# Totally nonnegative matrices, chain enumeration and zeros of polynomials

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joint work with
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# Totally nonnegative matrices

- A matrix with real entries is totally nonnegative $(\mathrm{TN})$ if all its minors are nonnegative.
- Important in linear algebra, statistics, analysis, geometry, algebraic combinatorics and physics.

Felix Gantmacher

Mark Krein

Samuel Karlin

# A general theorem about totally nonnegative matrices

- Let $R = (r_{n,k})_{n,k=0}^{N}$, where $N \in \mathbb{N} \cup \{\infty\}$, be a lower triangular matrix whose diagonal entries are all equal to one.
- Define a sequence of polynomial: $p_0(t) = 1$, and

$$p_n(t) = t \sum_{k=0}^{n-1} r_{n,k} \cdot p_k(t), \qquad n > 0.$$

- Theorem (B., Saud, 2024). If $R$ is $\mathrm{TN}$, then all zeros of $p_n(t)$ are real and located in the interval $[-1, 0]$.
- If $r_{n,k} = 1$, then $p_n(t) = t(1+t)^{n-1}$.
- If $r_{n,k} = \binom{n}{k}$, then $p_n(t) = \sum_{k=0}^{n} k! S(n,k) t^k$, where $S(n,k)$ are the Stirling numbers of the second kind.

# Face numbers of simplicial complexes

- Recall the *f-polynomial* of a $(d-1)$-dimensional simplicial complex $\Delta$:

$$f_\Delta(t) = \sum_{i=0}^{d} f_{i-1}(\Delta) t^i = \sum_{S \in \Delta} t^{|S|},$$

  where $f_{-1}(\Delta) = 1$, and

$$f_j(\Delta) = \text{ number of } j\text{-dimensional faces of } \Delta.$$

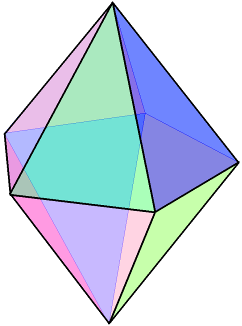
- The *h-vector* of $\Delta$ is the vector $(h_0(\Delta), \ldots, h_d(\Delta))$ for which

$$f_\Delta(t) = \sum_{i=0}^{d} h_i(\Delta) \cdot t^i (1+t)^{d-i}.$$

- We are interested in inequalities for $f$-vectors and $h$-vectors for various families of complexes.

# *h*-vectors of simplicial complexes

- For many important classes of simplicial complexes, the *h*-vectors are nonnegative:
- Cohen-Macaulay complexes, $\Leftarrow$ shellable complexes,
- for example face lattices of simplicial polytopes.



# Zeros and inequalities

Let $\mathcal{A} = a_0, a_1, \ldots, a_n$ be a sequence of positive numbers.

(U) $\mathcal{A}$ is unimodal if there is an index $m$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

(LC) $\mathcal{A}$ is log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i < n$.

(ULC) $\mathcal{A}$ is ultra log-concave if

$$\frac{a_i^2}{\binom{n}{i}^2} \geq \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}}$$

for all $1 \leq i < n$.

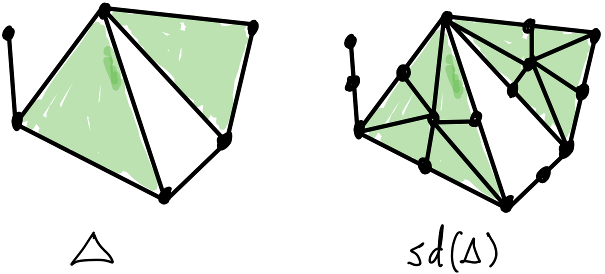
(RR) The polynomial $a_0 + a_1 t + \cdots + a_n t^n$ is real-rooted.

