Macdonald polynomials and the multispecies zero range process

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joint with Arvind Ayyer and James Martin,
`arXiv:2022.06117` + upcoming

overview

1. Motivation: Macdonald polynomials and interacting particle systems
2. A new combinatorial formula for $\widetilde{H}_\lambda(X; q, t)$
3. Multispecies Totally Asymmetric Zero Range Process (mTAZRP)
4. Markov chain on tableaux
5. Observables

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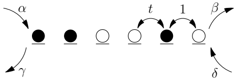

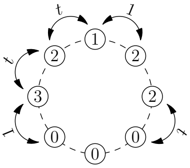

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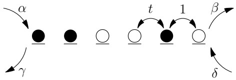

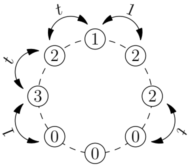

- we are interested in studying integrable systems whose exact solutions (stationary distributions) can be expressed in terms of combinatorial formulas or special functions (e.g. Macdonald polynomials)

Macdonald polynomials

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i. orthogonal with respect to $\langle , \rangle$

ii. upper triangular with respect to $\{m_\lambda\}$:

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\mu\lambda} m_\mu$$

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- $s_\lambda = \sum_\sigma x^\sigma$ where σ is a semi-standard filling of the Young diagram of shape λ

  E.g. the following are the fillings of shape $(2,1)$ on 3 letters:

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

$$s_{(2,1)} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 = m_{(2,1)} + m_{(1,1,1)}$$

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Let $\langle , \rangle_{q,t}$ be the inner product on $\Lambda(q, t)$ given by:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i \geq 1} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$

Then $\{P_\lambda\}$ is the unique basis of $\Lambda(q, t)$ that is uniquely determined by:

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- Example:

$$P_{(2,1)}(X; q, t) = m_{(2,1)} + \frac{(1 - t)(2 + q + t + 2qt)}{1 - qt^2} m_{(1,1,1)}.$$

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- obtained from a normalized form of $P_\lambda(X; q, t)$ by plethystic substitution:

$$\widetilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda \left[\frac{X}{1 - t^{-1}}; q, t^{-1} \right]$$

  where J_λ is a scalar multiple of P_λ.

  Example: $\widetilde{H}_{(2,1)}(X; q, t) = m_{(3)} + (1 + q + t) m_{(2,1)} + (1 + 2q + 2t + qt) m_{(1,1,1)}$

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- Haglund-Haiman-Loehr '04 gave formulas for P_λ and $\widetilde{H}_\lambda$ as sums over tableaux with statistics maj and (co)inv:
  - $P_\lambda(X;q,t) = \sum_{\substack{\sigma \in \mathrm{dg}(\lambda) \\ \sigma \text{ non-attacking}}} q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} x^\sigma \prod_u \frac{1-t}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}}$
  - $\widetilde{H}_\lambda(X;q,t) = \sum_{\sigma \in \mathrm{dg}(\lambda)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)} x^\sigma$

Combinatorial formulas

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- Garbali-Wheeler '20 gave a formula for $\widetilde{H}_\lambda$ using integrability, in terms of colored paths
- Corteel-Haglund-M-Mason-Williams '20 gave a "compressed" formula for $\widetilde{H}_\lambda$. Using multiline queues and the plethystic relationship between $\widetilde{H}_\lambda$ and P_λ, also conjectured a new formula for $\widetilde{H}_\lambda$ with statistics maj and a new statistic quinv:

$$\widetilde{H}_\lambda(X;q,t) = \sum_{\sigma \in \mathrm{dg}(\lambda)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^\sigma$$

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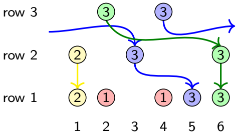
- a multiline queue (MLQ) of type λ, n is an arrangement and pairing of balls on a $n \times \lambda_1$ lattice, with λ'_j balls in row j.



