Combinatorics & Topology of Totally Positive Spaces

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- Slides available at:
  http://www4.ncsu.edu/~mplhersh/
  Brown-Colloq.pdf
- see "Regular cell complexes in total positivity", Inventiones Mathematicae, 197 (2014), 57-114 for details.
Topological Aspects of Total Positivity

- Lusztig initiated study of Totally nonnegative, real part of (matrix) Schubert varieties, flag varieties, ...
  (i.e. part with minors all nonnegative in spaces of matrices or of flags)
- Conjecturally/provably homeomorphic to closed balls (after deconving)
- Proving this:
  - puts restrictions on relations among (exponentiated) Chevalley generators,
  - reveals structure in canonical bases; a motivation for cluster algebras.
- **Main Result of Talk:** Proof of Fomin-Shapiro Conjecture via new tools exploiting interplay of combinatorial data & topological data.
**CW Complexes & their Face Posets**
(i.e. Partially Ordered Sets)

**Example:**

$K = $ ball

$K' = \mathbb{R}P^2$

$F(K) = e_1$

$F(K')$

“Closure poset” or “face poset”

$(u \leq v \iff u \leq \bar{v})$

**Recall:** A CW complex: cells $e_d \cong \mathbb{R}^{d(d)}$

characteristic maps $f_d: B^{\dim(e_d)} \rightarrow v \in e_0$

attaching maps $f_d|_{\partial B^{\dim(e_d)}}$
- A poset is **graded** if \( u \leq v \) in \( P \) implies all minimal paths upward from \( u \) to \( v \) have same length (i.e. \(#edges\))

  ![Diagram](image)

  e.g. is not graded

- A graded poset is **thin** if each rank 2 interval has exactly 4 elements.

**Recall**: A CW complex is **regular** if the attaching map \( f_\alpha \) for each cell \( e_\alpha \) is a homeomorphism, i.e. cell closures are closed balls.

**Recall**: ball \( e \) homeomorphic
• $K$ regular $\Rightarrow K \cong \Delta(F(K) \setminus \emptyset) = sdK$

nerve or order complex, i.e. simplicial complex whose faces are chains $u_i < \ldots < u_j$.

\[ K'' \rightarrow F(K) - \emptyset \rightarrow \Delta(F(K) - \emptyset) \]

**Defn (Björner):** A finite, graded poset $P$ is **CW poset** if

- $P$ has unique min’l elt. $\hat{0}$
- $P$ has additional element(s)
- $x \neq \hat{0} \Rightarrow \Delta(\hat{0}, x) \cong S^{\text{rank}(x) - 2}$

**Thm (Björner):** $P$ is CW poset $\iff$ there exists regular CW complex $K$ with $P = F(K)$. 
Some Examples of CW Posets

- Bruhat order (Björner & Wachs) of finite Coxeter group
- Closure poset for double Bruhat cleomp. of totally nonneg. part of flag variety (Williams)
- Thin, "shellable" posets (Danaraj & Klee)
- Closure poset of double suspension of triangulation of homology sphere with "big cell" glued in (due to work of J. Cannon & R. Edwards)

A Goal of Mine: Use combinatorics of F(K) + limited topological info (codim. one cell incidences) to understand K.
Finite Coxeter Groups & Finite Reflection Groups: Quick Review

Coxeter group: generators \( \{ s_i \mid i \leq 3 \} \)

\( t \) relns \( s_i^2 = e \neq (s_i s_j)^{m(i,j)} = e \)

e.g. \( S_3 = \langle s_1, s_2 | s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle \)

Symmetric group \( (1,2) (2,3) \) (and other Weyl gps)

geometrically:

A reduced expression for \( w \in W \) is an expression of minimal "length" e.g. \( s_1 s_2 \)
but not \( s_1 s_2 s_1 s_2 \) (which equals \( s_2 s_1 \))
The Bruhat order is a partial order on elements of Coxeter group $W$ with $u \leq v \iff$ there exist reduced expressions $r(u)$ and $r(v)$ for $u \neq v$, respectively, with $r(u)$ subexpression of $r(v)$.

