Schur functions in noncommuting variables

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Integer partitions

An integer partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ of n is a list of positive integers whose sum is n: $3221 \vdash 8$.

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell = 1^{r_1} 2^{r_2} \cdots n^{r_n}$. Then

$$\lambda! = \lambda_1! \lambda_2! \cdots \lambda_\ell!$$

and

$$\lambda^! = r_1! r_2! \cdots r_n!$$

Example

If $\lambda = 3221 = 1^1 2^2 3^1 4^0 5^0 6^0 7^0 8^0 \vdash 8$, then

$$\lambda! = 3!2!2!1! = 6 \times 2 \times 2 \times 1 = 24$$

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Set partitions

A set partition π of $[n] = \{1, 2, \ldots, n\}$ is a partitioning of $[n]$ into disjoint sets $B_1, B_2, \ldots, B_\ell$ called blocks so

- $B_i \neq \emptyset$
- $B_1 \cup B_2 \cup \cdots \cup B_\ell = [n]$.

$$\pi = B_1/B_2/\cdots/B_\ell \vdash [n]$$

Example

$$\{1, 3, 4\}, \{2, 5\}, \{6\}, \{7, 8\}$$

is a set partition of [8], or

$$\pi = 134/25/6/78 \vdash [8].$$

Partitions and permutations

If $\pi = B_1/B_2/\cdots/B_\ell \vdash [n]$, then

$$\lambda(\pi) = |B_1||B_2|\cdots|B_\ell|$$

with sizes weakly decreasing. If $\delta \in \mathfrak{S}_n$, then

$$\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).$$

Example

If $\pi = 134/25/6/78 \vdash [8]$, then

$$\lambda(134/25/6/78) = 3221$$

and $\delta = 14325876 \in \mathfrak{S}_8$

$$\begin{aligned}\delta(134/25/6/78) &= \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8)\\ &= 132/45/8/76 = 123/45/67/8.\end{aligned}$$

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Slash product

If

$$S + n = \{s + n : s \in S\}$$

then for $\pi \vdash [n]$ and $\sigma = B_1/B_2/\cdots/B_\ell \vdash [m]$ the slash product is

$$\pi \mid \sigma = \pi/(B_1 + n)/(B_2 + n)/\cdots/(B_\ell + n) \vdash [n + m].$$

Example

If $\pi = 134/25 \vdash [5]$ and $\sigma = 1/23 \vdash [3]$ then

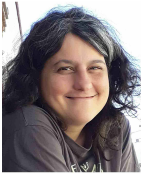
$$\pi \mid \sigma = 134/25/6/78 \vdash [8].$$

NCSym (Rosas-Sagan 2004)

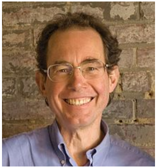
NCSym is the algebra of symmetric functions in noncommuting variables $x_1, x_2, x_3, \ldots$

$$\mathrm{NCSym} = \mathrm{NCSym}^0 \oplus \mathrm{NCSym}^1 \oplus \cdots \subset \mathbb{Q} \ll x_1, x_2, x_3, \ldots \gg$$

where $\mathrm{NCSym}^0 = \mathrm{span}\{1\}$ and for $n > 0$



$$\begin{aligned} \mathrm{NCSym}^n &= \mathrm{span}\{m_\pi \ : \ \pi \vdash [n]\} \\ &= \mathrm{span}\{p_\pi \ : \ \pi \vdash [n]\} \\ &= \mathrm{span}\{e_\pi \ : \ \pi \vdash [n]\} \\ &= \mathrm{span}\{h_\pi \ : \ \pi \vdash [n]\}. \end{aligned}$$



Note: The e_π defined by Wolf in 1936.

Monomial function in NCSym

The monomial symmetric function in NCSym for $\pi \vdash [n]$ is

$$m_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j = i_k$$

if and only if j and k are in the same block of π.

