On Counting by
Inclusion-Exclusion: Möbius
Functions, Shellability &
Discrete Morse Theory

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Counting by Inclusion-Exclusion

e.g. "counting" points in the \( \mathbb{R}^2 \) complement of \( \gamma \)
yields: counted \( \frac{1}{1-1-1-1+2} = 0 \) times

\[ \mathbb{R}^2 \]

* Coefficients \( 1, 1, 1, 1, 2 \) in such inclusion-exclusion counting formula
given by "Möbius function" \( \mu \) (upcoming).
Calculating these coefficients with Möbius function \( M \)

- we’ll want:
  \[
  M(\mathbb{R}^2, \mathbb{R}^2) = 1 = \text{wef. of } \mathbb{R}^2 \\
  M(\mathbb{R}^2, l_1) = -1 = \text{wef. of } l_1 \\
  M(\mathbb{R}^2, l_2) = -1 = \text{wef. of } l_2 \\
  M(\mathbb{R}^2, l_3) = -1 = \text{wef. of } l_3 \\
  M(\mathbb{R}^2, p) = 2 = \text{wef. of } p 
  \]
- how to achieve this:
  \[
  M(\mathbb{R}^2, u) = -\sum M(\mathbb{R}^2, v) \\
  \text{v contains } u 
  \]

½ more generally....
**Defn:** Given any finite partially ordered set (poset) \( P \), recursively define "Möbius function" \( M : P \times P \to \mathbb{Z} \) by:

\[
M_p(u, u) = 1
\]
\[
M_p(u, v) = -\sum_{u \leq z < v} M_p(u, z)
\]

*E.g.*

\[
P = e_1 \cap e_2 \cap e_3
\]
\[
P = \mathbb{R}^2
\]

\[
M_p(\mathbb{R}^2, -)
\]
**Defn:** The order complex of poset \( P \) is the abstract simplicial complex, denoted \( \Delta(P) \), whose \( i \)-dim' faces are the \((i+1)\)-chains \( u_0 < u_1 < \ldots < u_i \) in \( P \).

**Key Property (due to Hall & popularized by Rota):**

\[
M_p(x,y) = \sum (\Delta_p(x,y)) = -1 + \# \text{0-chains} - \# \text{1-chains} + \ldots
\]

\[
\Delta_p(x,y) = \Delta(\exists z \in P \mid x < z < y^3)
\]

- \#2-dim' hole baby's
(Reduced) Euler Characteristic

- The reduced Euler characteristic of $K$, denoted $\widetilde{\chi}(K) = -1 + \#\text{vertices} - \#\text{edges} + \#\text{triangles}$...

E.g., $\widetilde{\chi}(\triangle) = -1 + 4 - 6 + 4 = 1$ $\widetilde{\chi}(\square) = -1 + 5 - 9 + 6 = 1$ $\widetilde{\chi}(2\text{-sphere})$

Adding faces without changing "Topology" won't change $\widetilde{\chi}$!

\[ \chi = -1 + 3 - 3 + 1 = -1 + 4 - 5 + 2 = -1 + 5 - 8 + 4 \]
"Topological Proof" of Möbius Function for Poset of Subsets

poset of subsets of \( \{v_1, v_2, v_3\} \) = \( F(K) \)

\[ \Delta(\emptyset, S) = v_1 \]

\[ sd(2K) \]

\[ s_1 \]

\[ s_2 \]
Intersection Posets

e.g.,

\[ A = \mathbb{R}^2 \]

\[ l_1, l_2, l_3, l_4 \]

\[ P_{12}, P_{14}, P'_{12}, P''_{12} \]

\[ \sim \sim \sim \sim \]

\[ l_1, l_2, l_3, l_4 \]

\[ \mathbb{R}^2 \]

\[ \text{(intersection poset)} \]

\[ H, H_1, H_1 l_1, H_2, l_1 l_2, l_1 l_3 \]

\[ \mathbb{R}^3 \]
Intersection Poset \( L_A \) for
\[ A = \{ x_i = x_j \mid 1 \leq i < j \leq n \} \]
the "Partition Lattice"

\[ \hat{\tau} = 1234 \]

\[ \begin{align*}
123|4 & \quad 12|34 & \quad 1|234 & \quad 13|24 & \quad 124|3 & \quad 23|14 & \quad 2|134 \\
12|3|4 & \quad 13|2|4 & \quad 14|2|3 & \quad 23|1|4 & \quad 24|1|3 & \quad 34|1|2 \\
\tau_1 = x_2 & \quad \tau_1 = x_3 & \quad \tau_1 = x_4 & \quad \tau_2 = x_3 & \quad \tau_2 = x_4 & \quad \tau_3 = x_4
\end{align*} \]

\[ \Pi_4 = \hat{\tau} = 1234 \]

\[ M_{\Pi_4} (\hat{\tau}, \hat{\tau}) = -6 \]
Some Applications of Möbius Functions & "Shellability"