$\lambda = (3, 3, 2, 1, 1, 0)$

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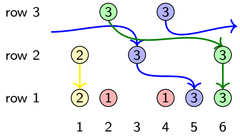

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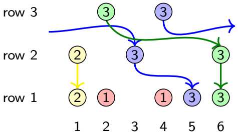

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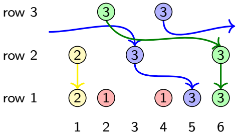

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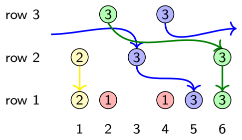

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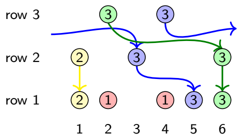

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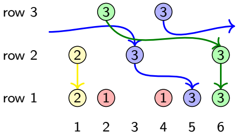

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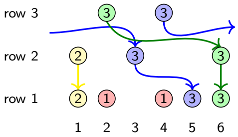

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6	3	1		
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  - skipped balls in the MLQ "correspond" to a coinv statistic in t
  - wrapping balls in the MLQ correspond to a maj statistic in q
- Can be represented by a non-attacking tableau, where each string is mapped to a column of the same height, recording the position of each ball in the MLQ.

From ASEP to Macdonald polynomials

Theorem (Martin '18, Corteel-M-Williams '18)

The (unnormalized) stationary probability of state α of the mASEP is

$$\widetilde{\Pr}(\alpha)(t) = \sum_{M:\ \text{row } 1=\alpha} wt(M)(1,\ldots,1;1,t)$$

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Theorem (Cantini–de Gier–Wheeler '15)

The partition function of ASEP(λ, n) *is a specialization of the Macdonald polynomial:*

$$P_{\lambda}(1,\ldots,1;1,t) = \mathcal{Z}_{\lambda,n}(t) = \sum_{\alpha\in S_n\cdot\lambda} \widetilde{\Pr}(\alpha)(t).$$

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$$P_\lambda(x_1,\ldots,x_n;q,t) = \sum_{M\in \mathrm{MLQ}(\lambda,n)} \mathrm{wt}(M)(x_1,\ldots,x_n;q,t)$$

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This formula essentially coincides with that of Lenart '09 for λ with distinct parts.

Example for $P_{(2,1)}(x_1, x_2, x_3; q, t)$

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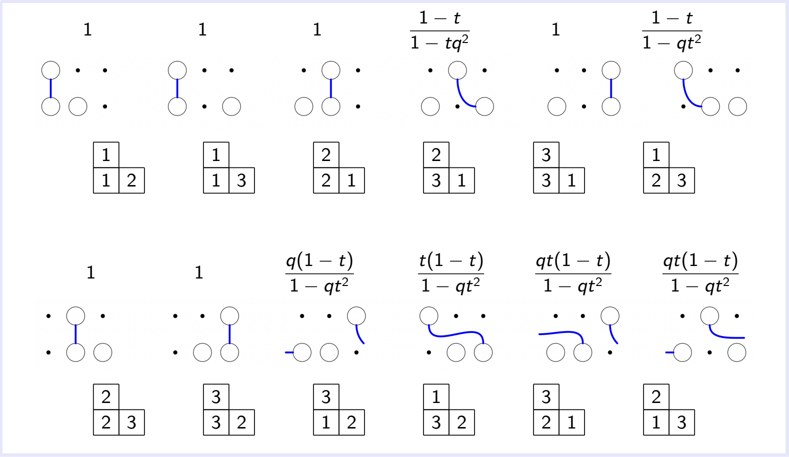

From multiline queues to a new formula for $\widetilde{H}_\lambda$

- Recall: $\widetilde{H}_\lambda(X; q, t)$ is obtained from the integral form of P_λ via plethysm:

$$\widetilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda \left[\frac{X}{1 - t^{-1}}; q, t^{-1} \right]$$
$$= f_\lambda(q, t)\ P_\lambda \left(x_1, x_1 t^{-1}, x_1 t^{-2}, \ldots, x_2, x_2 t^{-1}, x_2 t^{-2}, \ldots\ ;\ q, t^{-1} \right)$$

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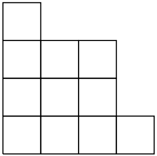
- the resulting objects are of the same flavor as multiline queues, except that multiple balls are allowed at each location. (This translates to removing the "non-attacking" condition from the corresponding tableaux)

tableaux formulas: notation and statistics

- dg(λ) (the diagram of $\lambda = (\lambda_1, \ldots, \lambda_k)$) consists of k bottom justified columns with λ_i boxes, from left to right

$\lambda = (4, 3, 3, 1)$

dg(λ) =



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$$\sigma = \begin{array}{|c|c|c|c|} \hline 4 & \multicolumn{3}{c}{} \\ \cline{1-3} 2 & 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 1 & 1 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 3 & 4 \\ \hline \end{array}$$