Example

$m_{13/2} = x_1 x_2 x_1 + x_2 x_1 x_2 + x_1 x_3 x_1 + x_2 x_3 x_2 + \cdots$

Power sum function in NCSym

The power sum symmetric function in NCSym for $\pi \vdash [n]$ is

$$p_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j = i_k$$

if j and k are in the same block of π.

Example

$p_{13/2} = x_1 x_2 x_1 + x_2 x_1 x_2 + \cdots + x_1^3 + x_2^3 + \cdots$

Elementary function in NCSym

The elementary symmetric function in NCSym for $\pi \vdash [n]$ is

$$e_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j \neq i_k$$

if j and k are in the same block of π.

Example

$e_{13/2} = x_1 x_1 x_2 + x_1 x_2 x_2 + x_2 x_2 x_1 + x_2 x_1 x_1 + \cdots + x_1 x_2 x_3 + \cdots$

Complete homogeneous function in NCSym

The complete homogeneous symmetric function in NCSym for $\pi \vdash [n]$ is

$$h_\pi = \sum_\sigma \lambda(\sigma \wedge \pi)! m_\sigma$$

where $\sigma \wedge \pi$ is the maximal set partition where every block is a subblock of σ and π.

Example

$$h_{13/2} = 2m_{123} + m_{12/3} + m_{1/23} + 2m_{13/2} + m_{1/2/3}$$

$$123 \wedge 13/2 = 13/2 \wedge 13/2 = 13/2 \quad \lambda(13/2)! = 2$$

$$12/3 \wedge 13/2 = 1/23 \wedge 13/2 = 1/2/3 \wedge 13/2 = 1/2/3 \quad \lambda(1/2/3)! = 1$$

Permutations and products

Fact: If $\pi \vdash [n]$, $\delta \in \mathfrak{S}_n$ and $\delta \circ m_\pi = m_{\delta(\pi)}$, then for $b = p, e, h$

$$\delta \circ b_\pi = b_{\delta(\pi)}.$$

Fact: If π, σ are set partitions, then for $b = p, e, h$

$$b_\pi b_\sigma = b_{\pi|\sigma}.$$

Example

$$132 \circ m_{12/3} = m_{13/2} \qquad p_{13/2} p_1 = p_{13/2|1} = p_{13/2/4}$$

What's in a name?

Let the variables commute:

$$\rho : \mathrm{NCSym} \to \mathrm{Sym}$$

Theorem (Rosas-Sagan 2004)

Let π be a set partition.

$$\rho(m_\pi) = \lambda(\pi)^! m_{\lambda(\pi)} \qquad \rho(p_\pi) = p_{\lambda(\pi)}$$

$$\rho(e_\pi) = \lambda(\pi)! e_{\lambda(\pi)} \qquad \rho(h_\pi) = \lambda(\pi)! h_{\lambda(\pi)}$$

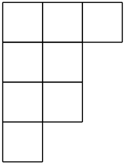
Note: The images are classical monomial, power sum, elementary and complete homogeneous symmetric functions.

Question: Where are the Schur functions?

Partitions and diagrams

An integer partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ of n is a list of positive integers whose sum is n: $3221 \vdash 8$.

The diagram $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ is the array of boxes with λ_i boxes in row i from the top.



3221

Semistandard Young tableaux

A semistandard Young tableau (SSYT) T of shape λ is a filling with $1, 2, 3, \ldots$ so rows weakly increase and columns increase.

1	1
2	

Given an SSYT T we have for commuting variables

$$x^T = x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \cdots .$$

$$x_1^2 x_2$$

Schur functions

The Schur function in Sym is

$$s_\lambda = \sum_{T \text{ SSYT of shape } \lambda} x^T.$$

Example

$$s_{21} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3 + \cdots$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Dotted Young tableaux

A dotted Young tableau (DYT) $\dot{T}$ of shape λ is a filling with $1, 2, 3, \ldots$ so rows weakly increase and columns increase, and $1, 2, \ldots, n$ dots appear exactly once.