(RR) $\Longrightarrow$ (ULC) $\Longrightarrow$ (LC) $\Longrightarrow$ (U)

# Barycentric subdivision

- The barycentric subdivision $\mathrm{sd}(\Delta)$ of $\Delta$ is the simplicial complex on $\Delta \setminus \{\varnothing\}$ with faces $\{S_1, \ldots, S_k\}$ for which

$$\varnothing \subset S_1 \subset S_2 \subset \cdots \subset S_k \in \Delta.$$




- $\Delta \longmapsto \mathrm{sd}(\Delta)$ defines a linear map $\mathrm{SD} : \mathbb{R}[t] \to \mathbb{R}[t]$ for which $\mathrm{SD}(f_\Delta(t)) = f_{\mathrm{sd}(\Delta)}(t)$:

$$t^n \longmapsto \sum_{k=0}^{n} k! S(n,k) t^k.$$

# Barycentric subdivision

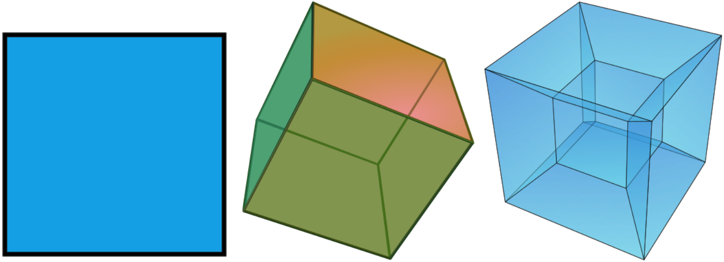
- Theorem (B., 2006). If $f(t)$ is a polynomial of the form

$$f(t) = \sum_{i=0}^{d} h_i \cdot t^i (1+t)^{d-i}, \qquad \text{where } h_i \geq 0 \text{ for all } i,$$

  then $\mathrm{SD}(f(t))$ is real-rooted.
- Corollary (Brenti, Welker, 2008). If the $h$-vector of $\Delta$ is nonnegative, then all zeros of the $f$-polynomial of $\mathrm{sd}(\Delta)$ are real.
- Corollary. If $\mathcal{P}$ is a simplicial polytope, then the $f$-polynomial of the barycentric subdivision of $\mathcal{P}$ is real-rooted.
- Conjecture (Brenti, Welker, 2008). The $f$-polynomial of the barycentric subdivision of a polytope is real-rooted.

# Barycentric subdivision

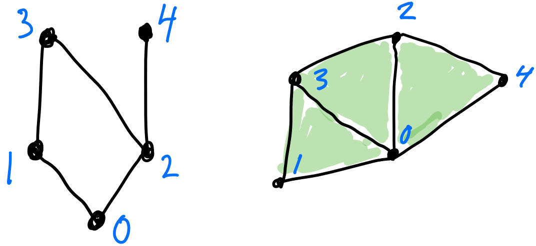
- Athanasiadis (2021) proved Brenti and Welker's conjecture for cubical polytopes, i.e., polytopes whose facets are $n$-dimensional (hyper-)cubes $[0,1]^n$.



- Can we generalize these results to other complexes such as $q$-complexes, cubical complexes, CW-complexes,...?

# Order complexes of posets

- A chain in a poset $P$ is a totally ordered subset of $P$.
- The order complex $\Delta(P)$ is the simplicial complex consisting of all chains of $P$.




- Notice that $\mathrm{sd}(\Delta)$ is the order complex of $P = \Delta \setminus \{\varnothing\}$.
- The chain polynomial of $P$ is the $f$-polynomial of $\Delta(P)$.

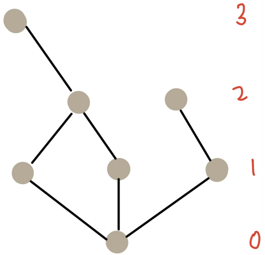
$$1 + 5t + 7t^2 + 3t^3 = (1 + 3t)(1 + t)(1 + t)$$

# Real-rooted chain polynomials

- Conjecture (Neggers, 1978). $P$ is a distributive lattice. (counterexamples by Stembridge, 2007).
- (Stanley, 1998) $P$ is $(3+1)$-free.
- (B., 2006, Brenti-Welker, 2008) $P$ is a simplicial complex with nonnegative $h$-vector.
- (Athanasiadis, 2021) $P$ is a cubical complex with nonnegative cubical $h$-vector.
- Conjecture (Athanasiadis, Kalampogia-Evangelinou, 2023). $P$ is a geometric lattice.

