Macdonald polynomials and the multispecies zero range process

Olya Mandelshtam
University of Waterloo

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joint with Arvind Ayyer and James Martin,
arXiv:2022.06117 + upcoming
Motivation: Macdonald polynomials and interacting particle systems

A new combinatorial formula for $\tilde{H}_\lambda(X; q, t)$

Multispecies Totally Asymmetric Zero Range Process (mTAZRP)

Markov chain on tableaux

Observables
integrable systems include a class of dynamical systems with a certain restricted structure, in particular making them solvable
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**canonical example**: the ASEP describes particles hopping on a finite 1D lattice with $\leq 1$ particle at each site

we are interested in studying integrable systems whose exact solutions (stationary distributions) can be expressed in terms of combinatorial formulas or special functions (e.g. Macdonald polynomials)
Macdonald polynomials

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...
Macdonald polynomials

- Let $X = x_1, x_2, \cdots$ be a family of indeterminates, and let $\Lambda = \Lambda_\mathbb{Q}$ be the algebra of symmetric functions in $X$ over $\mathbb{Q}$.
- $\Lambda$ has several nice bases: e.g. $\{m_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$, indexed by partitions $\lambda$. 

\[ s_\lambda = \sum_{\sigma} x_\sigma \] 

where $\sigma$ is a semi-standard filling of the Young diagram of shape $\lambda$. 

E.g. the following are the fillings of shape $(2,1)$ on 3 letters:

2 1 1
3 1 1
2 1 2
3 1 2
2 1 3
3 1 3
2 2 1
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Macdonald polynomials

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Let $\langle,\rangle$ be the standard inner product on $\Lambda$. Then $\{s_\lambda\}$ is the unique basis of $\Lambda$ that is:

i. orthogonal with respect to $\langle,\rangle$

ii. upper triangular with respect to $\{m_\lambda\}$:

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\mu \lambda} m_\mu$$

where $<$ is with respect to dominance order on partitions.
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E.g. the following are the fillings of shape $(2, 1)$ on 3 letters:

- $s_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 = m_{(2,1)} + m_{(1,1,1)}$
Let $\Lambda = \Lambda_Q(q, t)$, the algebra of symmetric functions with parameters $q, t$ over $\mathbb{Q}$. Macdonald polynomials $P_\lambda(X; q, t)$
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- Let $\Lambda = \Lambda_Q(q, t)$, the algebra of symmetric functions with parameters $q, t$ over $\mathbb{Q}$
- Macdonald '88 introduced a family of homogeneous symmetric polynomials $\{P_\lambda(X; q, t)\}$ in $\Lambda(q, t)$, simultaneously generalizing the Schur polynomials (at $q = t = 0$), Hall-Littlewood polynomials (at $q = 0$), and Jack polynomials (at $t = q^\alpha$ and $q \to 1$)

Example:

$$P_{(2,1)}(X; q, t) = m_{(2,1)}(X) + (1 - t)(2 + q + t + 2qt)$$
Macdonald polynomials $P_\lambda(X; q, t)$

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Let $\langle \cdot, \cdot \rangle_{q,t}$ be the inner product on $\Lambda(q, t)$ given by:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda, \mu} z_\lambda \prod_{i \geq 1} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$  

Then $\{P_\lambda\}$ is the unique basis of $\Lambda(q, t)$ that is uniquely determined by:

i. orthogonal basis for $\Lambda(q, t)$ with respect to $\langle \cdot, \cdot \rangle_{q,t}$

ii. upper triangular with respect to $\{m_\lambda\}$:

$$P_\lambda(X; q, t) = m_\lambda(X) + \sum_{\mu < \lambda} c_{\mu \lambda}(q, t)m_\mu(X)$$
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Example:

$$P_{(2,1)}(X; q, t) = m_{(2,1)} + \frac{(1 - t)(2 + q + t + 2qt)}{1 - qt^2} m_{(1,1,1)}.$$
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$$\tilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda \left[ \frac{X}{1 - t^{-1}}; q, t^{-1} \right]$$

where $J_\lambda$ is a scalar multiple of $P_\lambda$.

Example: $\tilde{H}_{(2,1)}(X; q, t) = m_{(3)} + (1 + q + t)m_{(2,1)} + (1 + 2q + 2t + qt)m_{(1,1,1)}$
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- Haglund-Haiman-Loehr '04 gave formulas for $P_\lambda$ and $\tilde{H}_\lambda$ as sums over tableaux with statistics $\text{maj}$ and $(\text{co})\text{inv}$:

  - $P_\lambda(X; q, t) = \sum_{\sigma \in \text{dg}(\lambda), \sigma \text{ non-attacking}} q^{\text{maj}(\sigma)} t^{\text{coinv}(\sigma)} X^\sigma \prod_u \frac{1 - t}{1 - q^{\text{leg}(u) + 1} t^{\text{arm}(u) + 1}}$

  - $\tilde{H}_\lambda(X; q, t) = \sum_{\sigma \in \text{dg}(\lambda)} q^{\text{maj}(\sigma)} t^{\text{inv}(\sigma)} X^\sigma$
Combinatorial formulas

- Corteel-M-Williams '18 gave a new formula for $P_\lambda$ in terms of multiline queues, which also gives formulas for the stationary distribution of the ASEP.
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Garbali-Wheeler '20 gave a formula for $\tilde{H}_\lambda$ using integrability, in terms of colored paths.
Combinatorial formulas

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- **Garbali-Wheeler '20** gave a formula for $\tilde{H}_\lambda$ using integrability, in terms of colored paths.

- **Corteel-Haglund-M-Mason-Williams '20** gave a "compressed" formula for $\tilde{H}_\lambda$. Using multiline queues and the plethystic relationship between $\tilde{H}_\lambda$ and $P_\lambda$, also conjectured a new formula for $\tilde{H}_\lambda$ with statistics $\text{maj}$ and a new statistic $\text{quinv}$:

$$
\tilde{H}_\lambda(X; q, t) = \sum_{\sigma \in \text{dg}(\lambda)} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} X^\sigma
$$
multiline queues and the ASEP

- a **multiline queue** (MLQ) of type $\lambda$, $n$ is an arrangement and pairing of balls on a $n \times \lambda_1$ lattice, with $\lambda_j'$ balls in row $j$.

![Diagram of multiline queue]

$\lambda = (3, 3, 2, 1, 1, 0)$

Angel '08, Ferrari-Martin '07 ($t = 0$ case), Martin '18 (for $q = x_1 = \cdots = x_n = 1$),
Corteel–M–Williams '18 (general)
multiline queues and the ASEP

- a multiline queue (MLQ) of type $\lambda$, $n$ is an arrangement and pairing of balls on a $n \times \lambda_1$ lattice, with $\lambda_j'$ balls in row $j$.
- It can be represented by a queueing system, or described as a coupled system of 1-ASEPs

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- The weight $wt(M)$ of a multiline queue depends on the parameters $t, q, x_1, \ldots, x_n$:
  - A string of length $\ell$ corresponds to an ASEP particle of species $\ell$. The labels of the balls in that string are $\ell$. 
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\[ \begin{array}{ccccccc}
\text{row 3} & \quad & \text{row 2} & \quad & \text{row 1} & \quad & \lambda = (3, 3, 2, 1, 1, 0) \\
& 3 & & 3 & & 3 & \\
2 & & 3 & & & & \\
1 & 2 & 3 & 4 & 5 & 6 & \\
& & & & & & \\
\end{array} \]

\[ \begin{array}{ccccccc}
\alpha = (2, 1, 0, 1, 3, 3) & \\
\end{array} \]

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multiline queues and the ASEP

- **Multiline queue (MLQ)**: A multiline queue (MLQ) of type \(\lambda\), \(n\) is an arrangement and pairing of balls on an \(n \times \lambda_1\) lattice, with \(\lambda_j'\) balls in row \(j\).
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- The \(j^{\text{th}}\) column of the MLQ corresponds to the variable \(x_j\) (a MLQ with \(n\) columns corresponds to an ASEP on \(n\) sites and uses \(n\) variables \(x_1, \ldots, x_n\)).