E.g. $W = S_3$ with generators $s_1 = (1,2)$, $s_2 = (2,3)$

$$321 = s_1 s_2 s_1 = s_2 s_1 s_2$$

- Reduced word $(i, \ldots, i,d)$ for $s_i s_j \ldots s_d$
- Closure poset for Schubert cell decompositions of flag varieties $G/B$ (all cells have even real dim)
Qn (Bernstein & Björner): Find regular \( CW \) complexes naturally arising from rep'n theory which are homeomorphic to closed balls and have the (lower) Bruhat intervals as closure posets.

Conjectural Solution (Fomin & Shapiro): The Bruhat stratification of \( \mathrm{lk} (\text{id}) \) in totally nonneg. real part of unipotent radical of Borel in semisimple, simply connected algebraic group defined and split over \( \mathbb{R} \).

i.e. a "slice" \( \text{id} \) within:

\[
\gamma_w = \left[ B^- w B^- \cap \text{(unipotent subgp of } B) \right]_{\text{ld}}
\]

upper-triang. w/ 1's on diagonal
Theorem (H.): Fomin-Shapiro
Conjecture indeed holds.

Special Case (Running Example for Talk): Space of totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed, positive constant, stratified by which minors are positive and which are 0.

Concrete Realization: products of certain elementary matrices, by results of Whitney & Lusztig.
The Totally Nonnegative Part of a Space of Matrices

- $x_i(t) = \exp(t e_i)$ (general finite type, expon'd Chevalley generator)
- $f_{(i_1, \ldots, i_d)} : \mathbb{R}^d_{\geq 0} \to M_{n \times n} \subseteq \mathbb{R}^{n^2}$

\[ (t_1, \ldots, t_d) \mapsto x_{i_1}(t_1) \cdots x_{i_d}(t_d) \]

E.g. $f_{(1,2,1)}(t_1, t_2, t_3) = x_1(t_1) x_2(t_2) x_1(t_3)$

\[ = \begin{pmatrix} 1 & t_1 \\ t_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ t_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 \\ t_3 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & t_1 + t_3 & t_1 t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \]
"Picture" of Map $f_{(1, 2, 1)}$

$$f_{(1, 2, 1)}(t_1, t_2, t_3) = \left( \begin{array}{c} 1 \\ t_1 \\ 1 \\ t_2 \\ 1 \\ t_3 \\ 1 \end{array} \right)$$

$t_2 = 0 \Rightarrow x_i(t_1) \cdot x_i(t_3)$

$$f_{(1, 2, 1)}(t_1, 0, t_3) = \left( \begin{array}{c} 1 \\ t_1 \\ 1 \\ t_3 \\ 1 \end{array} \right)$$

$$= \left( \begin{array}{c} 1 \\ t_1 + t_3 \\ 1 \\ 1 \end{array} \right) = x_i(t_1 + t_3)$$

**Non-injectivity:** results from "modified nil moves" $x_i(u)x_i(v) \Rightarrow x_i(u + v)$ directly & after "long braid moves" in Hecke algebra.
1st Key Idea to Proof of F-S Conjecture

O-Hecke Algebra Captures which Simplex Faces have Same Image under $f_{(i_1, i_2)}$

(1) $x_i(t_1) x_i(t_2) = x_i(t_1 + t_2)$  
   \[ \text{"nil-move"} \]
   \[ \text{suppress parameters} \]
   \[ x_i^2 = x_i \]  \( (O\text{-Hecke alg. reln, up to sign}) \)