1. Shellability of intersection posets of hyperplane arrangements due to shellability of "geometric lattices"

(Anders Björner) & "geometric semilattices"

(Michelle James Wachs Walker), yielding Möbius fns of "intersection posets" of hyperplane arrangements

\[ \text{useful e.g. for...} \]
2. Zaslavsky:

Region counting formulas for the complement of IR-hyperplane aren't A

\[ \# \text{regions} = \sum_{u \in L_A} |M(0, v)| \]
\[ \# \text{bdd regions} = \left| \sum_{u \in L_A} M(0, v) \right| \]

\( A = \{ H_1 \cap H_2 \cap H_3 \} \)

\( \text{e.g.} \quad \# \text{regions} = 1 + 3 + 2 \) \n\( \# \text{bdd regions} = 1 - 3 + 2 \)

\( L_A = \text{"intersection poset"} \)

\( M(\mathbb{R}^2, \mathbb{R}^2) = 1 \)
\( M(\mathbb{R}^2, H_i) = -1 \text{ for } i = 1, 2, 3 \)
\( M(\mathbb{R}^2, H_1 \cap H_2 \cap H_3) = 2 \)
3. **Björner-Lovász-Yao**: lower bound via Möbius funs for deciding if there are $k$ equal coordinates in $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ by pairwise coord. comparisons, i.e. deciding whether $\vec{a}$ lies on "$k$-equal arr't" of subspaces $x_i = \ldots = x_i$.
• lower bd on # leaves (and hence on \( \log_3(\text{depth}) \)) was given in terms of betti #’s (i.e. # holes in each dimension) in topological space \( IR^d \) – k-equal subspace arrangement

\[ x_1 = x_2 = x_3 \]

for \( k = 3 \)

• Mark Goresky & Robert MacPherson showed how to compute these betti #’s from poset order complexes

• Björner & Welker found shellings for these poset order complexes, namely intersection posets for “k-equal arrangement”
Techniques Yielding Möbius Functions (\$?\$ Poset Topology)

- (lexicographic) shellability
  - EL-labelings (Anders Björner)
  - CL-labelings (Anders Björner \& Michelle Wachs)

\[ \Delta(\mathcal{P}) \cong \hat{x} \] (telling us \( \hat{x} \) hence \( \mu \))

- Lexicographic discrete Morse functions (Babson-H.)
  (for other topol. types)
**Technique**: Shellability

- Simplicial complex is pure of dimension $d$ if all maximal faces ("facets") have dim. $d$.
- Simplicial complex is shellable if there is facet order $F_i,F_2,\ldots$ called a shelling s.t. $\forall j \geq 2$
  $$F_i \cap (\cup_{i < j} F_i)$$
  is pure of dimension one less than $F_j$.

**Feature**: Each attachment preserves topology, or completes a sphere.
Technique 1*: Lexicographic Shellability

(Anders Björner & Michelle Wachs)

A poset $P$ is EL-shellable if it admits labeling $\lambda$ of its cover relations $x < y$ w/ integers (called an EL-labeling) s.t. $u < v$ implies:

1) there is unique saturated chain $u < u_1 < \ldots < u_k < v$ s.t.
   \[ \lambda(u, u_1) \leq \lambda(u_1, u_2) \leq \ldots \leq \lambda(u_k, v) \]
   and

2) $(\lambda(u, u_1), \lambda(u_1, u_2), \ldots, \lambda(u_k, v))$
   is lexicographically smaller than the label sequences on all other saturated chains from $u$ to $v$. 
**Thm (Björner):** EL-labeling $\Rightarrow$ Shelling

**Idea:** Lexicographic order on maximal chains (breaking ties arbitrarily) induces shelling order on corresponding facets of $\Delta(P)$.

- "descents in $\text{codim one}$ labeling" $\iff$ overlap of facets
- "descending" $\iff$ facets attaching along entire bary (spheres)
- $M_P(u,v) = \pm \# \text{descending chains } u \to v$ (for $P$ graded)
Example: Intersection Posets of Hyperplane Arrangements

- Choose any total order \( H_1, H_2, \ldots, H_k \) on hyperplanes (resp. "atoms")

- Label \( u < v \) with
  \[ \min \{ i \mid H_i \neq u \text{ and } H_i \leq v \} \]

\[ \text{e.g.} \]

\[ A = H_1, H_2, H_3 \]

\[ L_4 = \frac{1}{2} \text{IR}^2 \]
Intersection Poset $L_A$ for $A = \{x_i = x_j | 1 \leq i < j \leq n\}$ the “Partition Lattice”

\[ \hat{i} = 1234 \]

\[ \begin{align*}
1234 & 1234 \quad 1234 & 1243 & 1324 & 1243 & 2314 & 2134 \\
\end{align*} \]

\[ \begin{align*}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{align*} \]

\[ \begin{align*}
x_1 &= x_2 & x_1 &= x_3 & x_1 &= x_4 & x_2 &= x_3 & x_2 &= x_4 & x_3 &= x_4 \\
\end{align*} \]

\[ \Pi_4 = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

\[ \hat{0} = 121314 \]

\[ M_{\Pi_4}(\hat{0}, \hat{i}) = -6 \]
Discrete Morse Theory
(due to Forman reformulated by Chari)

Given any regular CW complex $\Delta$, construct an acyclic matching, a.k.a. Morse matching on its face poset, i.e.,

an edge orientation s.t. "up edges" give a matching and directed graph has no cycles.