Angel '08, Ferrari-Martin '07 \((t = 0\) case), Martin '18 \((q = x_1 = \cdots = x_n = 1)\), Corteel–M–Williams '18 \(\text{(general)}\).
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```
\begin{align*}
\text{row 3} & : 3 \\
\text{row 2} & : 2, \ 3 \\
\text{row 1} & : 2, \ 1, \ 3, \ 3
\end{align*}
```

```
\begin{align*}
\lambda & = (3, 3, 2, 1, 1, 0) \\
\alpha & = (2, 1, 0, 1, 3, 3)
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- Skipped balls in the MLQ "correspond" to a coinv statistic in $t$. 
multiline queues and the ASEP

- a multiline queue (MLQ) of type $\lambda$, $n$ is an arrangement and pairing of balls on a $n \times \lambda_1$ lattice, with $\lambda'_j$ balls in row $j$.
- It can be represented by a queueing system, or described as a coupled system of $1$-ASEPs

\[
\begin{array}{ccccccc}
\text{row 1} & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{row 2} & 2 & 3 & 3 & 3 & & \\
\text{row 3} & 3 & 3 & & & & \\
\end{array}
\]

$\lambda = (3, 3, 2, 1, 1, 0)$
$\alpha = (2, 1, 0, 1, 3, 3)$

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  - wrapping balls in the MLQ correspond to a maj statistic in $q$
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![Diagram of a multiline queue](image)

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- The weight \( wt(M) \) of a multiline queue depends on the parameters \( t, q, x_1, \ldots, x_n \):
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  - the bottom row corresponds to a state of the ASEP of type \( \lambda, n \)
  - the \( j' \)th column of the MLQ corresponds to the variable \( x_j \) (a MLQ with \( n \) columns corresponds to an ASEP on \( n \) sites and uses \( n \) variables \( x_1, \ldots, x_n \))
  - skipped balls in the MLQ "correspond" to a coinv statistic in \( t \)
  - wrapping balls in the MLQ correspond to a maj statistic in \( q \)

- Can be represented by a **non-attacking** tableau, where each string is mapped to a column of the same height, recording the position of each ball in the MLQ.
Theorem (Martin ’18, Corteel-M-Williams ’18)

The (unnormalized) stationary probability of state $\alpha$ of the mASEP is

$$\Pr(\alpha)(t) = \sum_{M: \text{row 1} = \alpha} wt(M)(1, \ldots, 1; 1, t)$$
Theorem (Martin ’18, Corteel-M-Williams ’18)

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Theorem (Cantini–de Gier–Wheeler ’15)

The partition function of ASEP($\lambda$, $n$) is a specialization of the Macdonald polynomial:

$$P_\lambda(1, \ldots, 1; 1, t) = Z_{\lambda,n}(t) = \sum_{\alpha \in S_n \cdot \lambda} \tilde{\Pr}(\alpha)(t).$$
### Theorem (Martin '18, Corteel-M-Williams '18)

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### Theorem (Corteel–M–Williams '18)

$$P_{\lambda}(x_1, \ldots, x_n; q, t) = \sum_{M \in \text{MLQ}(\lambda, n)} \text{wt}(M)(x_1, \ldots, x_n; q, t)$$
From ASEP to Macdonald polynomials

**Theorem (Martin ’18, Corteel-M-Williams ’18)**

The (unnormalized) stationary probability of state $\alpha$ of the mASEP is

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**Theorem (Cantini–de Gier–Wheeler ’15)**

The partition function of ASEP($\lambda$, $n$) is a specialization of the Macdonald polynomial:

$$P_\lambda(1, \ldots, 1; 1, t) = \mathcal{Z}_\lambda,n(t) = \sum_{\alpha \in S_n \cdot \lambda} \tilde{\text{Pr}}(\alpha)(t).$$

**Theorem (Corteel–M–Williams ’18)**

$$P_\lambda(x_1, \ldots, x_n; q, t) = \sum_{M \in \text{MLQ}(\lambda, n)} \text{wt}(M)(x_1, \ldots, x_n; q, t)$$

This formula essentially coincides with that of Lenart ’09 for $\lambda$ with distinct parts.
Example for $P_{(2,1)}(x_1, x_2, x_3; q, t)$

$$P_{(2,1)}(x_1, x_2, x_3; q, t) = m_{(2,1)} + \frac{(2 + t + q + 2qt)(1 - t)}{(1 - qt^2)} m_{(1,1,1)}$$
Recall: $\tilde{H}_\lambda(X; q, t)$ is obtained from the integral form of $P_\lambda$ via plethysm:

$$\tilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda \left[ \frac{X}{1 - t^{-1}} ; q, t^{-1} \right]$$

$$= f_\lambda(q, t) P_\lambda \left( x_1, x_1 t^{-1}, x_1 t^{-2}, \ldots, x_2, x_2 t^{-1}, x_2 t^{-2}, \ldots ; q, t^{-1} \right)$$
From multiline queues to a new formula for $\tilde{H}_\lambda$

- Recall: $\tilde{H}_\lambda(X; q, t)$ is obtained from the integral form of $P_\lambda$ via plethysm:

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- $P_\lambda(x_1, x_1 t^{-1}, x_1 t^{-2}, \ldots, x_2, x_2 t^{-1}, x_2 t^{-2}, \ldots; q, t^{-1})$ should correspond to a multiline queue with countably many columns labeled by

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Recall: \( \tilde{H}_\lambda(X; q, t) \) is obtained from the integral form of \( P_\lambda \) via plethysm:

\[
\tilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda \left[ \frac{X}{1 - t^{-1}} ; q, t^{-1} \right]
\]

\[
= f_\lambda(q, t) P_\lambda \left( x_1, x_1 t^{-1}, x_1 t^{-2}, \ldots, x_2, x_2 t^{-1}, x_2 t^{-2}, \ldots ; q, t^{-1} \right)
\]

\( P_\lambda \left( x_1, x_1 t^{-1}, x_1 t^{-2}, \ldots, x_2, x_2 t^{-1}, x_2 t^{-2}, \ldots ; q, t^{-1} \right) \) should correspond to a multiline queue with countably many columns labeled by

\[x_1, x_1 t^{-1}, x_1 t^{-2}, \ldots, x_2, x_2 t^{-1}, x_2 t^{-2}, \ldots\]

this leads to a new “queue inversion” statistic for \( t \) that we call quinv

\( \text{(Corteel–Haglund–M–Mason–Williams '20, Ayyer–M–Martin '21) } \)
Recall: $\tilde{H}_\lambda(X; q, t)$ is obtained from the integral form of $P_\lambda$ via plethysm:

$$\tilde{H}_\lambda(X; q, t) = t^{\nu(\lambda)} \left[ \frac{X}{1 - t^{-1}} ; q, t^{-1} \right]$$

$$= f_\lambda(q, t) \ P_\lambda(x_1, x_1 t^{-1}, x_1 t^{-2}, \ldots, x_2, x_2 t^{-1}, x_2 t^{-2}, \ldots ; q, t^{-1})$$

$P_\lambda(x_1, x_1 t^{-1}, x_1 t^{-2}, \ldots, x_2, x_2 t^{-1}, x_2 t^{-2}, \ldots ; q, t^{-1})$ should correspond to a multiline queue with countably many columns labeled by

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this leads to a new “queue inversion” statistic for $t$ that we call $\text{quinv}$ (Corteel–Haglund–M–Mason–Williams ’20, Ayyer–M–Martin ’21)

the resulting objects are of the same flavor as multiline queues, except that multiple balls are allowed at each location. (This translates to removing the “non-attacking” condition from the corresponding tableaux)
**Tableaux Formulas: Notation and Statistics**

