Towards Hessenberg–Schubert calculus

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Outline

Chromatic symmetric functions

Let $A = (V, E)$ be a graph. A proper coloring of A is a function $\kappa : V \to \mathbb{Z}_{>0}$ such that, if $\{v_1, v_2\} \in E$, $\kappa(v_1) \neq \kappa(v_2)$. The chromatic symmetric function χ_A is the "generating function" for proper colorings of A; i.e.

$$\chi_A(\mathbf{x}) = \sum_{\kappa} \prod_{v \in V} x_{\kappa(v)}.$$

For example, if A is 
then $\chi_A(x_1, \ldots) = 6x_1x_2x_3 + \cdots + x_1^2x_2 + \cdots = e_21 + 3e_3$.

Stanley–Stembridge conjecture (Hikita theorem)

Pick n real numbers $a_1 < \cdots < a_n \in \mathbb{R}$, and form a graph A on the vertex set $\{1, \ldots, n\}$ where vertex i is connected to vertex j if $(a_i, a_i + 1) \cap (a_j, a_j + 1) \neq \emptyset$. Call these graphs **unit interval graphs**. (Picking $0 < 2/3 < 4/3$ gives the graph on the previous slide.)

Stanley and Stembridge conjectured (1993) that the chromatic symmetric function of unit interval graphs have positive expansions in terms of the elementary symmetric functions. (This statement is a reduction of the original due to Guay-Paquet.)

This conjecture was recently proved by Hikita by coming up with a probabilistic interpretation of the coefficients of the expansion.

Flags

Fix a positive integer n. A **(complete) flag** is a sequence of subspaces

$$V_\bullet = V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n$$

where $\dim(V_i) = i$.

The **flag variety** is the set of all complete flags.

We have a correspondence

$$g \in GL_n(\mathbb{C}) \leftrightarrow \mathrm{Span}(g_1) \subsetneq \mathrm{Span}(g_1, g_2) \subsetneq \cdots$$

where g_i is the i-th column of g. For any upper triangular matrix b, gb and b represent the same flag. So flag variety is G/B. ($G = GL_n$, B is subgroup of upper triangular matrices.)

Hessenberg varieties

Given a unit interval graph A, let $h_A : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be defined by letting $h_A(i)$ be the largest j such that i and j share an edge in A. By definition, $i \leq h_A(i)$ and $h_G(i) \leq h_A(i+1)$ for all i.

Pick some diagonal $n \times n$ matrix M with distinct entries and define

$$Y_{M,A} = \{V_\bullet \in G/B \mid MV_i \subseteq V_{h(i)}\}.$$

$Y_{M,A}$ is a **(regular semisimple) Hessenberg variety**.

If A is the complete graph, $Y_{M,A} = G/B$. If A is a path, $Y_{M,A}$ is isomorphic to the toric variety of the permutahedron.

Alternate definition of Hessenberg varieties

Alternatively, let $E(A)$ be the edges of A and let H be the subspace of $n \times n$ matrices $[m_{ij}]$ where $m_{ij} = 0$ when $i > j$ and $(j, i) \notin E(A)$. Then

$$Y_{M,A} = \{gB \in G/B \mid g^{-1}Mg \in H\}.$$

This gives $\binom{n}{2} - \#E(A)$ many equations defining $Y_{M,A}$ as a subvariety of G/B.

This also tells us what happens when we translate $Y_{M,A}$ by some element $g' \in G$:

$$g' \cdot Y_{M,A} = Y_{g'Mg'^{-1},A}.$$

Also, if t is a diagonal matrix, and $gB \in Y_{M,A}$, then $tgB \in Y_{M,A}$ (since $t^{-1}Mt = M$).

(Equivariant) cohomology of Hessenberg varieties

The equivariant cohomology ring $H^*_T(Y_{M,A})$ can be defined as follows.
Create a graph (**the GKM graph**) whose vertices are all the permutations in S_n, with an edge between $v, w \in S_n$ if $vr_{ij} = w$ for some transposition r_{ij} where vertices i and j are connected in G.
Label that edge by the polynomial $t_k - t_\ell$ where $r_{k\ell}v = w$.