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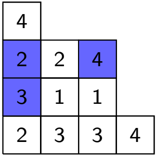
$\sigma =$ 

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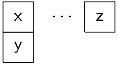

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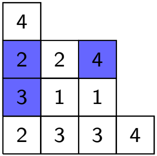
$\sigma =$ 

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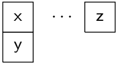

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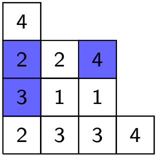
$\sigma =$ 

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$\mathrm{maj}(\sigma) = 6$

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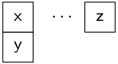

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a new statistic: queue-inversion

$\sigma =$

4			
2	2	4	
3	1	1	
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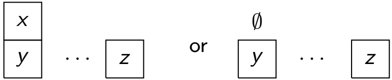
- an *L*-triple is a triple of cells in the configuration:



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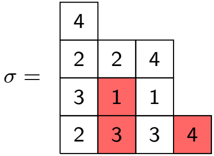

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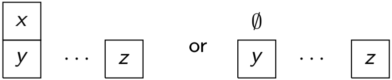

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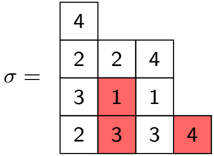

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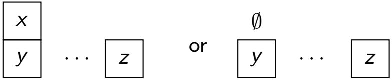

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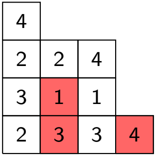

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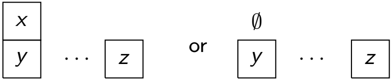

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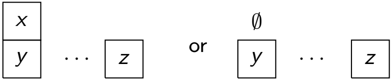

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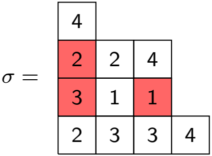

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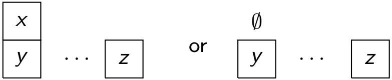

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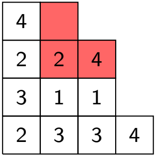

$\text{quinv}(\sigma) = 4$

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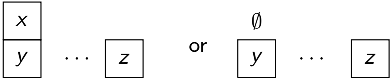

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Example: $\widetilde{H}_{(2,1)}(X;q,t)$

$$\widetilde{H}_{(2,1)}(x_1,x_2;q,t) = m_{(3)} + (1+t+q)m_{(2,1)} + (1+2t+2q+qt)m_{(1,1,1)}$$

- (AMM) $$\widetilde{H}_\lambda(X;q,t) = \sum_{\sigma:\mathrm{dg}(\lambda)\to\mathbb{Z}_+} q^{\mathrm{maj}(\sigma)} t^{\mathrm{quinv}(\sigma)} x^\sigma$$

1	2	1	1	1	2	3	2	1	3
1 1	1 1	1 2	2 1	2 3	1 3	1 2	3 1	3 2	2 1
m_3	$q\, m_{21}$	$t\, m_{21}$	m_{21}	$t\, m_{111}$	$q\, m_{111}$	$qt\, m_{111}$	$t\, m_{111}$	m_{111}	$q\, m_{111}$

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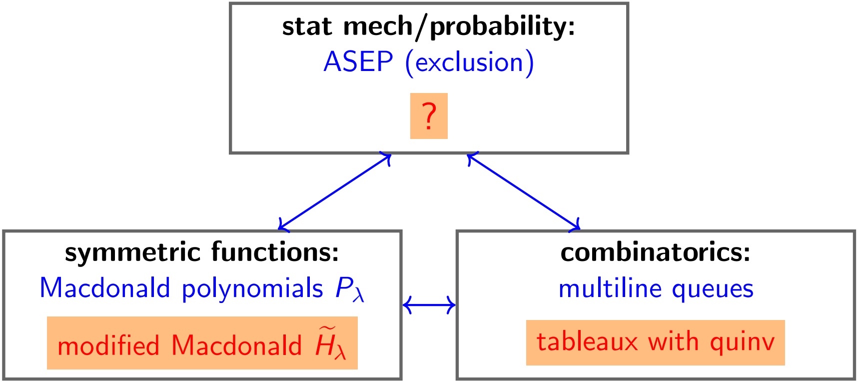
- while the inv and quinv statistics appear very similar, there does not seem to be an easy way to go from one to the other – is there a bijective proof?