$\dddot{1}$	$\dot{1}$
$\ddot{2}$	

Given a DYT $\dot{T}$ we have for noncommuting variables

$$x^{\dot{T}} = x_i \text{ in position } j \text{ iff } i \text{ has } j \text{ dots above it.}$$

$$x_1 x_2 x_1$$

Rosas-Sagan Schur functions

The Rosas-Sagan Schur function in NCSym is

$$S^{RS}_{\lambda} = \sum_{\dot{T} \text{ DYT of shape } \lambda} x^{\dot{T}}.$$

Example

$$S^{RS}_{21} = 2x_1x_1x_2 + 2x_1x_2x_1 + 2x_2x_1x_1 + \cdots$$

$$\begin{array}{|c|c|}\hline \dot{1} & \ddot{1} \\ \hline \dddot{2} & \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline \ddot{1} & \dot{1} \\ \hline \dddot{2} & \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline \dot{1} & \dddot{1} \\ \hline \ddot{2} & \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline \dddot{1} & \dot{1} \\ \hline \ddot{2} & \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline \ddot{1} & \dddot{1} \\ \hline \dot{2} & \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline \dddot{1} & \ddot{1} \\ \hline \dot{2} & \\ \cline{1-1} \end{array}$$

Rosas-Sagan Schur functions

Theorem (Rosas-Sagan 2004)

Let $\lambda \vdash n$.

- *The S_λ^{RS} are linearly independent.*
- *We have $\rho(S_\lambda^{RS}) = n! s_\lambda$.*

Note: However they are not a basis for NCSym because we only have one for each integer partition, not set partition.

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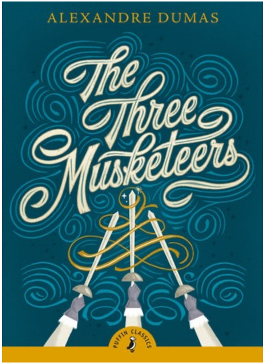
Yes there is!

The three musketeers



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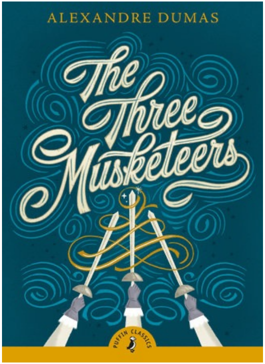

Schur functions revisited

Definition

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the Jacobi-Trudi identity

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le \ell}$$

where $h_0 = 1$ and $h_{\text{-ve}} = 0$.

Example

$$s_{21} = \det\begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = h_2 h_1 - h_3 h_0 = h_{21} - h_3$$

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"All for one ..." – Dumas, T3M

The noncommutative Leibniz formula for $A = (a_{ij})_{1 \le i,j \le n}$ with noncommuting entries a_{ij} is

$$\det(A) = \sum_{\varepsilon \in \mathfrak{S}_n} \operatorname{sgn}(\varepsilon) a_{1\varepsilon(1)} a_{2\varepsilon(2)} \cdots a_{n\varepsilon(n)}$$

- product of the entries is taken top row to the bottom row
- $\operatorname{sgn}(\varepsilon)$ is the sign of permutation ε.

" ... and one for all," – Dumas, T3M

Definition

Let $\lambda = \lambda_1\lambda_2\cdots\lambda_\ell$. Then the source Schur function

$$s_{[\lambda]} = \mathbf{det}\left(\frac{1}{(\lambda_i - i + j)!} h_{[\lambda_i - i + j]}\right)_{1\leq i,j\leq \ell}$$

where $h_{[0]} = h_\emptyset = 1$ and $h_{\text{-ve}} = 0$.