- \( \text{dg}(\lambda) \) (the diagram of \( \lambda = (\lambda_1, \ldots, \lambda_k) \)) consists of \( k \) bottom justified columns with \( \lambda_i \) boxes, from left to right.

\[
\begin{array}{cccc}
\Box & \Box & & \\
\Box & \Box & \Box & \\
\Box & \Box & \Box & \\
\Box & \Box & & \\
\end{array}
\]

\( \lambda = (4, 3, 3, 1) \)

Theorem (Haglund–Haiman–Loehr '05)

The modified Macdonald polynomial is given by

\[
\tilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma : \text{dg}(\lambda) \rightarrow [n]} q^{\text{maj}(\sigma)} t^{\text{inv}(\sigma)} x^\sigma.
\]
tableaux formulas: notation and statistics

- $\text{dg}(\lambda)$ (the diagram of $\lambda = (\lambda_1, \ldots, \lambda_k)$) consists of $k$ bottom justified columns with $\lambda_i$ boxes, from left to right

  $\lambda = (4, 3, 3, 1)$

  $\text{dg}(\lambda) = \begin{array}{cccc}
    4 \\
    2 & 2 & 4 \\
    3 & 1 & 1 \\
    2 & 3 & 3 & 4 \\
  \end{array}$

- A tableau of type $(\lambda, n)$ is a filling $\sigma : \text{dg}(\lambda) \to [n]$ of the cells
tableaux formulas: notation and statistics

- \( \text{dg}(\lambda) \) (the diagram of \( \lambda = (\lambda_1, \ldots, \lambda_k) \)) consists of \( k \) bottom justified columns with \( \lambda_i \) boxes, from left to right

\[
\begin{array}{cccc}
4 & & & \\
2 & 2 & 4 & \\
3 & 1 & 1 & \\
2 & 3 & 3 & 4
\end{array}
\]

\( \sigma = \lambda = (4, 3, 3, 1) \)

- a **tableau** of type \((\lambda, n)\) is a filling \( \sigma : \text{dg}(\lambda) \to [n] \) of the cells

- \( \text{inv}(\sigma) \) is the number of **inversions** in the configuration

\[
\begin{array}{ccc}
x & \cdots & z \\
y & & \\
\end{array}
\]

where \( x < y < z \) (cyclically mod \( n \))
**tableaux formulas: notation and statistics**

- $\text{dg}(\lambda)$ (the diagram of $\lambda = (\lambda_1, \ldots, \lambda_k)$) consists of $k$ bottom justified columns with $\lambda_i$ boxes, from left to right

  \[
  \begin{array}{cccc}
  & 4 & & \\
 2 & 2 & 4 & \\
 3 & 1 & 1 & \\
 2 & 3 & 3 & 4 \\
  \end{array}
  \]

  $\lambda = (4, 3, 3, 1)$

  $x^\sigma = x_1^2 x_2^3 x_3^3 x_4^3$

- A **tableau** of type $(\lambda, n)$ is a filling $\sigma : \text{dg}(\lambda) \to [n]$ of the cells

- $\text{inv}(\sigma)$ is the number of inversions in the configuration

  \[
  \begin{array}{ccc}
  x & \cdots & z \\
  y & & \\
  \end{array}
  \]

  where $x < y < z$ (cyclically mod $n$)
Tableaux Formulas: Notation and Statistics

- $\text{dg}(\lambda)$ (the diagram of $\lambda = (\lambda_1, \ldots, \lambda_k)$) consists of $k$ bottom justified columns with $\lambda_i$ boxes, from left to right.

\[
\begin{array}{cccc}
4 & & & \\
2 & 2 & 4 & \\
3 & 1 & 1 & \\
2 & 3 & 3 & 4
\end{array}
\]

\[
\lambda = (4, 3, 3, 1)
\]

\[
x^\sigma = x_1^2 x_2^3 x_3^3 x_4^3
\]

\[
\text{maj}(\sigma) = 6
\]

\[
\text{inv}(\sigma) = 1
\]

- A tableau of type $(\lambda, n)$ is a filling $\sigma : \text{dg}(\lambda) \rightarrow [n]$ of the cells.

- $\text{inv}(\sigma)$ is the number of inversions in the configuration:

\[
\begin{array}{ccc}
\times & \cdots & \times
\end{array}
\]

where $x < y < z$ (cyclically mod $n$)

**Theorem (Haglund–Haiman–Loehr ’05)**

The modified Macdonald polynomial is given by

\[
\tilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma : \text{dg}(\lambda) \rightarrow [n]} q^{\text{maj}(\sigma)} t^{\text{inv}(\sigma)} x^\sigma.
\]
a new statistic: queue-inversion

\[ \sigma = \begin{array}{ccc}
4 \\
2 & 2 & 4 \\
3 & 1 & 1 \\
2 & 3 & 3 & 4 \\
\end{array} \]

an \textit{L-triple} is a triple of cells in the configuration:

\begin{array}{cccc}
x & \ldots & z \\
y & \ldots & z \\
\emptyset & \ldots & z \\
\end{array}

\text{Theorem (Ayyer–M–Martin '20)}

Let \( \lambda \) be a partition. The modified Macdonald polynomial equals

\[ \tilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma: \text{dg}(\lambda) \to [n]} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} x_\sigma \]

(first conjectured by Corteel–Haglund–M–Mason–Williams '19)
a new statistic: queue-inversion

\[ \sigma = \begin{array}{ccc}
4 & & \\
2 & 2 & 4 \\
3 & 1 & 1 \\
2 & 3 & 3 & 4 \\
\end{array} \]

- an \textbf{L-triple} is a triple of cells in the configuration:
  \[
  \begin{array}{ccc}
  x & & \\
  y & \cdots & z \\
  \emptyset & \cdots & z \\
  \end{array}
  \]

- an \textbf{L-triple} forms a \textbf{quinv} (queue-inversion) if \( x < y < z \) cyclically mod \( n \) (ties are broken by a top-to-bottom and right-to-left reading order)
a new statistic: queue-inversion

\[
\sigma = \begin{array}{ccc}
4 \\
2 & 2 & 4 \\
3 & 1 & 1 \\
2 & 3 & 3 & 4
\end{array}
\]

- an **L-triple** is a triple of cells in the configuration:

  \[
  \begin{array}{ccc}
  x \\
y & \cdots & z
  \end{array}
  \quad \text{or} \quad
  \begin{array}{ccc}
  \emptyset \\
y & \cdots & z
  \end{array}
  \]

- an **L-triple** forms a **quinv** (queue-inversion) if \( x < y < z \) cyclically mod \( n \) (ties are broken by a top-to-bottom and right-to-left reading order)

- **quinv(\( \sigma \))** is the total number of queue-inversions in \( \sigma \).
a new statistic: queue-inversion

\[
\sigma = \begin{bmatrix}
4 \\
2 & 2 & 4 \\
3 & 1 & 1 \\
2 & 3 & 3 & 4 \\
\end{bmatrix}
\]

• an \textit{L-triple} is a triple of cells in the configuration:

\[
\begin{array}{ccc}
\mathbb{\lambda} & \cdots & \mathbb{\nu} \\
\mathbb{\xi} & \cdots & \mathbb{\upsilon} \\
\end{array}
\]

or

\[
\begin{array}{ccc}
\emptyset & \cdots & \mathbb{\upsilon} \\
\mathbb{\xi} & \cdots & \mathbb{\upsilon} \\
\end{array}
\]

• an \textit{L-triple} forms a \textit{quinv (queue-inversion)} if \(x < y < z\) cyclically mod \(n\) (ties are broken by a top-to-bottom and right-to-left reading order)

• \textit{quinv(\(\sigma\))} is the total number of queue-inversions in \(\sigma\).