Motivation



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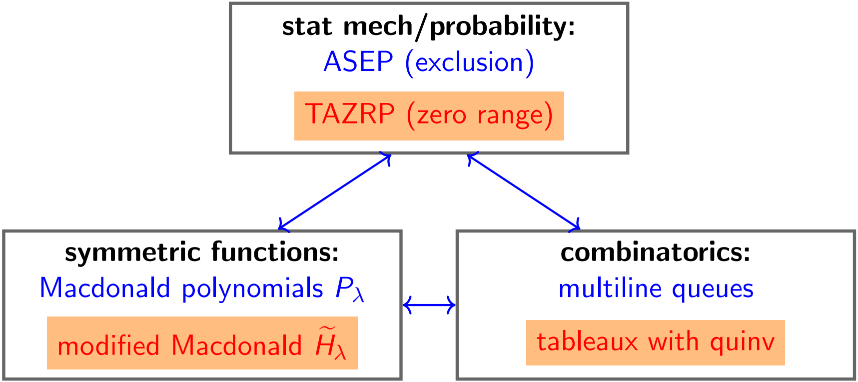
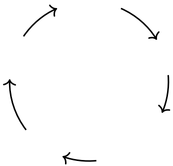


What is the analogous interacting particle system whose partition function is a specialization of $\widetilde{H}_\lambda$?

totally asymmetric zero range processes (TAZRP)

- continuous-time stochastic processes (Spitzer '70), can be defined on arbitrary graphs. In our case, we have a circular lattice with n sites.



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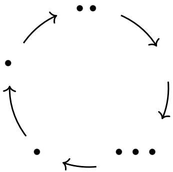

Here, $n = 5$, $k = 7$

$$\tau = (\ 11 \mid \cdot \mid 111 \mid 1 \mid 1\ )$$

- simplest case: there are k indistinguishable particles, moving counter-clockwise. A configuration $\tau = (\tau_1, \ldots, \tau_n)$ is any allocation of the k particles on the n sites.

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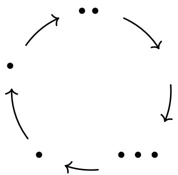

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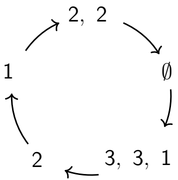

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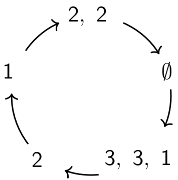

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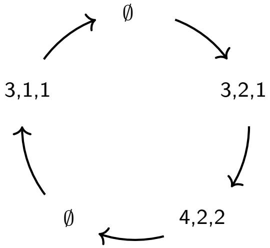
- multispecies variant: we now allow different particle types, labeled by integers (particles of the same type are still indistinguishable)
- Kuniba–Maruyama–Okado (2015+) (and others) have studied many multispecies variants of the TAZRP. **All of these are integrable!** The version we will describe was first studied by Takayama '15

the mTAZRP: states

- Fix a (circular 1D) lattice on n sites and a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$ for the particle types



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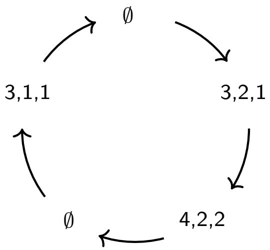
$\tau = (\ \cdot \mid 321 \mid 422 \mid \cdot \mid 311\ )$

the mTAZRP: states

- Fix a (circular 1D) lattice on n sites and a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$ for the particle types
- TAZRP(λ, n) is a Markov chain whose states are *multiset compositions* τ of type λ, with n (possibly empty) parts



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- Each particle is equipped with an exponential clock. Transitions are jumps from site j to site $j+1$

the mTAZRP: transition rates

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For example: If site j contains the particles $\{4, 3, 3, 1, 1, 1\}$, then:

$$k = 1: \quad d = 3, \quad c = 3, \quad f_j(1) = x_j^{-1} t^3 (1 + t + t^2).$$

$$k = 3: \quad d = 1, \quad c = 2, \quad f_j(3) = x_j^{-1} t (1 + t).$$

$$k = 4: \quad d = 0, \quad c = 1, \quad f_j(4) = x_j^{-1}.$$

Lumping of tableaux to mTAZRP

- Given a filling σ, read the state $\tau \in \mathrm{TAZRP}(\lambda, n)$ from the bottom row of σ as follows:

$$\tau_j \text{ is the multiset } \{\lambda_i : \sigma(1, i) = j\}$$

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- For example, for $\lambda = (2,1,1)$ and $n = 3$, the following are all the tableaux that correspond to the state $\tau = \left(\, 21 \,\middle|\, \cdot \,\middle|\, 1 \,\right)$:

$$\begin{array}{|c|cc} \hline 1 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 2 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & & \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 2 & & \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