Example

$$s_{[21]} = \mathbf{det}\begin{pmatrix} \frac{1}{2!}h_{12} & \frac{1}{3!}h_{123} \\ \frac{1}{0!}h_\emptyset & \frac{1}{1!}h_1 \end{pmatrix} = \frac{1}{2!}h_{12}\frac{1}{1!}h_1 - \frac{1}{3!}h_{123}\frac{1}{0!}h_\emptyset$$
$$= \frac{1}{2}h_{12|1} - \frac{1}{6}h_{123} = \frac{1}{2}h_{12/3} - \frac{1}{6}h_{123}.$$

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$$= \frac{1}{2}h_{12|1} - \frac{1}{6}h_{123} = \frac{1}{2}h_{12/3} - \frac{1}{6}h_{123}.$$

"… and one for all," – Dumas, T3M

We now create a set partition $\pi \vdash [n]$ from an integer partition $\lambda \vdash n$ where

$$\lambda(\pi) = \lambda$$

using tableaux T_π of shape λ such that

- every element $1, 2, \ldots, n$ appears exactly once
- rows increase left to right
- first column of rows of same length increase top to bottom.

$$\text{rows} \longleftrightarrow \text{blocks}$$

Example

$$T_\pi = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \qquad \longleftrightarrow \qquad \pi = 134/25/78/6$$

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We now create a permutation $\delta_\pi \in \mathfrak{S}_n$ using T_π.

read by row $\longleftrightarrow$ one line notation

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Definition

Let $\pi = B_1/B_2/\cdots/B_\ell$. Then the Schur function in NCSym is

$$s_\pi = \delta_\pi \circ s_{[\lambda(\pi)]} = \delta_\pi \circ \mathbf{det}\left(\frac{1}{(\lambda_i - i + j)!} h_{[\lambda_i - i + j]}\right)_{1\le i,j\le \ell}.$$

Example

If $\pi = 13/2$ then $T_\pi = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$ and $\delta_\pi = 132$.

$$s_{13/2} = 132 \circ s_{[21]} = 132 \circ \left(\frac{1}{2}h_{12/3} - \frac{1}{6}h_{123}\right) = \frac{1}{2}h_{13/2} - \frac{1}{6}h_{123}$$

A refined basis

Theorem (Aliniaeifard-Li-vW 2022)

$$\mathrm{NCSym}^n = \mathrm{span}\{s_\pi \ : \ \pi \vdash [n]\} \qquad \rho(s_\pi) = s_{\lambda(\pi)}$$

$$S^{RS}_\lambda = \sum_{\delta \in \mathfrak{S}_n} \delta \circ s_{[\lambda]}$$

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Note: $\delta \circ s_\pi \neq s_{\delta(\pi)}$ in general.

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Theorem (Aliniaeifard-Li-vW 2022)

For $n \geq 5$ we have $n!$ different bases:

$$\mathrm{NCSym}^n = \mathrm{span}\{\delta \circ s_\pi \ : \ \pi \vdash [n]\} \qquad \rho(\delta \circ s_\pi) = s_{\lambda(\pi)}$$

Young tableaux

A Young tableau (YT) t of shape λ is a filling with $1, 2, 3, \ldots$ so each number appears exactly once.

We now create a permutation $\delta_t \in \mathfrak{S}_n$ using t.

read by row $\longleftrightarrow$ one line notation

Then

$$s_t = \delta_t \circ s_{[\lambda]}.$$

Example

$$t = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} \qquad \delta_t = 213$$

$$s_t = 213 \circ s_{[21]} = 213 \circ \left(\frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}\right) = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}$$

Young tableaux

The row-equivalence class $[t]$ of t consists of $\tilde{t}$

- same shape as t
- same set of row entries as t.

Example

$$t = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad [t] = \left\{ \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \right\}$$

$$\delta_t = 213 \quad \delta_{\tilde{t}} = 123$$

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A tabloid basis

The tabloid Schur function in NCSym is

$$s_{[t]} = \sum_{\tilde{t}} s_{\tilde{t}}$$

where $\tilde{t}$ and t are row equivalent.

Example

$$s\left[\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array}\right] = 213 \circ s_{[21]} + 123 \circ s_{[21]}$$

$$= \frac{1}{2}h_{12/3} - \frac{1}{6}h_{123} + \frac{1}{2}h_{12/3} - \frac{1}{6}h_{123} = h_{12/3} - \frac{1}{3}h_{123}$$

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Theorem (Aliniaeifard-Li-vW 2022)

We have basis

$$\mathrm{NCSym}^n = \mathrm{span}\{s_{[t]} \,:\, t \leftrightarrow \pi, \pi \vdash [n]\}.$$

If t has shape λ then

$$\rho(s_{[t]}) = \lambda! s_\lambda.$$

Specht modules

Given Young tableau t

$$e_t = \left(\sum_\delta \operatorname{sgn}(\delta)\delta\right) \circ [t] \qquad \mathbf{e}_t = \left(\sum_\delta \operatorname{sgn}(\delta)\delta\right) \circ s_{[t]}$$

where the sums are over all column-stabilizers of t that permute elements within each column.