\textbf{Theorem (Ayyer–M–Martin '20)}

Let \(\lambda\) be a partition. The modified Macdonald polynomial equals

\[
\tilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma: \lambda \rightarrow [n]} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} x^\sigma
\]

(first conjectured by Corteel–Haglund–M–Mason–Williams '19)
a new statistic: queue-inversion

\[ \sigma = \begin{array}{cccc}
4 & & & \\
2 & 2 & 4 & \\
3 & 1 & 1 & \\
2 & 3 & 3 & 4 \\
\end{array} \]

- an **L-triple** is a triple of cells in the configuration:
  \[
  x \\
  y \quad \cdots \quad z \\
  \emptyset \quad \cdots \\
\]

- an L-triple forms a **quinv** (queue-inversion) if \( x < y < z \) cyclically mod \( n \) (ties are broken by a top-to-bottom and right-to-left reading order)

- \( \text{quinv}(\sigma) \) is the total number of queue-inversions in \( \sigma \).

**Theorem (Ayyer–M–Martin '20)**

Let \( \lambda \) be a partition. The modified Macdonald polynomial equals

\[
\tilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma : \text{dg}(\lambda) \to [n]} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} x^\sigma
\]

(first conjectured by Corteel–Haglund–M–Mason–Williams '19)
a new statistic: queue-inversion

\[ \sigma = \begin{array}{cccc}
4 & & & \\
2 & 2 & 4 & \\
3 & 1 & 1 & \\
2 & 3 & 3 & 4 \\
\end{array} \]

- An **L-triple** is a triple of cells in the configuration:
  
  \[
  \begin{array}{cccc}
  x & \cdots & z & \\
  y & \cdots & z & \\
  \emptyset & \cdots & z & \\
  \end{array} 
  \]

- An L-triple forms a **quinv** (queue-inversion) if \( x < y < z \) cyclically mod \( n \) (ties are broken by a top-to-bottom and right-to-left reading order).

- \( \text{quinv}(\sigma) \) is the total number of queue-inversions in \( \sigma \).

**Theorem (Ayyer–M–Martin ’20)**

*Let \( \lambda \) be a partition. The modified Macdonald polynomial equals*

\[
\tilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma : \text{dg}(\lambda) \to [n]} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} x^\sigma
\]

(first conjectured by Corteel–Haglund–M–Mason–Williams ’19)
a new statistic: queue-inversion

\[ \sigma = \begin{array}{cccc}
4 & & & \\
2 & 2 & 4 & \\
3 & 1 & 1 & \\
2 & 3 & 3 & 4 \\
\end{array} \]

\[ \text{quinv}(\sigma) = 4 \]

- an \textbf{L-triple} is a triple of cells in the configuration:
  \[
  \begin{array}{ccc}
  x & \cdots & z \\
  y & & \emptyset \\
  \end{array}
  \quad \text{or} \quad
  \begin{array}{ccc}
  \emptyset & \cdots & z \\
  y & & \\
  \end{array}
  \]

- an \textbf{L-triple} forms a \textbf{quinv} (queue-inversion) if \( x < y < z \) cyclically mod \( n \) (ties are broken by a top-to-bottom and right-to-left reading order)

- \textbf{quinv}(\sigma) is the total number of queue-inversions in \( \sigma \).

Theorem (Ayyer–M–Martin '20)

\[ \widetilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma : \text{dg}(\lambda) \to [n]} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} x^\sigma \]

(first conjectured by Corteel–Haglund–M–Mason–Williams '19)
Example: \( \tilde{H}_{(2,1)}(X; q, t) \)

\[
\tilde{H}_{(2,1)}(x_1, x_2; q, t) = m_{(3)} + (1 + t + q)m_{(2,1)} + (1 + 2t + 2q + qt)m_{(1,1,1)}
\]

- (AMM) \( \tilde{H}_\lambda(X; q, t) = \sum_{\sigma: \text{dg}(\lambda) \rightarrow \mathbb{Z}_+} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} x^\sigma \)

- (HHL) \( \tilde{H}_\lambda(X; q, t) = \sum_{\sigma: \text{dg}(\lambda) \rightarrow \mathbb{Z}_+} q^{\text{maj}(\sigma)} t^{\text{inv}(\sigma)} x^\sigma \)
Example: \( \tilde{H}(2,1)(X; q, t) \)

\[
\tilde{H}(2,1)(x_1, x_2; q, t) = m_{(3)} + (1 + t + q)m_{(2,1)} + (1 + 2t + 2q + qt)m_{(1,1,1)}
\]

\( \bullet \) (AMM) \( \tilde{H}_\lambda(X; q, t) = \sum_{\sigma: \text{dg}(\lambda) \to \mathbb{Z}_+} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} x^\sigma \)

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1
\end{array}
\]

\[
\begin{array}{cccccccc}
m_3 & q m_{21} & t m_{21} & m_{21} & t m_{111} & q m_{111} & q t m_{111} & t m_{111} & m_{111}
\end{array}
\]

\( \bullet \) (HHL) \( \tilde{H}_\lambda(X; q, t) = \sum_{\sigma: \text{dg}(\lambda) \to \mathbb{Z}_+} q^{\text{maj}(\sigma)} t^{\text{inv}(\sigma)} x^\sigma \)

\[
\begin{array}{cccccccc}
1 & 2 & 1 & 1 & 1 & 2 & 3 & 1 & 2
\end{array}
\]

\[
\begin{array}{cccccccc}
m_3 & q m_{21} & m_{21} & t m_{21} & m_{111} & q m_{111} & q m_{111} & t m_{111} & q t m_{111}
\end{array}
\]

- while the inv and quinv statistics appear very similar, there does not seem to be an easy way to go from one to the other – is there a bijective proof?
Motivation

What is the analogous interacting particle system whose partition function is a specialization of $\tilde{H}_\lambda$?
What is the analogous interacting particle system whose partition function is a specialization of $\tilde{H}_\lambda$?
totally asymmetric zero range processes (TAZRP)

- continuous-time stochastic processes (Spitzer '70), can be defined on arbitrary graphs. In our case, we have a circular lattice with \( n \) sites.

\[
\tau = \left( \begin{array}{c}
\tau_1 \\
\vdots \\
\tau_n
\end{array} \right)
\]

simplest case: there are \( k \) indistinguishable particles, moving counter-clockwise. A configuration \( \tau = (\tau_1, \ldots, \tau_n) \) is any allocation of the \( k \) particles on the \( n \) sites.

transitions: a particle jumps from site \( j \) to site \( j+1 \mod n \) with rate \( f(\tau_j) \) for some \( f : \mathbb{N} \to \mathbb{R}^+ \).

multispecies variant: we now allow different particle types, labeled by integers (particles of the same type are still indistinguishable). Kuniba–Maruyama–Okado (2015+) (and others) have studied many multispecies variants of the TAZRP. All of these are integrable!

The version we will describe was first studied by Takayama '15.
totally asymmetric zero range processes (TAZRP)

- continuous-time stochastic processes (Spitzer ’70), can be defined on arbitrary graphs. In our case, we have a circular lattice with \( n \) sites.

Here, \( n = 5, \ k = 7 \)

\[ \tau = (11 \mid \cdot \mid 111 \mid 1 \mid 1) \]

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Here, $n = 5$, $k = 7$

$$\tau = (11 \mid \cdot \mid 111 \mid 1 \mid 1)$$

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totally asymmetric zero range processes (TAZRP)

- continuous-time stochastic processes (Spitzer ’70), can be defined on arbitrary graphs. In our case, we have a circular lattice with \( n \) sites.

\[
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\uparrow \\
3, 3, 1 \\
\downarrow \\
2,
\end{array}
\]

Here, \( n = 5, \ k = 7 \)

\[
\tau = ( 2, 2 \mid \cdot \mid 3, 3, 1 \mid 2 \mid 1 )
\]

- simplest case: there are \( k \) indistinguishable particles, moving counter-clockwise. A configuration \( \tau = (\tau_1, \ldots, \tau_n) \) is any allocation of the \( k \) particles on the \( n \) sites.