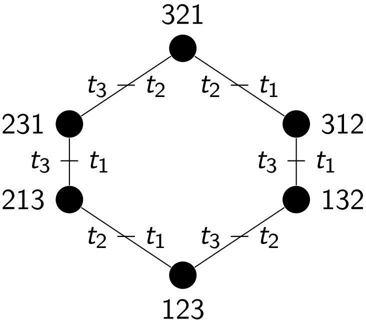
A labelling $(\sigma\mid_w)_{w\in S_n}$ of the vertices by polynomials in $\mathbb{C}[t_1, \ldots, t_n]$ is an element of $H^*_T(Y_{M,A})$ if, whenever v and w share an edge e in this graph, $\sigma\mid_w - \sigma\mid_v$ is divisible by the label on e.

Multiplication of classes is pointwise.

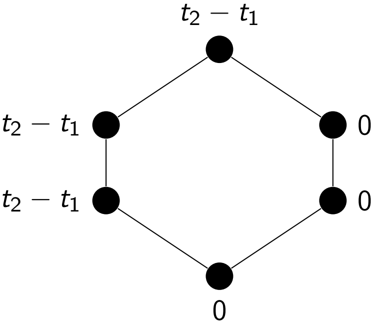
Examples of cohomology classes

If A is 

the GKM graph is

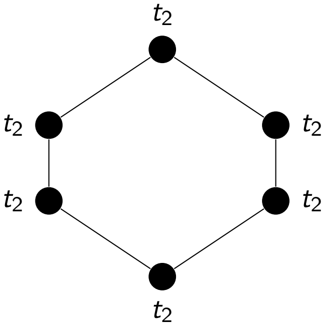


a class is

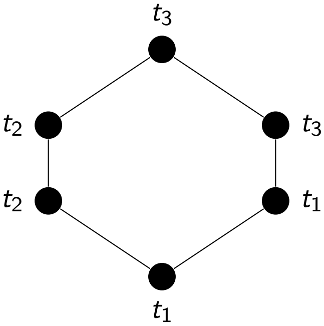


Two important classes

The constant class $\mathbf{t}_2$



the class $\mathbf{x}_1$ $(\mathbf{x}_i \mid_w = t_{w(i)})$



Ordinary cohomology

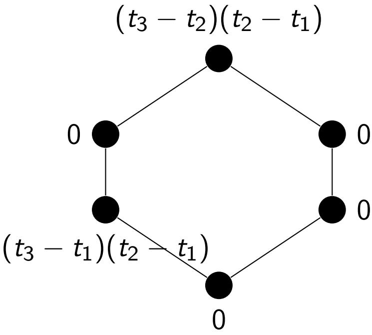
Ordinary cohomology is equivariant cohomology modulo $\langle \mathbf{t}_1, \ldots, \mathbf{t}_n \rangle$. Equivariant cohomology is a free $\mathbb{C}[\mathbf{t}_1, \ldots, \mathbf{t}_n]$-module over ordinary cohomology.

One can also look at equivariant cohomology modulo $\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle$, and equivariant cohomology is a free $\mathbb{C}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ over this ring. There is an equivalent statement of the Stanley–Stembridge conjecture in terms of this unicellular LLT polynomials which is about this ring.

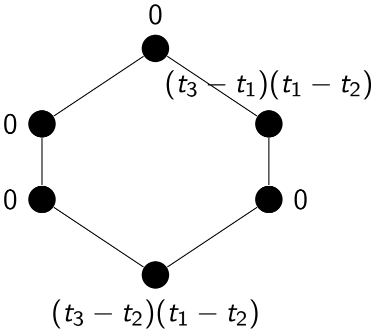
Dot action on cohomology

The group S_n acts on $H_T^*(Y_{M,A})$ on the left by both permuting the variables t_i and moving the labels around.

A class σ



The class $(213) \cdot \sigma$



Connection to Stanley–Stembridge conjecture

Shareshian–Wachs conjectured, and Brosnan–Chow and Guay-Paquet (independently) proved that

$$\omega\chi_A(\mathbf{x}) = \mathcal{F}(H^*(Y_{M,A}))$$

Here, $\mathcal{F}$ is the map that sends the irrep V^λ to the symmetric function s_λ (and direct sum to addition).

Shareshian–Wachs also defined $\chi_A(\mathbf{x}, t)$ which gives a graded version of the Stanley–Stembridge conjecture; this has also been proved.

