,1\text{-cells} + \#\,2\text{-cells} - \cdots$$

$$= -1 + \beta_0 - \beta_1 + \beta_2 - \cdots$$

<u>For Posets</u>: $\mu_P(x,y) = \tilde{\chi}(\Delta(x,y)) = \tilde{\chi}(\Delta^M(x,y))$

3. Morse Inequalities:

1. $\tilde{\beta}_i(\Delta) \leq \tilde{m}_i(\Delta) = \#$ i-dim'l critical cells

2. $\sum_{i \leq j} (-1)^{j-i} \tilde{\beta}_i(\Delta) \leq \sum_{i \leq j} (-1)^{j-i} \tilde{m}_i(\Delta)$

(for each $j \leq \dim(\Delta)$)

<u>Rk</u>: "Greedy" matchings tend to satisfy acyclicity requirement.

Question (H.): Is there a good way to "complete the square":

$$\begin{tikzcd}
\text{lexicographic shelling} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & ?? \arrow[d, Rightarrow] \\
\text{shelling} \arrow[r, Rightarrow] & \text{discrete Morse function}
\end{tikzcd}$$

to understand posets that fail to be shellable (e.g. not wedge of spheres)?

Proposed Answer (Eric Babson & P.H.)

"lexicographic discrete Morse fn's"

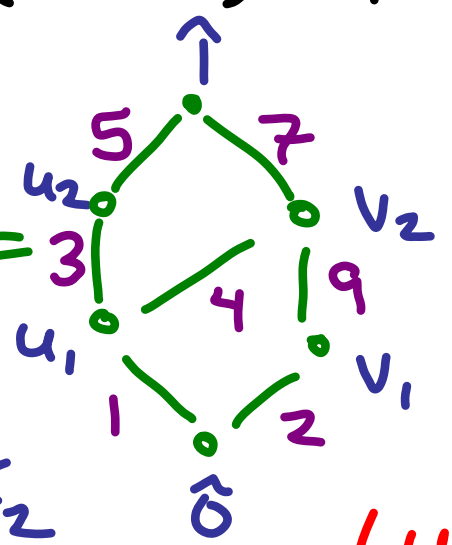
Lexicographic Discrete Morse Functions: A General Construction

(partly joint work with E. Babson)

Step 1: Any edge labeling on poset P induces lexicographic order $F_1, F_2, \ldots, F_m$ on maximal faces (facets) of $\Delta(P)$

e.g.

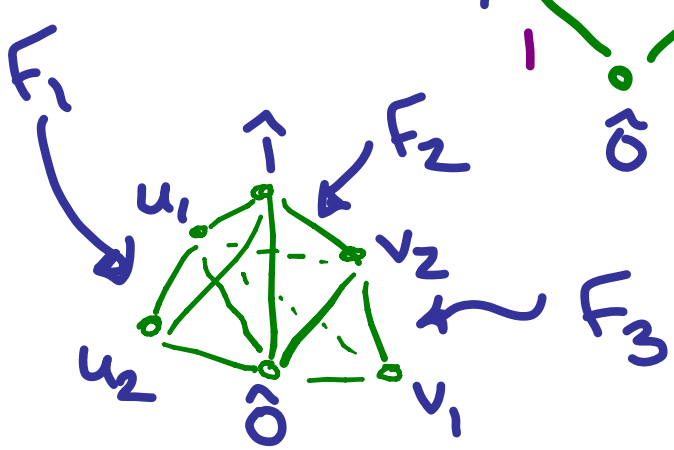
$P =$



$F_1 = 135$

$F_2 = 147$

$F_3 = 297$

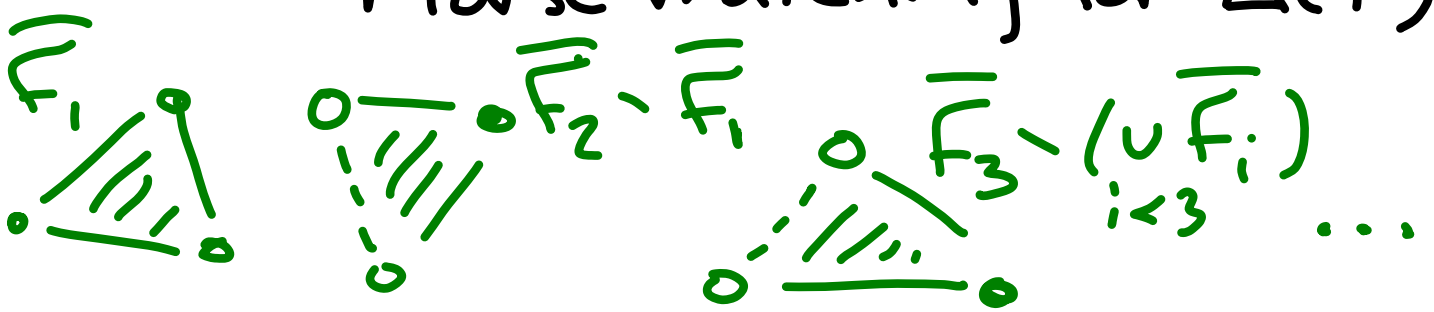


(Usually not EL-labeling!)

Step 2: Morse matching on each $\overline{F}_j \setminus (\bigcup_{i<j} \overline{F}_i)$ s.t.