(A matching is a collection of graph edges s.t. no vertex is in more than one edge)
**Theorem (Forman):** $\Delta^N \cong \Delta^M$ a CW complex comprised of the unmatched cells, called critical cells.

e.g., $\sim$

\[ \text{same topological structure (same homology groups + more!)} \]

**Idea:** Find pairs of faces where one can be "pulled across" the other eliminating both without changing topology, via moves called "collapses".
First Examples

1. Boolean algebra of subsets of $\xi_1, \ldots, \eta_3$, face poset of simplex, matching $5 \cup \xi_13$ with $SU \xi_13$ A S

Matching edge in "reduced homology" version of discrete Morse theory
2. Any union of acyclic matchings on $F(\Delta_2 \setminus \Delta_1), F(\Delta_3 \setminus \Delta_2), \ldots$, for $\Delta_1 \leq \Delta_2 \leq \ldots \leq \Delta_k = \Delta$, a filtration of subcomplexes is an acyclic matching for $\Delta$

c.g. $\overline{F}_1 \subseteq \overline{F}_1 \cup \overline{F}_2 \subseteq \overline{F}_1 \cup \overline{F}_2 \cup \overline{F}_3$

3. Shelling $\Rightarrow$ Discrete Morse fn whose critical cells are the maximal faces attaching along their entire boundary
Explanation for $\Delta \simeq \Delta^m$: Matching edges specify (internal) elementary collapses preserving homotopy type.

Some Consequences of $\Delta \simeq \Delta^m$:

1. If $F(\Delta)$ has complete acyclic matching (w/ $\emptyset \in F(\Delta)$) then $\Delta$ is collapsible.

Recall: Some contractible complexes are not collapsible.

  e.g. dunæ cap
2. \( \tilde{\chi}(\Delta) = \tilde{\chi}(\Delta^m) \)

\[
= -1 + \# \text{0-cells} - \# \text{1-cells} \\
\quad + \# \text{2-cells} - \ldots \\
= -1 + \beta_0 - \beta_1 + \beta_2 - \ldots \\
\]

For Posets: \( M_p(x,y) = \tilde{\chi}(\Delta(x,y)) = \tilde{\chi}(\Delta^m(x,y)) \)

3. Morse Inequalities:

1. \( \tilde{\beta}_i(\Delta) \leq \tilde{m}_i(\Delta) = \# \text{i-dim \ 'critical cells} \)

\[
\sum_{i \leq j} \tilde{\beta}_i(\Delta) \leq \sum_{i \leq j} \tilde{m}_i(\Delta) \\
\]

(\text{for each } j \leq \text{dim}(\Delta))

Rk: "Greedy" matchings tend to satisfy acyclicity requirement.
**Question (H.):** Is there a good way to "complete the square":

- lexicographic shelling

\[ \Rightarrow ?? \]

\[ \Rightarrow \text{shelling} \Rightarrow \text{discrete Morse function} \]

... to understand posets that fail to be shellable (e.g. not wedge of spheres)?

**Proposed Answer (Eric Babson & P.H.):**

"lexicographic discrete Morse fn's"
**Lexicographic Discrete Morse Functions: A General Construction**

(partly joint work with E. Babson)

**Step 1:** Any edge labeling on poset $P$ induces lexicographic order $F_1, F_2, \ldots, F_m$ on maximal faces (facets) of $\Delta(P)$

*Example:*

$P = \{3\}$

$F_1 = 135$

$F_2 = 147$

$F_3 = 297$

(Usually not EL-labeling!)
Step 2: Morse matching on each $F_j \setminus \bigcup_{i < j} F_i$ s.t.

1. Each $F_j \setminus \bigcup_{i < j} F_i$ has 0 or 1 unmatched (critical) cells
2. Union of these matchings is Morse matching for $\Delta(D)$

Theorem (Babson-H.) Any edge labeling on any finite poset gives rise to a lexicographic discrete Morse fn s.t. critical cells $\leftrightarrow$ facets whose attachment changes the homotopy type of complex.
Description of Critical Cells

"interval system"

$I \rightarrow J$

[Faces in $F_j \setminus \left( \bigcup_{i \leq j} F_i \right)$] $\rightarrow$ [Subsets of ranks in $F_j$ that "hit" all intervals in $I$-system]

- No critical cell unless truncated interval system $J$ fully covers $F_j$