Given partition λ we have Specht module; submodule of NCSym.

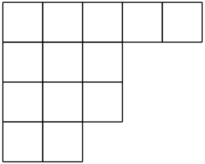
$$S^\lambda = \text{span}\{e_t \ : \ t \text{ has shape } \lambda\} \qquad \mathbf{S}^\lambda = \text{span}\{\mathbf{e}_t \ : \ t \text{ has shape } \lambda\}$$

Theorem (Aliniaeifard-Li-vW 2022)

$$\begin{array}{ccc} S^\lambda & \cong & \mathbf{S}^\lambda \\ e_t & \mapsto & \mathbf{e}_t \end{array}$$

Skew diagrams

For $\lambda \vdash n, \mu \vdash m$ the skew diagram λ/μ is the array of $n - m$ boxes contained in λ but not in μ.



5332/221

A skew shape λ/μ is a ribbon if it is

connected with no 2×2 square.

A ribbon can be denoted by row lengths α:

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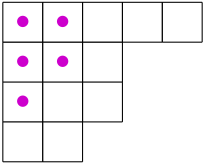

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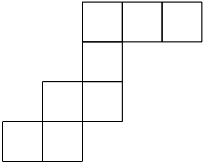

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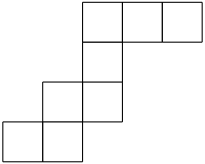

5332/221

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A ribbon can be denoted by row lengths α: 3122.

skew Schur functions

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the skew Schur function in Sym is

$$s_{\lambda/\mu} = \det \left(h_{\lambda_i - \mu_j - i + j} \right)_{1 \leq i,j \leq \ell}$$
$$= \sum_{T \text{ SSYT of shape } \lambda/\mu} x^T$$
$$= \sum_{\nu} c^{\lambda}_{\mu\nu} s_\nu$$

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&= \sum_{T \text{ SSYT of shape } \lambda/\mu} x^T \\
&= \sum_{\nu} c^{\lambda}_{\mu\nu} s_\nu
\end{aligned}
$$

and the $c^{\lambda}_{\mu\nu}$ are the Littlewood-Richardson coefficients.

Let λ/μ be a ribbon α. Then the ribbon Schur function in Sym is

$$r_\alpha = s_{\lambda/\mu}.$$

Skew Schur functions in noncommuting variables

Let $\lambda = \lambda_1\lambda_2\cdots\lambda_\ell$. Then the source skew Schur function

$$s_{[\lambda/\mu]} = \mathbf{det}\left(\frac{1}{(\lambda_i-\mu_j-i+j)!}h_{[\lambda_i-\mu_j-i+j]}\right)_{1\le i,j\le \ell}$$

and the skew Schur function in NCSym is

$$s_{(\delta,\lambda/\mu)} = \delta\circ s_{[\lambda/\mu]} = \delta\circ\mathbf{det}\left(\frac{1}{(\lambda_i-\mu_j-i+j)!}h_{[\lambda_i-\mu_j-i+j]}\right)_{1\le i,j\le \ell}.$$