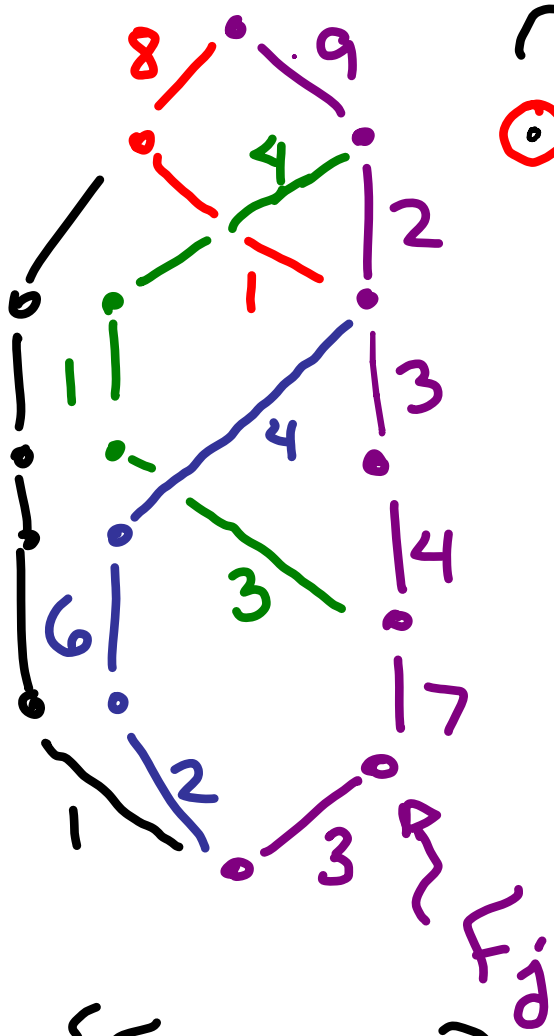
(1) Each $\overline{F}_j \setminus (\bigcup_{i<j} \overline{F}_i)$ has 0 or 1 unmatched (critical) cells

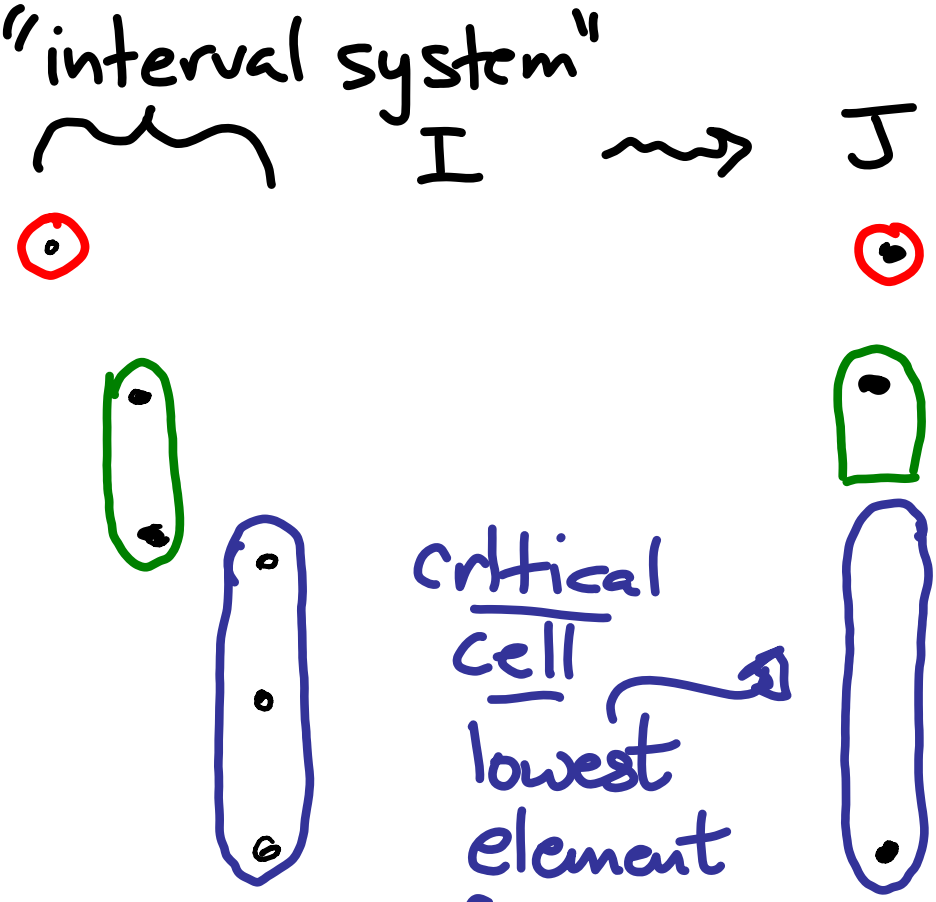
(2) Union of these matchings is Morse matching for $\Delta(P)$



Theorem (Babson-H.) Any edge labeling on any finite poset gives rise to a lexicographic discrete Morse fn s.t. critical cells $\leftrightarrow$ facets whose attachment changes the homotopy type of complex.

Description of Critical Cells





critical cell
lowest element of each (truncated) interval

- $\left\{ \begin{array}{l} \text{Faces in} \\ \overline{F}_j \setminus \left(\bigcup_{i<j} \overline{F}_i \right) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Subsets of ranks in} \\ F_j \text{ that "hit" all} \\ \text{intervals in I-system} \end{array} \right\}$
- No critical cell unless truncated interval system J fully covers F_j