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- multispecies variant: we now allow different particle types, labeled by integers (particles of the same type are still indistinguishable)
totally asymmetric zero range processes (TAZRP)

- continuous-time stochastic processes \( (\text{Spitzer '70}) \), can be defined on arbitrary graphs. In our case, we have a **circular lattice with** \( n \) **sites.**

\[
\begin{array}{c}
\tau = (2, 2 | 3, 3, 1 | 2 | 1) \\
\text{Here, } n = 5, \ k = 7
\end{array}
\]

- simplest case: there are \( k \) indistinguishable particles, moving counter-clockwise. A configuration \( \tau = (\tau_1, \ldots, \tau_n) \) is any allocation of the \( k \) particles on the \( n \) sites.
- transitions: a particle jumps from site \( j \) to site \( j + 1 \mod n \) with rate \( f(\tau_j) \) for some \( f : \mathbb{N} \to \mathbb{R}_+ \)
- **multispecies variant:** we now allow different particle types, labeled by integers (particles of the same type are still indistinguishable)
- Kuniba–Maruyama–Okado (2015+) (and others) have studied many multispecies variants of the TAZRP. **All of these are integrable!** The version we will describe was first studied by Takayama '15
Fix a (circular 1D) lattice on $n$ sites and a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$ for the particle types

$$
\begin{align*}
&\emptyset &\emptyset & 3,1,1 & 3,2,1 \\
&\emptyset & 4,2,2
\end{align*}
$$

$n = 5$

$$\lambda = (4, 3, 3, 2, 2, 1, 1, 1)$$

$$\tau = (\cdot | 321 | 422 | \cdot | 311)$$
Fix a (circular 1D) lattice on \( n \) sites and a partition \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0) \) for the particle types

\[ \text{TAZRP}(\lambda, n) \] is a Markov chain whose states are multiset compositions \( \tau \) of type \( \lambda \), with \( n \) (possibly empty) parts

\[ \lambda = (4, 3, 3, 2, 2, 1, 1, 1) \]

\[ n = 5 \]

\[ \tau = (\cdot | 321 | 422 | \cdot | 311) \]
Each particle is equipped with an exponential clock. Transitions are jumps from site $j$ to site $j + 1$. The rates depend on a fixed parameter $0 \leq t < 1$, and on the content of the site containing the particle. For $1 \leq j \leq n$ and $k \in \lambda$, call $f_j(k)$ the rate of the jump of particle $k$ from site $j$ to site $j + 1$. If site $j$ has $d$ particles larger than $k$ and $c$ particles of type $k$, then $f_j(k) = x - 1_j t^{d-c+1} \sum_{u=0}^{t} u$.

For example: If site $j$ contains the particles $\{4, 3, 3, 1, 1, 1\}$, then:

- $k = 1$ : $d = 3$, $c = 3$, $f_j(1) = x - 1_j t^3 (1 + t + t^2)$.
- $k = 3$ : $d = 1$, $c = 2$, $f_j(3) = x - 1_j t (1 + t)$.
- $k = 4$ : $d = 0$, $c = 1$, $f_j(4) = x - 1_j t^0$.
Each particle is equipped with an exponential clock. Transitions are jumps from site $j$ to site $j+1$.

The rates depend on a fixed parameter $0 \leq t < 1$, and on the content of the site containing the particle.
Each particle is equipped with an exponential clock. Transitions are jumps from site $j$ to site $j + 1$.

The rates depend on a fixed parameter $0 \leq t < 1$, and on the content of the site containing the particle.

For $1 \leq j \leq n$ and $k \in \lambda$, call $f_j(k)$ the rate of the jump of particle $k$ from site $j$ to site $j + 1$. If site $j$ has $d$ particles larger than $k$ and $c$ particles of type $k$, then

$$f_j(k) = x_j^{-1} t^d \sum_{u=0}^{c-1} t^u$$
the mTAZRP: transition rates

- Each particle is equipped with an exponential clock. Transitions are jumps from site $j$ to site $j + 1$.

- The rates depend on a fixed parameter $0 \leq t < 1$, and on the content of the site containing the particle.

- For $1 \leq j \leq n$ and $k \in \lambda$, call $f_j(k)$ the rate of the jump of particle $k$ from site $j$ to site $j + 1$. If site $j$ has $d$ particles larger than $k$ and $c$ particles of type $k$, then

$$f_j(k) = x_j^{-1} t^d \sum_{u=0}^{c-1} t^u$$
the mTAZRP: transition rates

- Each particle is equipped with an exponential clock. Transitions are jumps from site $j$ to site $j + 1$

- The rates depend on a fixed parameter $0 \leq t < 1$, and on the content of the site containing the particle

- For $1 \leq j \leq n$ and $k \in \lambda$, call $f_j(k)$ the rate of the jump of particle $k$ from site $j$ to site $j + 1$. If site $j$ has $d$ particles larger than $k$ and $c$ particles of type $k$, then

$$f_j(k) = x_j^{-1} t^d \frac{c - 1}{u=0} t^u$$

For example: If site $j$ contains the particles $\{4, 3, 3, 1, 1, 1\}$, then:

- $k = 1: \quad d = 3, \quad c = 3, \quad f_j(1) = x_j^{-1} t^3 (1 + t + t^2)$.
- $k = 3: \quad d = 1, \quad c = 2, \quad f_j(3) = x_j^{-1} t (1 + t)$.
- $k = 4: \quad d = 0, \quad c = 1, \quad f_j(4) = x_j^{-1}$. 

Given a filling $\sigma$, read the state $\tau \in \mathrm{TAZRP}(\lambda, n)$ from the bottom row of $\sigma$ as follows:

$$\tau_j \text{ is the multiset } \{\lambda_i : \sigma(1, i) = j\}$$
Lumping of tableaux to mTAZRP

Given a filling $\sigma$, read the state $\tau \in \text{TAZRP}(\lambda, n)$ from the bottom row of $\sigma$ as follows:

$$\tau_j$$ is the multiset $\{\lambda_i : \sigma(1, i) = j\}$

For example, for $\lambda = (2, 1, 1)$ and $n = 3$, the following are all the tableaux that correspond to the state $\tau = (21 \mid \cdot \mid 1)$:

1\begin{tabular}{|c|c|c|}
\hline
1 & 1 & 3 \\
\hline
\end{tabular}  2\begin{tabular}{|c|c|c|}
\hline
2 & 1 & 1 \\
\hline
\end{tabular}  3\begin{tabular}{|c|c|c|}
\hline
3 & 1 & 1 \\
\hline
\end{tabular}  1\begin{tabular}{|c|c|c|}
\hline
1 & 1 & 3 \\
\hline
\end{tabular}  2\begin{tabular}{|c|c|c|}
\hline
2 & 1 & 3 \\
\hline
\end{tabular}  3\begin{tabular}{|c|c|c|}
\hline
3 & 1 & 3 \\
\hline
\end{tabular}
Theorem (Ayyer–M–Martin ’21)

Fix $\lambda, n$. The (unnormalized) stationary probability of $\tau \in \text{TAZRP}(\lambda, n)$ is

$$\tilde{\Pr}(\tau) = \sum_{\sigma: \text{dg}(\lambda) \to [n]} x^{\sigma} t^{\text{quinv}(\sigma)}.$$

Corollary

The so-called partition function of $\text{TAZRP}(\lambda, n)$ is

$$\mathcal{Z}_{\lambda, n}(x_1, \ldots, x_n; t) = \tilde{H}_\lambda(x_1, \ldots, x_n; 1, t).$$
an example for $\lambda = (2, 1, 1)$ and $n = 2$

The stationary distribution is:

$$
\begin{align*}
(211 | \cdot) & \quad x_1^3(x_1 + x_2) \\
(11 | 2) & \quad x_1^2x_2(t^2x_2 + x_1) \\
(21 | 1) & \quad x_1^2x_2(tx_1 + x_2)(1 + t) \\
(1 | 21) & \quad x_1x_2^2(x_1 + tx_2)(1 + t) \\
(2 | 11) & \quad x_1x_2^2(t^2x_1 + x_2) \\
(\cdot | 211) & \quad x_2^3(x_1 + x_2)
\end{align*}
$$

Example computation for $(21 | 1)$:

$$
\begin{align*}
\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
\end{array} & : t^2, \\
\begin{array}{c}
2 \\
1 \\
1 \\
2 \\
\end{array} & : t, \\
\begin{array}{c}
1 \\
1 \\
2 \\
1 \\
\end{array} & : t, \\
\begin{array}{c}
2 \\
1 \\
2 \\
1 \\
\end{array} & : 1
\end{align*}
$$

the total is: $\tilde{\Pr}(21|1) = x_1^2x_2(tx_1 + x_2)(1 + t)$. 