TAZRP probabilities and tableaux

Theorem (Ayyer–M–Martin '21)

Fix λ, n. The (unnormalized) stationary probability of $\tau \in \mathrm{TAZRP}(\lambda, n)$ is

$$\widetilde{\Pr}(\tau) = \sum_{\substack{\sigma:\mathrm{dg}(\lambda)\to[n] \\ \sigma \text{ has type } \tau}} x^{\sigma} t^{\mathrm{quinv}(\sigma)}.$$

Corollary

The so-called partition function of $\mathrm{TAZRP}(\lambda, n)$ *is*

$$\mathcal{Z}_{\lambda,n}(x_1, \ldots, x_n; t) = \widetilde{H}_{\lambda}(x_1, \ldots, x_n; 1, t).$$

an example for $\lambda = (2,1,1)$ and $n = 2$

The stationary distribution is:

$(\,211 \mid \cdot\,)$	$x_1^3(x_1+x_2)$
$(\,11 \mid 2\,)$	$x_1^2x_2(t^2x_2+x_1)$
$(\,21 \mid 1\,)$	$x_1^2x_2(tx_1+x_2)(1+t)$
$(\,1 \mid 21\,)$	$x_1x_2^2(x_1+tx_2)(1+t)$
$(\,2 \mid 11\,)$	$x_1x_2^2(t^2x_1+x_2)$
$(\,\cdot \mid 211\,)$	$x_2^3(x_1+x_2)$

Example computation for $(\,21 \mid 1\,)$:

$\begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & 1 & 2 \\ \hline\end{array}$: t^2, $\quad$ $\begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & 1 & 2 \\ \hline\end{array}$: t, $\quad$ $\begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & 2 & 1 \\ \hline\end{array}$: t, $\quad$ $\begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & 2 & 1 \\ \hline\end{array}$: 1

the total is: $\widetilde{\Pr}(21|1) = x_1^2x_2(tx_1+x_2)(1+t)$.

why queue inversions? multiline diagrams

The tableaux are actually representing a queueing system which is an arrangement of lattice paths/strings: the lattice paths are representing the coupling of individual single species TAZRPs

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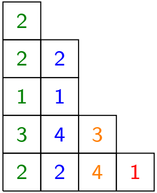
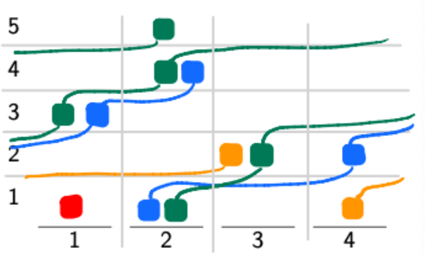


⟷



"plethystic version" of certain non-attacking fillings ⟷ "plethystic version" of multiline queues

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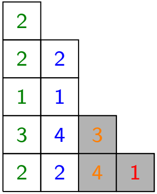
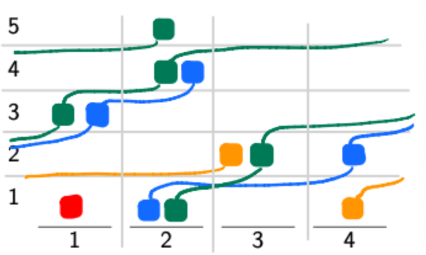


$\longleftrightarrow$



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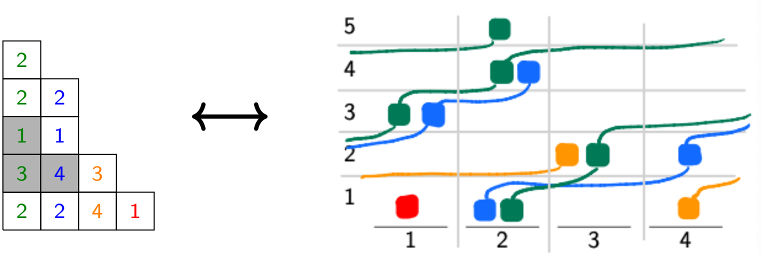