# Graded posets

- A poset $P$ with a minimal element $\hat{0}$ is ranked if there exists a function $\rho : P \to \mathbb{N}$ such that
  - $\rho(\hat{0}) = 0$, and
  - $\rho(y) = \rho(x) + 1$ whenever $y$ covers $x$ in $P$.




- $\rho$ is the rank function of $P$, and $\rho(x)$ is the rank of $x$.
- The $f$-polynomial of a finite ranked poset $P$ is

$$f_P(t) = \sum_{x \in P} t^{\rho(x)}.$$

- What is the $h$-vector of a poset?

# Rank uniform posets

- A locally finite ranked poset $P$ is called rank uniform if for all $x, y \in P$ with $\rho(x) = \rho(y)$,

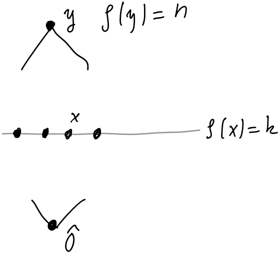
$$\{z \in P : z \leq x\} \cong \{z \in P : z \leq y\}.$$

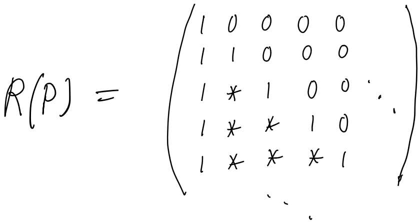
- Infinite boolean algebra: $\mathbb{B}$ is the set of all finite subsets of $\mathbb{N}$. Partial order: containment.
- Its $q$-analog: $\mathbb{B}(q)$ is the set of all finite dimensional subspaces of $\oplus_{k=1}^{\infty}\mathbb{F}_q$, where $\mathbb{F}_q$ is a finite field. Partial order: containment.
- Cubical posets $C$, i.e., $\{z \in C : z \leq x\}$ is isomorphic to a hypercube for each $x \in C$.
- Dual partition lattice $\Pi'$ of all partitions of $\mathbb{N}$ with finitely many blocks.

# Rank uniform posets

- Define a matrix $R(P) = (r_{n,k}(P))_{n,k}$ by: Let $y \in P, \rho(y) = n$ be fixed

$$r_{n,k}(P) = |\{x : x \leq y \text{ and } \rho(x) = k\}|.$$



# Rank uniform posets

- $r_{n,k}(P) = |\{x : x \leq y \text{ and } \rho(x) = k\}|$, where $\rho(y) = n$.

$$r_{n,k}(\mathbb{B}) = \binom{n}{k}, \quad r_{n,k}(\mathbb{B}(q)) = \binom{n}{k}_q, \quad r_{n,k}(\Pi') = S(n+1, k+1),$$

$$r_{n,k}(C) = \begin{cases} 1 & \text{if } k = 0, \\ 2^{n-k}\binom{n-1}{k-1} & \text{if } k \geq 1. \end{cases}$$

- What do these matrices have in common?
- They are totally nonnegative $(\mathrm{TN})$.
- We say that a rank uniform poset $P$ is $\mathrm{TN}$ if $R(P)$ is $\mathrm{TN}$.
- Examples: Chains, $\mathbb{B}$, $\mathbb{B}(q)$, face lattice of a hypercube, $\Pi'$.