The tableaux are actually representing a **queueing system** which is an arrangement of lattice paths/strings: the lattice paths are representing the **coupling of individual single species TAZRPs**
why queue inversions? multiline diagrams

The tableaux are actually representing a queueing system which is an arrangement of lattice paths/strings: the lattice paths are representing the coupling of individual single species TAZRPs

```
2
2 2
1 1
3 4 3
2 2 4 1
```

“plethystic version” of certain non-attacking fillings ←→ “plethystic version” of multiline queues
why queue inversions? multiline diagrams

The tableaux are actually representing a queueing system which is an arrangement of lattice paths/strings: the lattice paths are representing the coupling of individual single species TAZRPs

\[
\begin{array}{cccc}
2 & 2 & 2 & 3 \\
2 & 1 & 1 & 4 \\
3 & 4 & 3 & 2 \\
2 & 2 & 4 & 1
\end{array}
\]

quinv ←→ "refusal"

"plethystic version" of certain non-attacking fillings ←→ "plethystic version" of multiline queues
why queue inversions? multiline diagrams

The tableaux are actually representing a queueing system which is an arrangement of lattice paths/strings: the lattice paths are representing the coupling of individual single species TAZRPs

\[
\begin{array}{c}
2 \\
2 \hspace{0.2cm} 2 \\
1 \hspace{0.2cm} 1 \\
3 \hspace{0.2cm} 4 \hspace{0.2cm} 3 \\
2 \hspace{0.2cm} 2 \hspace{0.2cm} 4 \hspace{0.2cm} 1 \\
\end{array}
\]

\[
\begin{array}{c}
5 \\
4 \\
3 \\
2 \\
1
\end{array}
\]

\[
\begin{array}{cccc}
\text{quinv} & \leftrightarrow & \text{“refusal”} & \leftrightarrow \\
\text{“plethystic version” of certain non-attacking fillings} & & \text{“plethystic version” of multiline queues}
\end{array}
\]
the arm of a cell, denoted by \( \text{arm}(c) \),
- the arm of a cell, denoted by $\text{arm}(c)$,

$$\text{arm}(\sigma, c) = 3$$

- $\text{arm}(\sigma, c) =$ the number of cells in $\text{arm}(c)$ with the same content as $c$. 

If $c = (1, j)$ is in the bottom row, then $\text{arm}(\sigma, c)$ is equal to the number of particles larger than or equal to $\lambda_j$ at site $\sigma(c)$ of the corresponding state of the TAZRP. Thus $f(\sigma, c)$ is equal to the rate of the corresponding TAZRP jump.
the arm of a cell, denoted by \( \text{arm}(c) \),

\[
\begin{array}{c}
\text{k} \\
\text{k} \\
\text{k} \\
\text{k} \\
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\end{array}
\]

\( \text{arm}(\sigma, c) = 3 \)

\( \text{arm}(\sigma, c) = \) the number of cells in \( \text{arm}(c) \) with the same content as \( c \).

each cell \( c \) such that \( \sigma(\text{South}(c)) \neq \sigma(c) \) is equipped with an exponential clock with rate

\[
f(\sigma, c) = t^{\text{arm}(\sigma,c)} X_{\sigma(c)}^{-1}
\]
• the arm of a cell, denoted by $\text{arm}(c)$,

\[
\begin{array}{cccc}
\text{k} \\
\text{k} \\
\text{k} \\
\text{k} \\
\text{k}
\end{array}
\]

$\text{arm}(\sigma, c) = 3$

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\[
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A transition $M_c$ triggered by a cell $c$: if $\sigma(c) \neq \sigma(\text{South}(c))$, take the maximal contiguous (cyclically) increasing chain of cells weakly above $c$ in its column, and increment the content of each cell by 1. (This is sometimes called a ringing path)
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\[
\begin{array}{c c c c}
4 & & & \\
2 & 1 & & \\
1 & 1 & & \\
3 & 4 & 3 & \\
3 & 3 & 4 & 1 \\
\end{array}
\quad t x_3^{-1}

\begin{array}{c c c c}
4 & & & \\
2 & 1 & & \\
1 & 2 & & \\
3 & 1 & 3 & \\
3 & 4 & 4 & 1 \\
\end{array}
\]

Theorem (Ayyer–M–Martin '21)

The stationary distribution of the Markov process on the tableaux is

\[\text{wt}(\sigma) = x^\sigma t^{\text{quinv}(\sigma)}\]
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**Theorem (Ayyer–M–Martin '21)**

The stationary distribution of the Markov process on the tableaux is

$$\text{wt}(\sigma) = x^{\sigma} t^{\text{quinv}(\sigma)}$$

- if $c = (1, j)$ is in the bottom row, the rate $f(\sigma, c)$ matches the transition rate $f_{\sigma(c)}(\lambda_j)$ of the corresponding particle in the TAZRP.
- (when $\lambda$ has repeated parts, we need to do some more work!)
$M(\sigma) = \{ M_c(\sigma) : c \in \text{dg}(\lambda), \sigma(c) \neq \sigma(\text{South}(c)) \}$
a Markov chain on tableaux: proof

\[ M(\sigma) = \{ M_c(\sigma) : c \in \text{dg}(\lambda), \sigma(c) \neq \sigma(\text{South}(c)) \} \]

\[ \text{wt}(\sigma)(x_1^{-1} + tx_2^{-1} + x_2^{-1}) \]
a Markov chain on tableaux: proof

\[ M(\sigma) = \left\{ M_c(\sigma) : c \in \text{dg}(\lambda), \sigma(c) \neq \sigma(\text{South}(c)) \right\} \]

\[ R(\sigma) = \left\{ \sigma' \in \text{PQT}(\lambda, n) : T \in M(\sigma') \right\} \]

**Balance equation**

if each \( \sigma \in \text{dg}(\lambda) \to [n] \) satisfies:

\[
\sum_{\sigma' \in R(\sigma)} \text{wt}(\sigma') \text{rate}(\sigma' \to \sigma) = \text{wt}(\sigma) \sum_{\sigma' \in M(\sigma)} \text{rate}(\sigma \to \sigma'),
\]

then the stationary distribution of the M.C. on tableaux is \( \text{wt}(\sigma) \).
The partition function of the mTAZRP of type \( \lambda, n \) is defined to be the normalizing constant, or the sum of the unnormalized stationary probabilities:

\[
Z_{\lambda, n} = \sum_{\tau} \tilde{\Pr}(\tau)
\]
The partition function of the mTAZRP of type $\lambda, n$ is defined to be the normalizing constant, or the sum of the unnormalized stationary probabilities:

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We have

$$Z_{\lambda, n} = \tilde{H}_\lambda(x_1, \ldots, x_n; 1, t) = \prod_{j=1}^{\lambda_1} \tilde{H}_{(1^{\chi'_j})}(x_1, \ldots, x_n; 1, t)$$

$$= \prod_{j=1}^{\lambda_1} \sum_{\mu \vdash \lambda'_j} \left[ \frac{\chi'_j}{\mu} \right]_t m_\mu(x_1, \ldots, x_n).$$

Notice that $Z$ might have extra factors, e.g. when the gcd of the probabilities $\tilde{Pr}(\tau)$ is not 1.
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$$= \prod_{j=1}^{\lambda_1} \sum_{\mu \vdash \lambda_j'} \left[ \frac{\lambda_j'}{\mu} \right]_t m_\mu(x_1, \ldots, x_n).$$

Notice that $Z$ might have extra factors, e.g. when the gcd of the probabilities $\tilde{Pr}(\tau)$ is not 1.