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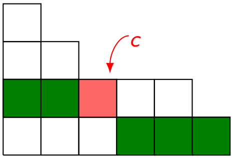
a Markov chain on tableaux: notation

- the arm of a cell, denoted by $\mathrm{arm}(c)$,



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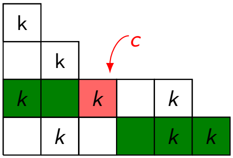

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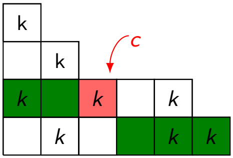

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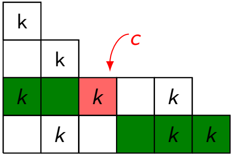

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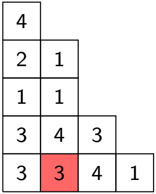
- If $c = (1, j)$ is in the bottom row, then $\text{arm}(\sigma, c)$ is equal to the **number of particles larger than or equal to λ_j at site** $\sigma(c)$ of the corresponding state of the TAZRP. Thus $f(\sigma, c)$ is equal to the rate of the correponding TAZRP jump.

a Markov chain on tableaux: transitions

A transition M_c triggered by a cell c: if $\sigma(c) \neq \sigma(\text{South}(c))$, take the maximal contiguous (cyclically) increasing chain of cells weakly above c in its column, and increment the content of each cell by 1. (This is sometimes called a ringing path)



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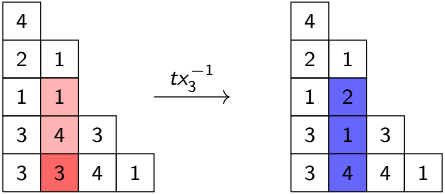

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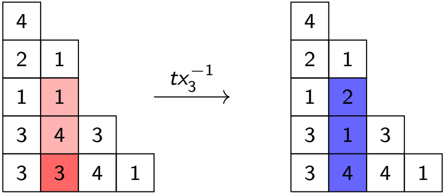

Theorem (Ayyer–M–Martin '21)

The stationary distribution of the Markov process on the tableaux is

$$\text{wt}(\sigma) = x^{\sigma}\, t^{\text{quinv}(\sigma)}$$

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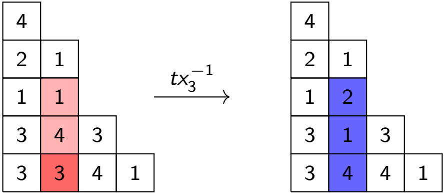

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- if $c = (1, j)$ is in the bottom row, the rate $f(\sigma, c)$ matches the transition rate $f_{\sigma(c)}(\lambda_j)$ of the corresponding particle in the TAZRP.
- (when λ has repeated parts, we need to do some more work!)

a Markov chain on tableaux: proof

$$M(\sigma) = \left\{ M_c(\sigma) \; : \; c \in \mathrm{dg}(\lambda),\ \sigma(c) \neq \sigma(\mathrm{South}(c)) \right\}$$

2	
1	2

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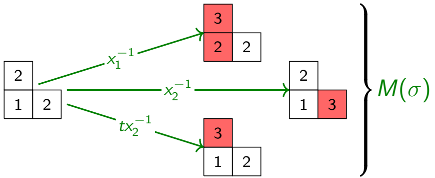
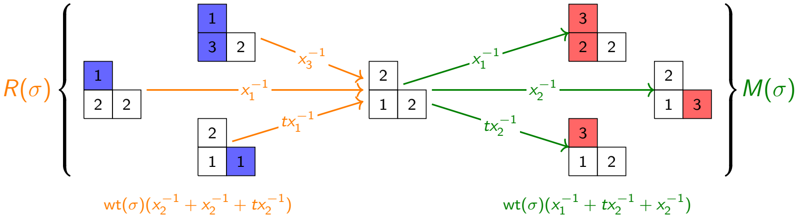