(2) $x_i(t_1) x_{i+1}(t_2) x_i(t_3) = x_{i+1} \left( \frac{t_2 + t_3}{t_1 t_3} \right) x_i(t_1 + t_3) x_{i+1} \left( \frac{t_1 + t_2}{t_1 t_3} \right)$  
   \[ \text{type A} \]
   \[ \text{assumes } t_1 + t_3 > 0 \]

$\quad x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$  
\[ \text{similar relation holds outside type A} \]
\[ \text{"long braid move"} \]
\[ \text{with enrichment from parameters} \]

Fibers as Curves:
\[ x_i^2 \rightarrow x_i \]
\[ t_1 + t_2 = k \]
\[ \text{OR after a braid move: } \]
Indexing Faces of Preimage by Words in O-Hecke Algebra

Key Observation About $f_{(i_1, \ldots, i_d)}$:

$\text{im}(F_1) = \text{im}(F_2) \iff x(F_1) = x(F_2)$

equal as $O$-Hecke algebra elements

Thm (Lusztig): If $(i_1, \ldots, i_d)$ is reduced, then $f_{(i_1, \ldots, i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

Observation: "non-reduced" subwords give redundant faces covered by curves, each in single fiber of $f_{(i_1, \ldots, i_d)}$
Properties of Change-of-Coordinates Map Given by Braid Moves

e.g. \((t_1, t_2, t_3) \mapsto \left(\frac{t_2 t_3}{t_1 + t_3}, t_1 + t_3, \frac{t_1 t_2}{t_1 + t_3}\right)\)
in type A

- Tropicalizes to change-of-basis map for Lusztig's canonical bases:
  \((a, b, c) \mapsto (b + c - \min(a, c), \min(a, c), a + b - \min(a, c))\)

- A motivation for development of cluster algebras (\& mutation)

Exercise: check this is an involution.
Proof Strategy (Phrased for Possible Future Application too)

**Set-up:** Continuous, surjective fn

\[ f : P \rightarrow \gamma \]

from convex polytope \( P \) (e.g. \( \Delta_n \)) s.t. \( f \) maps \( \text{int}(P) \) homeomorphically to \( \text{int}(\gamma) \).

**Step 1:** Perform "collapses" on \( \partial P \), each preserving regularity and homeomorphism type, via continuous, surjective collapsing functions \( P \rightarrow P \) yielding \( P/\sim \) with fewer cells s.t. \( x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2) \)

(to accomplish IDs we know are needed by collapsing all non-reduced faces)

**Step 2:** Prove \( \bar{f} : P/\sim \rightarrow \gamma \) is homeomorphism by new regularity criterion

(to prove no further IDs are needed)
Step 1: Collapsing cell $\tilde{\sigma}$ onto cell $p \subseteq \partial \bar{\Sigma}$ within $\partial \bar{\Sigma}$

Thm (M. Brown; Connelly): Any topological manifold with boundary $\partial M$ has a collar (i.e. a nbhd homeomorphic to $\partial M \times [0,1]$).

E.g. $\partial \Sigma \setminus \tilde{\sigma} = \bigcirc$

Plan: Collapse $\tilde{\sigma}$ onto $\bar{\rho} \subseteq \partial \sigma$, stretching collar for $\partial \Sigma \setminus \tilde{\sigma}$ to cover $\bar{\sigma} \setminus \bar{\rho}$, preserving top'l manifold structure ($\exists$ homeomorphism, type: regularity!)

Cell $\tilde{\sigma}$ Being Collapsed

$p \subseteq \partial (\tilde{\sigma})$

Collar for $\partial \Sigma \setminus \tilde{\sigma}$

Cell $\Sigma$

$\omega / \sigma \leq \bar{\Sigma}$

$\uparrow \dim \Sigma = \dim \sigma + 1$
(Mainly Combinatorial) Conditions Allowing Such Face Collapses Across Curves

\[ K_0 \xrightarrow{g_1} K_1 \xrightarrow{g_2} K_2 \rightarrow \ldots \rightarrow K_i \]

polytope!