# $\mathrm{TN}$-matrices and $\mathrm{TN}$-posets

- Recall the definition: $p_0(t) = 1$, and

$$p_n(t) = t \sum_{k=0}^{n-1} r_{n,k} \cdot p_k(t), \qquad n > 0.$$

- If $R = R(P)$, then

$$p_n(t) = \sum_{j \geq 1} |\{\hat{0} = x_0 < x_1 < \cdots < x_{j-1} < x_j = x\}| \cdot t^j,$$

  where $x$ is a fixed element of rank $n$.
- Theorem (B., Saud, 24). If $P$ is $\mathrm{TN}$, then all zeros of $p_n(t)$ are real and located in the interval $[-1, 0]$.
- Corollary (B., Saud, 24). The chain polynomial of any finite $\mathrm{TN}$-poset is real-rooted.

# Resolvable matrices

- Let $R = (r_{n,k})_{n,k=0}^{N}$, where $N \in \mathbb{N} \cup \{\infty\}$, be a lower triangular matrix whose diagonal entries are all equal to one.
- We call $R$ resolvable if there are polynomials $R_{n,k}(t)$, $0 \leq k \leq n \leq N$, and nonnegative real numbers $\lambda_{n,k}$ for which
  - $R_{n,0}(t) = \sum_{k=0}^{n} r_{n,k} t^k$ and $R_{n,n}(t) = t^n$,
  - $R_{n,k}(t)$ is monic of degree $n$,
  - $t^k$ divides $R_{n,k}(t)$,
  - $R_{n+1,k}(t) = R_{n+1,k+1}(t) + \lambda_{n,k} R_{n,k}(t)$.
- Example. If $r_{n,k} = \binom{n}{k}$, then $R_{n,k}(t) = t^k(1+t)^{n-k}$.
- Theorem(B., Saud, 24). $R$ is resolvable if and only if $R$ is $\mathrm{TN}$.
- Example. For $R(C)$ where $C$ is the hypercube:

$$R_{n,k}(t) = \begin{cases} 1 + t(2+t)^{n-1} & \text{if } k = 0, \\ t^k(1+t)(2+t)^{n-k-1} & \text{if } 0 < k < n, \\ t^n & \text{if } k = n. \end{cases}$$

# Generalized $h$-vectors for general "complexes"

- Let $P$ be a fixed $\mathrm{TN}$-poset.
- A poset $Q$ is a $P$-poset if

$$\{z \in Q : z \leq_Q x\} \cong \{z \in P : z \leq_P y\}$$

whenever $x \in Q$ and $y \in P$ have the same rank.

- For example if $\Delta$ is a simplicial complex, then $\Delta$ is a $\mathbb{B}$-poset.
- We may now extend the notion of $h$-vectors to $\mathrm{TN}$-posets:
- If $Q$ is a $P$-poset of rank $n$, then the $h$-vector of $Q$ is the vector $(h_0(Q), \ldots, h_n(Q))$ for which

$$f_Q(t) = \sum_{x \in Q} t^{\rho(x)} = \sum_{k=0}^{n} h_k(Q) \cdot R_{n,k}(t).$$

- This notion extends the usual notion of $h$-vectors as well as cubical $h$-vectors (Adin, 1996).

# Generalized $h$-vectors

- Let $P$ be a fixed $\mathrm{TN}$-poset.
- Theorem (B., Saud, 2024). If $Q$ is a $P$-poset with nonnegative $h$-vector, then the chain polynomial of $Q$ is real-rooted.
- This is a vast generalization of the theorems of Brenti-Welker and Athanasiadis.
- Applies to boolean posets, cubical posets, $q$-analogs of boolean algebras, partition lattices,...
- Corollary. If $P$ is a cubical poset with nonnegative $h$-vector, then the chain polynomial of $P$ is real-rooted.
- First proved by other methods by Athanasiadis, 2021.