$\mathrm{wt}(\sigma)(x_1^{-1} + tx_2^{-1} + x_2^{-1})$

a Markov chain on tableaux: proof

$$M(\sigma) = \{M_c(\sigma) \ : \ c \in \mathrm{dg}(\lambda),\ \sigma(c) \neq \sigma(\mathrm{South}(c))\}$$

$$R(\sigma) = \{\sigma' \in \mathrm{PQT}(\lambda, n) \ : \ T \in M(\sigma')\}$$



balance equation

if each $\sigma \in \mathrm{dg}(\lambda) \to [n]$ satisfies:

$$\sum_{\sigma' \in R(\sigma)} \mathrm{wt}(\sigma')\,\mathrm{rate}(\sigma' \to \sigma) = \mathrm{wt}(\sigma) \sum_{\sigma' \in M(\sigma)} \mathrm{rate}(\sigma \to \sigma'),$$

then the stationary distribution of the M.C. on tableaux is $\mathrm{wt}(\sigma)$.

Observables: partition function

- The partition function of the mTAZRP of type λ, n is defined to be the normalizing constant, or the sum of the unnormalized stationary probabilities:

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$$\begin{aligned} Z_{\lambda,n} &= \widetilde{H}_{\lambda}(x_1, \ldots, x_n; 1, t) = \prod_{j=1}^{\lambda_1} \widetilde{H}_{\left(1^{\lambda'_j}\right)}(x_1, \ldots, x_n; 1, t) \\ &= \prod_{j=1}^{\lambda_1} \sum_{\mu \vdash \lambda'_j} \begin{bmatrix} \lambda'_j \\ \mu \end{bmatrix}_t m_{\mu}(x_1, \ldots, x_n). \end{aligned}$$

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- Sanity check: from the point of view of the TAZRP, having three species of particles labeled $1, 2, 3$ is the same process as having three species labeled $2, 13, 27$. Thus we should expect their stationary probabilities to be proportional.
- At the very least, we need $\widetilde{H}_{(2,13,27)}(x_1,\ldots,x_n;1,t)$ to be divisible by $\widetilde{H}_{(1,2,3)}(x_1,\ldots,x_n;1,t)$. This is indeed true, since $(3,2,1)' \subset (27,13,2)'$.

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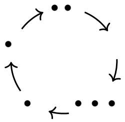

Here, $n = 5$, $m = 7$

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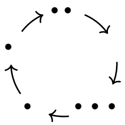

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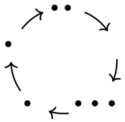

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Proposition (Current for the single species TAZRP)

For the single-species TAZRP on n sites with m particles, the current is given by

$$J = [m]_t \, \frac{\widetilde{H}_{\langle 1^{m-1} \rangle}(x_1, \ldots, x_n; 1, t)}{\widetilde{H}_{\langle 1^m \rangle}(x_1, \ldots, x_n; 1, t)}.$$

Current

Theorem (Ayyer-M-Martin '22+)

Let $\lambda = (1^{m_1}, \ldots, k^{m_k})$, and let $1 \le j \le k$. The current of the particle of type j of the TAZRP of type λ on n sites is given by

$$J = \left[m_j + \cdots + m_k\right]_t \; \frac{\widetilde{H}_{\left(1^{m_j+\cdots+m_k-1}\right)}}{\widetilde{H}_{\left(1^{m_j+\cdots+m_k}\right)}} - \left[m_{j+1} + \cdots + m_k\right]_t \; \frac{\widetilde{H}_{\left(1^{m_{j+1}+\cdots+m_k-1}\right)}}{\widetilde{H}_{\left(1^{m_{j+1}+\cdots+m_k}\right)}}$$

Densities

- Take TAZRP(λ, n) with content $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$.
- Define $z_j^{(\ell)}$ to be the random variable counting the number of particles of type ℓ at site j in a configuration of TAZRP(λ, n).
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Suppose $\langle z_1^{(\ell)} \rangle = r(x_1, \dots, x_n)$. *Then for any* j,

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- Thus it suffices to compute the densities of all species of particles at site 1.

We begin with the special case of $\lambda = 1^m$.

Theorem (Densities for the single species TAZRP)

The density at site 1 on TAZRP$(1^m, n)$ *is given by*

$$\langle z_1^{(1)} \rangle = x_1 \partial_{x_1} \log \widetilde{H}_{(1^m)}(x_1, \dots, x_n; 1, t).$$

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- Thus it suffices to compute the densities of all species of particles at site 1.

We begin with the special case of $\lambda = 1^m$.