- collapse face in \( K_i \) across images of parallel line segments in \( K_0 \) satisfying:
  - Distinct endpoints condition (DE):

\[ x \neq x' \]

- Distinct initial points condition (DIP):

\[ x \neq x' \]

- Least upper bound condition (LU3)

  to preclude

(Conditions checkable via 0-Hecke algebra)
Example of a Face Collapse

e.g. $f_{(1,2,3,1,2)}$ face with $t_3 = 0$

collapse across curves defined below

$$(t_1, t_2, 0, t_4, t_5) \rightarrow\begin{cases} x_1(t_1)x_2(t_2)x_1(t_4)x_2(t_5) \\ x_2(t_1')x_1(t_2')x_2(t_4')x_2(t_5) \\ x_2(t_1')x_1(t_2')x_2(t_4't_5)\end{cases}$$

for $t_1 + t_4 > 0$

$\text{Curves: } t_1' = k_1, t_2' = k_2, t_4' + t_5 = k_3$
Step 2: Proving that induced map $\tilde{f}_{(i_1, \ldots, i_d)} : \Delta_n \to \{\text{space of matrices}\}$ on quotient space is a homeomorphism.
New Regularity Criterion for Finite CW Complexes

Preparatory Lemma (H.): Let \( K \) be a finite CW complex w/ characteristic maps \( \{ f_\alpha \} \). Suppose:

1. \( \forall \alpha, f_\alpha(2B^{\dim \alpha}) \) is a union of open cells (surjectivity)

Non-Example:

2. \( \forall f_\alpha \), the preimages of the open cells of codim. one in \( \overline{f_\alpha} \) are dense in \( \overline{2(B^{\dim \alpha})} \)

Non-Example:

Then \( F(K) \) is graded by cell dimension.

Insightful feedback (Quinn): Next theorem “spreads around” injectivity requirement.
Thm (H.) Let $K$ be finite CW complex w.r.t. characteristic maps $\exists f_0, f_3$. Then $K$ is regular w.r.t. $\exists f_0, 3$ $\iff$

(1) $K$ meets requirements of prop'n for $F(K)$ to be graded by cell $\dim$.

(2) $F(K)$ is thin and each open interval $(u, v)$ for $\dim(v) - \dim(u) > 2$ is connected (as graph)

(combinatorial condition)

Non-Example

$\Delta^2 = P_2$

$P_1 \sim P_2$

$\gamma$

$\delta$

$\zeta$

$\zeta = \Pi$

contain $P_1$

$(p, \gamma) = \Pi$

\[ (p, \gamma) = \Pi \]
(3) For each $\alpha$, the restriction of $f_\alpha$ to preimages of codim. one cells in $\bar{e}_\alpha$ is injective.

(topological condition)

Non-Example:

(4) $\forall e_c \subseteq \bar{e}_\alpha$, $f_\alpha$ factors as continuous inclusion $i : B^{\dim \sigma} \rightarrow B^{\dim \alpha}$ followed by $f_\alpha$.

Non-Example:

Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one.

Proof: Induction on difference in dim.
Injectivity of Attaching Maps in Codimension One via Coxeter group strong exchange axiom

\[ S_1S_2S_1S_3 \]

\[ S_1S_2S_1, \quad S_1S_2S_3, \quad -S_2S_1S_3, \quad S_1S_3S_3 \]

Not reduced

Reduced subexpressions of reduced expression obtained by deleting one letter give distinct Coxeter group elements.