Dream proof of (graded) Stanley–Stembridge conjecture

Under the map $\mathcal{F}$, we have

$$\mathcal{F}(\mathrm{Ind}_{S_\mu}^{S_n} 1) = h_\mu.$$

This means the Stanley–Stembridge conjecture could be proved by finding a basis of $H^*(Y_{M,A})$ on which S_n acts as it does on (the set!) $\oplus S_n/S_\mu$.

The rest of this talk will be about trying to write a natural geometrically defined basis for $H^*(Y_{M,A})$ as labellings of GKM graphs.

Schubert cells

Given a permutation $w \in S_n$, we can think of w as a permutation matrix in GL_n, and let $C_w = B_- wB/B$, the orbit of the coset $wB \in G/B$ under left multiplication by the group B_- of lower triangular matrices.

The **opposite Schubert cell** C_w is isomorphic to $\mathbb{C}^{\binom{n}{2}-\ell(w)}$. For example, if $w = 41523$, C_w can be identified with the set of matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & b & 0 & 1 & 0 \\ 0 & c & 0 & e & 1 \\ 1 & 0 & 0 & 0 & 0 \\ a & d & 1 & 0 & 0 \end{bmatrix}.$$

Hessenberg-Schubert cells and varieties

The **opposite Schubert variety** X_w is $\overline{C_w}$ It turns out

$$X_w = \bigcup_{v \geq w} C_v$$

The **Hessenberg-Schubert cell** $C_{w,M,A}$ is $C_w \cap Y_{M,A}$.

The **Hessenberg-Schubert variety** is $X_{w,M,A} = \overline{C_{w,M,A}}$.

Unfortunately, $X_{w,M,A} \neq X_w \cap Y_{M,A}$ in general. This also means Hessenberg-Schubert varieties are not unions of Hessenberg-Schubert cells.

A $\mathbb{C}[\mathbf{t}_1, \ldots, \mathbf{t}_n]$-basis for $H_T^*(Y_{M,A})$ is given by the classes of the Hessenberg-Schubert varieties.

Hessenberg–Schubert example

If M is any diagonal matrix with distinct entries, and A is a path, then $Y_{M,A}$ is the set of flags $F_\bullet$ where $MF_i \subseteq F_{i+1}$ for all i.
The **Hessenberg-Schubert cell** $C_{41523,M,A}$ is isomorphic to $\mathbb{C}^2$; it can be identified with the set of matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & b & 0 & 1 & 0 \\ 0 & c & 0 & e & 1 \\ 1 & 0 & 0 & 0 & 0 \\ a & d & 1 & 0 & 0 \end{bmatrix}$$

where $a = b = c = 0$.

Billey's formula

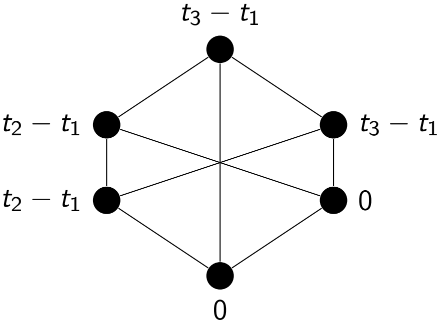
For $G/B = Y_{X,A}$ where A is the complete graph, there is a formula due to Andersen–Jantzen–Soergel and Billey for the classes $[X_w]$. Fix a reduced word $b_1 \cdots b_{\ell(v)}$ for v; this means each b_j is the adjacent transposition $s_k = (k\ \ k+1)$ for some k and $v = b_1 \cdots b_{\ell(v)}$. (Here, $\ell(v)$ is the length of the shortest such expression(s) which is the number of inversions.)

$$[X_w]\mid_v = \sum_{w = b_{i_1} \cdots b_{i_{\ell(w)}}} \prod_{j=1}^{\ell(w)} r_{i_j}.$$

Here we sum over all subwords of $b_1 \cdots b_{\ell(v)}$ that multiply to w. The polynomial $r_i = b_1 \cdots b_{i-1}(t_{k+1} - t_k)$, where $b_i = (k\ \ k+1)$. One goal of this project is to find an analogous formula for arbitrary A.