Theorem (Densities for the single species TAZRP)

The density at site 1 on TAZRP$(1^m, n)$ *is given by*

$$\langle z_1^{(1)} \rangle = x_1 \partial_{x_1} \log \widetilde{H}_{(1^m)}(x_1, \ldots, x_n; 1, t).$$

- In particular, when $x_1 = \cdots = x_n = 1$, the density is $\langle z_1^{(1)} \rangle = \frac{m}{n}$.

Densities

Theorem (Ayyer-M-Martin '22+)

For $1 \leq \ell \leq k$, the density of the ℓ'th species at site 1 is given by

$$\langle z_1^{(\ell)} \rangle = x_1 \partial_{x_1} \log \left(\frac{\widetilde{H}_{(1^{m_\ell + \cdots + m_k})}(x_1, \ldots, x_n; 1, t)}{\widetilde{H}_{(1^{m_{\ell+1} + \cdots + m_k})}(x_1, \ldots, x_n; 1, t)} \right).$$

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$\langle z_1^{(\ell)} \rangle$ *is symmetric in the variables* $\{x_2, \ldots, x_n\}$.

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$\langle z_1^{(\ell)} \rangle$ *is symmetric in the variables* $\{x_2, \ldots, x_n\}$.

Proof via coloring argument:

- true for base case $\lambda = (1^m)$
- transitions of particles of species $\ell, \ldots, k$ at site 1 are independent of the number of lower species particles at site 1. Thus we can ignore the particles of types $1, \ldots, \ell - 1$.
- the density of particles of species $\ell, \ldots, k$ at site 1 is equivalent to the density at site 1 of a TAZRP of type $\lambda = (1^{m_\ell + \cdots + m_k})$.
- to isolate species ℓ we subtract the density of species $\ell + 1, \ldots, k$ from the density of species $\ell, \ldots, k$.

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  - Let $\mathbb{P}_{\lambda,n}(\overline{w})$ be the stationary probability of having exactly the content $w_1, \ldots, w_\ell$ on sites $1, \ldots, \ell$.
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- Example: let $\lambda = (2, 2, 1, 1)$, $n = 4$, $\ell = 2$, and $w = (2|1)$.
  - Configurations contributing to $\mathbb{P}_{\lambda,n}(\overline{w})$ are

$$(2|1|12|\cdot), \quad (2|1|1|2), \quad (2|1|2|1), \quad (2|1|\cdot|12)$$

  - Additional configurations contributing to $\mathbb{P}_{\lambda,n}(\hat{w})$ are

$$(12|1|2|\cdot), \quad (2|11|2|\cdot), \quad (22|1|1|\cdot), \quad (2|12|1|\cdot), \quad (12|1|\cdot|2), \quad (2|11|\cdot|2)$$

$$(22|1|\cdot|1), \quad (2|12|\cdot|1), \quad (122|1|\cdot|\cdot), \quad (2|112|\cdot|\cdot), \quad (22|11|\cdot|\cdot), \quad (12|12|\cdot|\cdot)$$

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Theorem (Ayyer-M-Martin '22+)

Both $\mathbb{P}_{\lambda,n}(\overline{w})$ and $\mathbb{P}_{\lambda,n}(\hat{w})$ are symmetric in the variables $\{x_{\ell+1}, \dots, x_n\}$.

final remarks

- Explicit bijection from the inv to the quinv statistic?

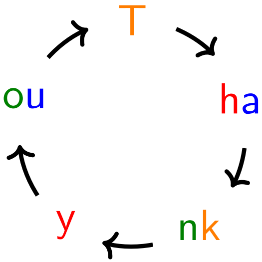
- Can we find a dynamical process that incorporates the q as a parameter?

  This seems difficult because

  - We lose factorization of $\widetilde{H}_\lambda$
  - We lose translation invariance

- Using multiline queues (for the ASEP on a circle), Corteel-Haglund-M-Mason-Williams '20 defined quasisymmetric Macdonald polynomials which refine P_λ. Can we use a parallel construction to define an interesting family of quasisymmetric polynomials that refine $\widetilde{H}_\lambda$?

- Same as above, but for nonsymmetric Macdonald polynomials



- **Modified Macdonald polynomials and the multispecies zero range process: I,** (with A. Ayyer and J. B. Martin), arXiv:2011.06117
- **A Markov chain on tableaux that projects to the multispecies TAZRP, and applications,** (with A. Ayyer and J. B. Martin), in preparation