# Rank selection of ranked posets

- For $S \subset \mathbb{N}$, let $P_S = \{x \in P : \rho(x) \in S\}$.
- If $P$ is $\mathrm{TN}$ and $0 \in S$, then $P_S$ is $\mathrm{TN}$.
- Recall the flag $f$-vector of $P$ is $\alpha_P : 2^{\{1,2,\ldots\}} \to \mathbb{Z}$,

$$\alpha_P(S) = \#\text{ maximal chains of } P_S,$$

- and the flag $h$-vector, $\beta_P : 2^{\{1,2,\ldots\}} \to \mathbb{Z}$,

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_P(T).$$

# Rank selection

- Theorem (B., Saud, 2024). Let $P$ be a bounded rank uniform poset of rank $n$. If $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq [n-1]$, then

$$\beta_P(S) = \det(R[\{s_1, \ldots, s_k, n\}, \{0, s_1, \ldots, s_k\}]),$$

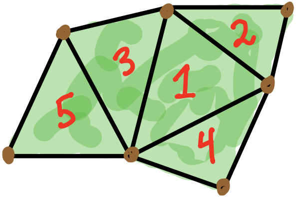
  where $R = R(P)$.
- The flag $h$-vector of any Cohen-Macaulay poset is nonnegative. The same is true for $\mathrm{TN}$-posets.
- Corollary (B., Saud, 2024). The flag $h$-vector of a $\mathrm{TN}$-poset is nonnegative.

# Shellings of simplicial complexes

- Let $\Delta$ be a pure simplicial complex of dimension $d$.
- A total order $F_1, F_2, \ldots, F_m$ of the facets of $\Delta$ is called a shelling if for all $j > 1$:

$$\langle F_j \rangle \cap \bigcup_{i<j} \langle F_i \rangle, \qquad \langle F \rangle = \{S : S \subseteq F\},$$

is pure of dimension $d - 1$.




- Shellable complexes are Cohen-Macaulay and have nonnegative $h$-vectors.

# $q$-posets

- Recall that $\mathbb{B}(q)$ is the set of all finite dimensional subspaces of $\oplus_{k=1}^{\infty}\mathbb{F}_q$.
- In our terminology, a $q$-poset $P$ is a $\mathbb{B}(q)$-poset, i.e., $\{y \in P : y \leq x\}$ is isomorphic to the lattice of subspaces of $\mathbb{F}_q^n$, where $n = \rho(x)$.
- $q$-posets were introduced by Rota, and recently studied by Alder, Ghorpade, Pratihar, Randrianarisoa, Verdure,...
- Recall $R(\mathbb{B}(q)) = \left(\binom{n}{k}_q\right)_{n,k=0}^{\infty}$.
- $R(\mathbb{B}(q))$ is resolvable $(\mathrm{TN})$ with

$$R_{n,k}(t) = q^{k(n-k)}t^k \sum_{i=0}^{n-k} q^{-ki}\binom{n}{k}_q t^i.$$

# Shellings of $q$-posets

- Let $P$ be a pure $q$-poset of rank $d$.
- A total order $F_1, F_2, \ldots, F_m$ of the facets (maximal elements) of $P$ is called a shelling if for all $j > 1$:

$$\langle F_j \rangle \cap \bigcup_{i<j} \langle F_i \rangle, \qquad \langle F \rangle = \{x : x \leq F\},$$

  is pure of rank $d - 1$.
- Question (Alder, 2010). Is there a natural notion of $h$-vectors of $q$-posets such that
  - it reduces to the usual one when $q = 1$, and
  - shellable $q$-posets have nonnegative $h$-vectors?
- We have a notion of $h$-vector: If $P$ has rank $n$, then