Billey's formula example

Let $v = 321 = s_1 s_2 s_1$ and $w = 213 = s_1$. There are two subwords that give w; the first gives $(t_2 - t_1)$ and the second gives $s_1 s_2 (t_2 - t_1) = (t_3 - t_2)$, so $[X_{213}] |_{321} = (t_2 - t_1) + (t_3 - t_2) = (t_3 - t_1)$. The entire class $[X_{213}]$ is



Calculating classes: the local Gysin map

A general result of Brion (see Anderson–Fulton) tells us that, for any $X \subseteq Y_{M,A}$,

$$[X]^{Y_{M,A}} = [X]^{G/B}/[Y_{M,A}]^{G/B},$$

where division is pointwise, and

$$[Y_{M,A}]^{G/B} = \prod_{(i,j)\notin E(A)} (\mathbf{x}_i - \mathbf{x}_j).$$

Reduction to admissible elements

A permutation is **admissible** if, for all i, $w^{-1}(i+1) < w^{-1}(i)$ or $(w^{-1}(i), w^{-1}(i+1)) \in E(A)$.

For the graph  $u = 1324$ is not admissible but $v = 2314$ is.

Given an arbitrary w, Sommers–Tymoczko show there is a unique admissible $\widehat{w}$ such that $Inv(\widehat{w}) \cap E(A) = Inv(w) \cap E(A)$. For example, if $w = 1324$, then $\widehat{w} = 2314$.

Let $v = \widehat{w}w^{-1}$. Then $X_{\widehat{w}, vMv^{-1}, A} = vX_{w,M,A}$, and $[X_{w,M,A}] = v^{-1}[X_{\widehat{w},M,A}]$.

(The cohomology class statement is due to Cho–Hong–Lee; the geometric statement is new.)

Tymoczko dimension formula

Tymoczko gives a combinatorial rule for finding a set $D(w, A)$ of non-inversions of w such that $\dim(C_{w,M,A}) = D(w, A)$. One can also use this rule to find a set $N(w, A) \subseteq \widehat{E(A)}$ so that $C_{w,M,A}$ is cut out from C_w by the equations in $N(w, A)$.

If $D(v, A) \leq D(w, A)$ for all $v \geq w$, then

$$[X_w] \prod_{(i,j)\in N(w,A)} (\mathbf{x}_i - \mathbf{x}_j) = \sum_{v: v\geq w, D(v,A)=D(w,A)} [X_{v,M,A}],$$

and this can be used to calculate $[X_{w,M,A}]$.

Reduction to partial flags

In at least some of the cases where this doesn't work, the definition of $Y_{M,A}$ only depends on some of the subspaces in the flag $F_\bullet$, and there are other tricks coming from working with the partial flag variety.

The permutahedral case

This is the special case where A is a path, and $Y_{M,A}$ is the toric variety of the permutahedron.

The admissible permutations are of the form $w_0 w_{0,I}$; we take $w_0 = n \cdots 1$ and reverse some disjoint blocks. The set $E(A)$ captures precisely the descents of w and $\dim(X_{w,M,A})$ is the number of ascents of w. For each permutation w, the permutation $\widehat{w}$ is the longest one with the same ascents/descents.

Classes in permutahedral case

Using the formula of Anderson–Tymoczko,

$$[X_{w,M,A}]^{G/B} = v \cdot \left(\frac{|W_P|}{n!} \prod_{i+1<j} (\mathbf{x}_i - \mathbf{x}_j) \prod_{i:w(i)<w(i+1)} (\mathbf{x}_i - \mathbf{x}_{i+1})\right),$$

where P is the parabolic subgroup generated by the ascents of w.
In this case (and probably all cases?), results extend to all regular M, meaning that the Jordan blocks of M all have different eigenvalues.
Cho–Hong–Lee have given a "dream proof" of Stanley–Stembridge in this case.

Cell closures in permutahedral case

In the permutahedral case, $X_{w,M,A}$ is a product of smaller permutahedral varieties (according to the ascents of w). In particular, if w is admissible,

$$X_{w,M,A} = \bigcup_{v \geq w} C_{w,M,A}.$$

This means, for general w, $X_{w,M,A} \cap C_{v,M,A} \neq \emptyset$ if and only if v is obtained from w by undoing ascents of w, and $\dim(X_{w,M,A} \cap C_{v,M,A})$ is the number of ascents of v that are also ascents of w.

Cell closure example

For $w = 41523$, $X_{w,M,A}$ intersects

- $C_{41532,M,A}$ in dimension 1
- $C_{45123,M,A}$ in dimension 1 (but $C_{45123,M,A}$ has dimension 2).
- $C_{45132,M,A}$ in dimension 0 (but $C_{45132,M,A}$ has dimension 2).

The End

Thank you!