In contrast: fails in higher codimension.
Additional Key Challenges

1. O-Hecke algebra lacks inverses
   lacks cancellation law.

   **Key Idea:** find ways to transfer
   properties from Coxeter gp

2. Need change-of-coords for braid
   moves as homeomorphisms on closed cells

   **Key Idea:** induction by embedding
   smaller instance

   \[ \text{im}(f_{i_1i_2\cdots i_{d-1}i_d}) \]

   all possible collapses done
   not in any collapses yet

3. Need maps to extend to full complex

4. Tricky combinatorics to verify
   LUB at each collapsing step.
Other Stratified Spaces with Seemingly Similar Features

1. Totally nonnegative part of Grassmannian

Postnikov: polytope of planar graphs w/ “measurement map” to $Gr_{20}$
+ elaborate theory of planar graphs

Postnikov-Speyer-Williams: $Gr_{20}$ is CW complex (via attaching maps that are not homeomorphisms)

2. Closed cells in totally nonnegative part of loop group

Lam-Polyvyskyy: developed theory of these spaces
3. Totally nonnegative part of flag variety

Rietsch: poset of closure reln's
Marsh-Rietsch: parametrization
Williams: poset is CW poset
Rietsch-Williams: CW complex w/ attaching maps via canonical bases.

Note: our spaces arise as links of cells

4. Stratified spaces of electrical networks

Kenyon-Propp-Wilson, Lam, Curtis-Ingerman-Morrow, Kenyon,...

Open qn: homeomorphism type & other topological structure for these spaces?
A Follow-up Project:
(with Jim Davis & Ezra Miller)

Conjecture (Davis-H-Miller): $f^{-1}_{(i_1,\ldots,i_d)}(p)$
for each $p \in Y_\omega^*$ is a regular CW complex
homeomorphic to a ball with closure
poset dual to face poset for interior of
"subword complex" $\Delta (((i_1,\ldots,i_d), \omega)$.

Remark: Subword complexes introduced by
Knutson & E. Miller as Stanley-Reisner
complexes of initial ideals of coordinate
rings associated to matrix Schubert varieties.
A Poset Map (on Face Posets) induced by \( f(c_{12}, \ldots, c_{1d}) \) (\( e \) implicit Def’n of Subword Complexes)

Boolean Algebra \( B_n \) Bruhat Order

- Apply braid moves \( \triangleleft x_i \mapsto x_i \) to get reduced expression; replace \( x_i \)'s by \( s_i \)'s
- Fibers \( f_{c_2}^{-1}(u) = \exists x \in B_n \mid f(x) > u \) are dual to face posets of subword complexes (fibers as in Quillen's Lemma A)
Description of Fibers via Flow (Based on Collapses) to Base Point

e.g. \[ x_1 \rightarrow x_1 x_2 x_1 \rightarrow x_2 x_3 \rightarrow x_3 \]
\[ x_1 (t_1) x_1 (t_3) x_2 (t_4) x_1 (t_5) x_2 (t_7) x_3 (t_8) x_3 (t_{10}) \]
\[ x_1 (t_1 + t_3) \]
\[ x_2 (t_1') x_1 (t_2') x_2 (t_3' + t_7) \]

Remarks:
- DTM Conjecture \( \Rightarrow \) FS Conjecture (Via CE-Approx. Thm)
- Our proof factors out \( f_{(i_1, \ldots, i_d)} \) as product of "nice" maps
Future (Future) Further Questions:

1. Homeomorphism type for images of fibers of maps yielding:
   - totally nonnegative part of Grassmannian
   - totally nonnegative part of a loop group
   - totally nonnegative part of the flag variety
   - spaces of electrical networks
2. Combinatorics of generalized subword complexes as combinatorial model for fibers
   - Grassmannian?
   - loop group?

Vertex decomposability?
Gallery connectedness?

3. Combinatorics of generalized Bruhat order as combinatorial model for images
   - loop group?
   - electrical networks?

\[ \text{Lam: proved thin} \]

In particular: are these CW posets?
4. Explicit polytope & map for flag variety (perhaps by developing combinatorics of suitable reduced & non-reduced objects for canonical bases explicitly)

5. Explanation for subword complexes arising in seemingly disparate settings?

6. More general setting explaining very strong analogies between reduced words, reduced plabic graphs, etc.

Thank you!