$$f_P(t) = \sum_{k=0}^{n} h_k(P) \cdot R_{n,k}(t).$$

# $q$-matroids

- Theorem (B., Saud, 2024). Shellable $q$-posets have nonnegative $h$-vectors.
- Corollary. Chain polynomials of shellable $q$-posets are real-rooted.
- A $q$-matroid on $\mathbb{B}_n(q)$ is map $\varphi : \mathbb{B}_n(q) \to \mathbb{N}$ such that
  (a) $\varphi(x) \leq \dim(x)$ for all $x$,
  (b) $\varphi(x) \leq \varphi(y)$ whenever $x \leq y$,
  (c) $\varphi(x \vee y) + \varphi(x \wedge y) \leq \varphi(x) + \varphi(y)$ for all $x, y$.
- Studied by Crapo, Jurrius, Pellikaan, Johnsen, Verdure,...
- The set $I(\varphi) = \{x \in \mathbb{B}_n(q) : \varphi(x) = \dim(x)\}$ of "independent spaces" is a pure $q$-poset.
- Theorem (Ghorpade *et al.*, 2022). $I(\varphi)$ is shellable.
- Corollary (B., Saud, 2024). The chain polynomial of $I(\varphi)$ is real-rooted.

# Characteristic polynomials of hyperplane arrangements

- Let $\mathcal{H} = \{H_1, \dots, H_m\}$ be a collection of hyperplanes in $\mathbb{F}_q^n$.
- Recall that the characteristic polynomial of $\mathcal{H}$ is

$$\chi_{\mathcal{H}}(t) = \sum_{A \subseteq [m]} (-1)^{|A|} t^{\dim(\cap_{i \in A} H_i)} = \sum_{i=0}^{n} w_i t^i.$$

- Problem. Describe all linear inequalities that are satisfied by the coefficients of characteristic polynomials of hyperplane arrangements in $\mathbb{F}_q^n$.
- $w_n = 1$,
- $(-1)^{n-i} w_i \geq 0$, ...

# Characteristic polynomials of hyperplane arrangements

- Let $\chi^q_{n,0}(t) = t^n$ and $\chi^q_{n,k}(t) = t^{n-k}(t-1)(t-q)\cdots(t-q^{k-1})$, $0 < k \le n$.
- Theorem (B., Saud, 2024). There are unique nonnegative numbers $\theta_i(\mathcal{H})$ such that $\theta_0(\mathcal{H}) + \cdots + \theta_n(\mathcal{H}) = 1$, and

$$\chi_{\mathcal{H}}(t) = \sum_{i=0}^{n} \theta_i(\mathcal{H}) \cdot \chi^q_{n,i}(t).$$

- Theorem (B., Saud, 2024). The convex hull of the set of all characteristic polynomials of hyperplane arrangements in $\mathbb{F}_q^n$ is equal to the simplex

$$\left\{ \sum_{k=0}^{n} \theta_k \cdot \chi^q_{n,k}(t) : \theta_k \geq 0 \text{ for all } k, \text{ and } \theta_0 + \theta_1 + \cdots + \theta_n = 1 \right\}.$$

# Geometric lattices

- Conjecture (Athanasiadis, Kalampogia-Evangelinou, 2023). The chain polynomial of any geometric lattice is real-rooted
- Proved for the subspace-lattice of $\mathbb{F}_q^n$ and partition lattices.
- Using our theorem we prove that the conjecture is true for Dowling-lattices and paving matroids.

# Pólya frequency sequences

- A sequence $\{a_i\}_{i=0}^{\infty}$ is a Pólya frequency sequence if $(a_{i-j})_{i,j=0}^{\infty}$ is $\mathrm{TN}$.
- How does the main theorem translate for Pólya frequency sequences?

  Theorem (B., Saud, 2024). Suppose

  $$f(x) = Cx^N e^{\gamma x} \prod_{i=1}^{\infty} \frac{1+\alpha_i x}{1-\beta_i x},$$

  $C, \gamma, \alpha_i, \beta_i \geq 0$, and consider

  $$\frac{1}{1-t(f(x)-f(0))} = \sum_{n=0}^{\infty} f_n(t)x^n \in \mathbb{R}[[x,t]].$$

  Then $f_n(t)$ is real-rooted for each $n$.

# A conjecture of Forgács and Tran

- The case when

$$g(x) = \frac{x^r}{\prod_{i=1}^{n}(1-\beta_i x)}, \quad \text{where } \beta_i > 0 \text{ for all } i \text{ and } r \in \mathbb{Z}_{>0}.$$

was conjectured by Forgács and Tran (2016).