Sanity check: from the point of view of the TAZRP, having three species of particles labeled 1, 2, 3 is the same process as having three species labeled 2, 13, 27. Thus we should expect their stationary probabilities to be proportional.
Observables: partition function

The partition function of the mTAZRP of type $\lambda$, $n$ is defined to be the normalizing constant, or the sum of the unnormalized stationary probabilities:

$$Z_{\lambda,n} = \sum_{\tau} \tilde{\Pr}(\tau)$$

We have

$$Z_{\lambda,n} = \tilde{H}_\lambda(x_1, \ldots, x_n; 1, t) = \prod_{j=1}^{\lambda_1} \tilde{H}_{(1^{\lambda_j})}(x_1, \ldots, x_n; 1, t)$$

$$= \prod_{j=1}^{\lambda_1} \sum_{\mu \vdash \lambda'_j} \left[ \frac{\lambda'_j}{\mu} \right] t^m \mu(x_1, \ldots, x_n).$$

Notice that $Z$ might have extra factors, e.g. when the gcd of the probabilities $\tilde{Pr}(\tau)$ is not 1.

Sanity check: from the point of view of the TAZRP, having three species of particles labeled 1, 2, 3 is the same process as having three species labeled 2, 13, 27. Thus we should expect their stationary probabilities to be proportional.

At the very least, we need $\tilde{H}_{(2,13,27)}(x_1, \ldots, x_n; 1, t)$ to be divisible by $\tilde{H}_{(1,2,3)}(x_1, \ldots, x_n; 1, t)$. This is indeed true, since $(3, 2, 1)' \subset (27, 13, 2)'$. 
The current of particle $\ell$ across the edge $j$ is defined as the number of particles of type $\ell$ traversing the edge $j$ per unit of time in the large time limit.
The current of particle \( \ell \) across the edge \( j \) is defined as the number of particles of type \( \ell \) traversing the edge \( j \) per unit of time in the large time limit.

Let us first look at the single species case: \( \lambda = (1^m) \) on \( n \) sites.

Here, \( n = 5, \ m = 7 \)

\( \tau = (2, 0, 3, 1, 1) \)

Each configuration can be written as a weak composition \( \tau = (\tau_1, \ldots, \tau_n) \).
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The stationary probability of the configuration \( \tau \) is:

\[
\pi(\tau) = \frac{1}{\tilde{\mathcal{H}}(1^m)(x_1, \ldots, x_n; 1, t)} \left[ \tau_1, \ldots, \tau_n \right]_t \prod_{i=1}^{n} x_i^{\tau_i}.
\]
Current

- The current of particle $\ell$ across the edge $j$ is defined as the number of particles of type $\ell$ traversing the edge $j$ per unit of time in the large time limit.
- Let us first look at the single species case: $\lambda = (1^m)$ on $n$ sites.

Here, $n = 5$, $m = 7$

$\tau = (2, 0, 3, 1, 1)$

Each configuration can be written as a weak composition $\tau = (\tau_1, \ldots, \tau_n)$.
- The stationary probability of the configuration $\tau$ is:

$$\pi(\tau) = \frac{1}{\tilde{H}_{(1^m)}(x_1, \ldots, x_n; 1, t)} \left[\begin{array}{c} m \\ \tau_1, \ldots, \tau_n \end{array}\right]_t \prod_{i=1}^{n} x_i^{\tau_i}$$

Proposition (Current for the single species TAZRP)

For the single-species TAZRP on $n$ sites with $m$ particles, the current is given by

$$J = [m]_t \frac{\tilde{H}_{(1^{m-1})}(x_1, \ldots, x_n; 1, t)}{\tilde{H}_{(1^m)}(x_1, \ldots, x_n; 1, t)}.$$
Current

Theorem (Ayyer-M-Martin '22+)

Let $\lambda = (1^{m_1}, \ldots, k^{m_k})$, and let $1 \leq j \leq k$. The current of the particle of type $j$ of the TAZRP of type $\lambda$ on $n$ sites is given by

$$J = \left[ m_j + \cdots + m_k \right]_t \frac{\tilde{H}(1^{m_j+\cdots+m_k-1})}{\tilde{H}(1^{m_j+\cdots+m_k})} - \left[ m_{j+1} + \cdots + m_k \right]_t \frac{\tilde{H}(1^{m_{j+1}+\cdots+m_k-1})}{\tilde{H}(1^{m_{j+1}+\cdots+m_k})}$$
Densities

- Take TAZRP($\lambda$, $n$) with content $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$.
- Define $z_j^{(\ell)}$ to be the random variable counting the number of particles of type $\ell$ at site $j$ in a configuration of TAZRP($\lambda$, $n$).
- Denote the expectation in the stationary distribution by $\langle z_j^{(\ell)} \rangle$.

Proposition (Translation invariance)
Suppose $\langle z_1^{(\ell)} \rangle = r(x_1, \ldots, x_n)$. Then for any $j$, $\langle z_j^{(\ell)} \rangle = r(x_j, \ldots, x_n, x_1, \ldots, x_{j-1})$.
Thus it suffices to compute the densities of all species of particles at site 1.

We begin with the special case of $\lambda = 1^{m_1}$.

Theorem (Densities for the single species TAZRP)
The density at site 1 on TAZRP($1^{m_1}$, $n$) is given by $\langle z_1^{(1)} \rangle = x_1 \frac{\partial}{\partial x_1} \log \tilde{H}(1^{m_1})(x_1, \ldots, x_n; 1, t)$.
In particular, when $x_1 = \cdots = x_n = 1$, the density is $\langle z_1^{(1)} \rangle = m/n$. 

Densities

- Take TAZRP($\lambda$, $n$) with content $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$.

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Densities

- Take $\text{TAZRP}(\lambda, n)$ with content $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$.

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**Proposition (Translation invariance)**

Suppose \( \langle z_1^{(\ell)} \rangle = r(x_1, \ldots, x_n) \). Then for any \( j \),

\[
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\]

Thus it suffices to compute the densities of all species of particles at site 1.

We begin with the special case of \( \lambda = 1^m \).

**Theorem (Densities for the single species TAZRP)**

The density at site 1 on TAZRP(1^m, n) is given by

\[
\langle z_1^{(1)} \rangle = x_1 \partial_{x_1} \log \tilde{H}_{(1^m)}(x_1, \ldots, x_n; 1, t).
\]
Densities

- Take $\text{TAZRP}(\lambda, n)$ with content $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$.

- Define $z_j^{(\ell)}$ to be the random variable counting the number of particles of type $\ell$ at site $j$ in a configuration of $\text{TAZRP}(\lambda, n)$.