Theorem (Aliniaeifard-Li-vW 2022)

$$\rho(s_{(\delta,\lambda/\mu)}) = s_{\lambda/\mu}$$

Skew Schur functions in noncommuting variables

$$s_{[22/1]} = \mathbf{det}\begin{pmatrix} \frac{1}{1!}h_1 & \frac{1}{3!}h_{123} \\ \frac{1}{0!}h_\emptyset & \frac{1}{2!}h_{12} \end{pmatrix} = \frac{1}{1!}h_1\frac{1}{2!}h_{12} - \frac{1}{3!}h_{123}\frac{1}{0!}h_\emptyset$$

$$= \frac{1}{2}h_{1|12} - \frac{1}{6}h_{123} = \frac{1}{2}h_{1/23} - \frac{1}{6}h_{123}$$

$$s_{(132,22/1)} = 132 \circ s_{[22/1]} = 132 \circ \left(\frac{1}{2}h_{1/23} - \frac{1}{6}h_{123}\right) = \frac{1}{2}h_{1/23} - \frac{1}{6}h_{123}$$

Rosas-Sagan skew Schur functions

The Rosas-Sagan skew Schur function in NCSym is

$$S^{RS}_{\lambda/\mu} = \sum_{\dot{T} \text{ DYT of shape } \lambda/\mu} x^{\dot{T}}.$$

Theorem (Aliniaeifard-Li-vW 2022)

Let $\lambda \vdash n$ and $\mu \vdash m$.

$$\rho(S^{RS}_{\lambda/\mu}) = (n-m)!s_{\lambda/\mu}$$

$$S^{RS}_{\lambda/\mu} = \sum_{\delta \in \mathfrak{S}_{(n-m)}} s_{(\delta,\lambda/\mu)}$$

$$= \sum_{\nu \vdash (n-m)} c^{\lambda}_{\mu\nu} S^{RS}_{\nu}$$

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Noncommutative symmetric functions

Take the noncommutative symmetric functions (Gelfand-Krob-Lascoux-Leclerc-Thibon) with map $\mathcal{J}$ (Bergeron-Reutenauer-Rosas-Zabrocki).

$$\mathcal{J} : \sum_{j_1 \leq j_2 \leq \cdots \leq j_n} x_{j_1} x_{j_2} \cdots x_{j_n} \mapsto \frac{1}{n!} h_{[n]} \in \mathrm{NCSym}$$

Theorem (Aliniaeifard-Li-vW 2022)

For the immaculate function $\mathfrak{S}_\lambda$ of Berg-Bergeron-Saliola-Serrano-Zabrocki

$$\mathcal{J}(\mathfrak{S}_\lambda) = s_{[\lambda]}.$$

For the noncommutative ribbon Schur function $\mathbf{r}_\alpha$ of Gelfand et al.

$$\mathcal{J}(\mathbf{r}_\alpha) = r_{[\alpha]}$$

the ribbon source Schur function: $r_{[\alpha]} = s_{[\lambda/\mu]}$.

Further avenues

- Find product rules

$$s_\pi s_\sigma = \sum_\tau c^\tau_{\pi\sigma} s_\tau$$

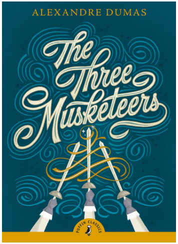
$$S^{RS}_\lambda S^{RS}_\mu = \sum_\nu c^\nu_{\lambda\mu} S^{RS}_\nu .$$

- Find a coproduct rule

$$\Delta(s_\pi) = \sum_{\sigma,\tau} d^\pi_{\sigma\tau} s_\sigma \otimes s_\tau .$$

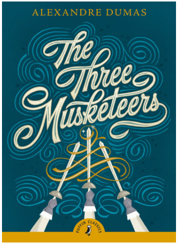
- Generalize s_π, for example to MacMahon symmetric functions.
- Find the dual basis to Schur functions in NCSym.
- Relationship to **x**-basis of Bergeron-Hohlweg-Rosas-Zabrocki.

"NEVER FEAR QUARRELS, BUT SEEK HAZARDOUS ADVENTURES"
– ALEXANDRE DUMAS, THE THREE MUSKETEERS



Schur functions in noncommuting variables
Farid Aliniaeifard, Shu Xiao Li, SvW, Adv. Math. 406 37pp (2022)

“Never fear quarrels, but seek hazardous adventures”
– Alexandre Dumas, The Three Musketeers



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Thank you very much,
and a big thank you to the organizers for a wonderful conference!