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Suppose $\langle z_1^{(\ell)} \rangle = r(x_1, \ldots, x_n)$. Then for any $j$,

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$$\langle z_1^{(1)} \rangle = x_1 \partial_{x_1} \log \tilde{H}_{(1^m)}(x_1, \ldots, x_n; 1, t).$$

In particular, when $x_1 = \cdots = x_n = 1$, the density is $\langle z_1^{(1)} \rangle = \frac{m}{n}$. 
### Densities

**Theorem (Ayyer-M-Martin ’22+)**

For $1 \leq \ell \leq k$, the density of the $\ell$'th species at site 1 is given by

$$
\langle z_{1}^{(\ell)} \rangle = x_{1} \partial_{x_{1}} \log \left( \frac{\widetilde{H}_{(1^{m_{\ell}}+\cdots+m_{k})}(x_{1}, \ldots, x_{n}; 1, t)}{\widetilde{H}_{(1^{m_{\ell+1}}+\cdots+m_{k})}(x_{1}, \ldots, x_{n}; 1, t)} \right).
$$

**Corollary**

$\langle z_{1}^{(\ell)} \rangle$ is symmetric in the variables $\{x_{2}, \ldots, x_{n}\}$.

**Proof via coloring argument:**

True for base case $\lambda = (1^{m_{\ell}})$ transitions of particles of species $\ell,...,k$ at site 1 are independent of the number of lower species particles at site 1. Thus we can ignore the particles of types $1,...,\ell-1$.

The density of particles of species $\ell,...,k$ at site 1 is equivalent to the density at site 1 of a TAZRP of type $\lambda = (1^{m_{\ell}}+\cdots+m_{k})$.

To isolate species $\ell$ we subtract the density of species $\ell+1,...,k$ from the density of species $\ell,...,k$. 

Densities

Theorem (Ayyer-M-Martin '22+)

For $1 \leq \ell \leq k$, the density of the $\ell$'th species at site 1 is given by

$$\langle z_1^{(\ell)} \rangle = x_1 \partial_{x_1} \log \left( \frac{\tilde{H}_{(1^m \ell + \cdots + m_k)}(x_1, \ldots, x_n; 1, t)}{\tilde{H}_{(1^m \ell + 1 + \cdots + m_k)}(x_1, \ldots, x_n; 1, t)} \right).$$

Corollary

$\langle z_1^{(\ell)} \rangle$ is symmetric in the variables $\{x_2, \ldots, x_n\}$. 
Theorem (Ayyer-M-Martin ’22+)

For \(1 \leq \ell \leq k\), the density of the \(\ell\)'th species at site 1 is given by

\[
\langle z_1^{(\ell)} \rangle = x_1 \partial_{x_1} \log \left( \frac{\tilde{H}_{(1^m)\ell+\cdots+mk)}(x_1, \ldots, x_n; 1, t)}{\tilde{H}_{(1^m)\ell+1+\cdots+mk)}(x_1, \ldots, x_n; 1, t)} \right).
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\(\langle z_1^{(\ell)} \rangle\) is symmetric in the variables \(\{x_2, \ldots, x_n\}\).

Proof via coloring argument:

- true for base case \(\lambda = (1^m)\)
- transitions of particles of species \(\ell, \ldots, k\) at site 1 are independent of the number of lower species particles at site 1. Thus we can ignore the particles of types \(1, \ldots, \ell - 1\).
- the density of particles of species \(\ell, \ldots, k\) at site 1 is equivalent to the density at site 1 of a TAZRP of type \(\lambda = (1^{m_\ell+\cdots+mk})\).
- to isolate species \(\ell\) we subtract the density of species \(\ell + 1, \ldots, k\) from the density of species \(\ell, \ldots, k\).
Local correlations

- Fix $\lambda$, $n$, and $0 \leq \ell \leq n$, and let $w$ be a configuration of the TAZRP on the first $\ell$ sites of type $\mu$, where $\mu \subseteq \lambda$. 

Theorem (Ayyer-M-Martin '22+)

Both $P_{\lambda,n}(w)$ and $P_{\lambda,n}(\hat{w})$ are symmetric in the variables $\{x_{\ell+1}, \ldots, x_n\}$. 
Fix $\lambda, n$, and $0 \leq \ell \leq n$, and let $w$ be a configuration of the TAZRP on the first $\ell$ sites of type $\mu$, where $\mu \subseteq \lambda$.

We consider two kinds of local correlations:

- Let $P_{\lambda, n}(w)$ be the stationary probability of having exactly the content $w_1, \ldots, w_{\ell}$ on sites $1, \ldots, \ell$.
- Let $P_{\lambda, n}(\hat{w})$ be the stationary probability of having at least the content $w_1, \ldots, w_{\ell}$ on sites $1, \ldots, \ell$.

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Fix $\lambda, n$, and $0 \leq \ell \leq n$, and let $w$ be a configuration of the TAZRP on the first $\ell$ sites of type $\mu$, where $\mu \subseteq \lambda$.

We consider two kinds of local correlations:

- Let $P_{\lambda, n}(\overline{w})$ be the stationary probability of having \textbf{exactly} the content $w_1, \ldots, w_\ell$ on sites $1, \ldots, \ell$.
- Let $P_{\lambda, n}(\hat{w})$ be the stationary probability of having \textbf{at least} the content $w_1, \ldots, w_\ell$ on sites $1, \ldots, \ell$.

Example: let $\lambda = (2, 2, 1, 1)$, $n = 4$, $\ell = 2$, and $w = (2|1)$.

- Configurations contributing to $P_{\lambda, n}(\overline{w})$ are
  $$(2|1|12|\cdot), \quad (2|1|1|2), \quad (2|1|2|1), \quad (2|1| \cdot |12)$$

- Additional configurations contributing to $P_{\lambda, n}(\hat{w})$ are
  $$(12|1|2|\cdot), \quad (2|11|2|\cdot), \quad (22|1|1|\cdot), \quad (2|12|1|\cdot), \quad (12|1| \cdot |2), \quad (2|11| \cdot |2)$$
  $$(22|1| \cdot |1), \quad (2|12| \cdot |1), \quad (122|1| \cdot |), \quad (2|112| \cdot |), \quad (22|11| \cdot |), \quad (12|12| \cdot |)$$
Local correlations

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- Additional configurations contributing to $P_{\lambda,n}(\hat{w})$ are
  
  $$(12|1|2|\cdot), \quad (2|11|2|\cdot), \quad (22|1|1|\cdot), \quad (2|12|1|\cdot), \quad (12|1|\cdot|2), \quad (2|11|\cdot|2)$$

  $$(22|1|\cdot|1), \quad (2|12|\cdot|1), \quad (122|1|\cdot|\cdot), \quad (2|112|\cdot|\cdot), \quad (22|11|\cdot|\cdot), \quad (12|12|\cdot|\cdot)$$

---

**Theorem (Ayyer-M-Martin ’22+)**

Both $P_{\lambda,n}(w)$ and $P_{\lambda,n}(\hat{w})$ are symmetric in the variables $\{x_{\ell+1}, \ldots, x_n\}$. 

Explicit bijection from the \textit{inv} to the \textit{quinv} statistic?

\[
\begin{array}{cccccc}
\times & \cdots & \times \\
y & \cdots & y \\
z & & z
\end{array}
\quad \text{vs} \quad 
\begin{array}{cccccc}
x & \cdots & x \\
y & \cdots & y \\
z & & z
\end{array}
\]

Can we find a dynamical process that incorporates the \( q \) as a parameter?

This seems difficult because
- We lose factorization of \( \widetilde{H}_\lambda \)
- We lose translation invariance

Using multiline queues (for the ASEP on a circle), Corteel-Haglund-M-Mason-Williams '20 defined \textit{quasisymmetric Macdonald polynomials} which refine \( P_\lambda \). Can we use a parallel construction to define an interesting family of \textit{quasisymmetric polynomials} that refine \( \widetilde{H}_\lambda \)?

Same as above, but for \textit{nonsymmetric} Macdonald polynomials
Modified Macdonald polynomials and the multispecies zero range process: I,  
(with A. Ayyer and J. B. Martin), arXiv:2011.06117

A Markov chain on tableaux that projects to the multispecies TAZRP, and  
applications, (with A. Ayyer and J. B. Martin), in preparation