Schur functions in noncommuting variables

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Integer partitions

An integer partition \( \lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0 \) of \( n \) is a list of positive integers whose sum is \( n \): \( 3221 \vdash 8 \).

Let \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell = 1^{r_1}2^{r_2} \cdots n^{r_n} \). Then

\[
\lambda! = \lambda_1!\lambda_2!\cdots\lambda_\ell!
\]

and

\[
\lambda^! = r_1!r_2!\cdots r_n!
\]

**Example**

If \( \lambda = 3221 = 1^12^23^14^05^06^07^08^0 \vdash 8 \), then

\[
\lambda! = 3!2!2!1! = 6 \times 2 \times 2 \times 1 = 24
\]

\[
\lambda^! = 1!2!1!0!0!0!0!0!0! = 2.
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**Integer partitions**

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If $\lambda = 3221 = 1^1 2^2 3^1 4^0 5^0 6^0 7^0 8^0 \vdash 8$, then

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**Set partitions**

A set partition \( \pi \) of \([n] = \{1, 2, \ldots, n\} \) is a partitioning of \([n]\) into disjoint sets \(B_1, B_2, \ldots, B_\ell\) called blocks so

- \(B_i \neq \emptyset\)
- \(B_1 \cup B_2 \cup \cdots \cup B_\ell = [n]\).

\[
\pi = B_1/B_2/\cdots/B_\ell \vdash [n]
\]

**Example**

\{1, 3, 4\}, \{2, 5\}, \{6\}, \{7, 8\}

is a set partition of \([8]\), or

\[
\pi = 134/25/6/78 \vdash [8].
\]
Partitions and permutations

If \( \pi = B_1/B_2/\cdots/B_\ell \vdash [n] \), then

\[
\lambda(\pi) = |B_1||B_2|\cdots|B_\ell|
\]

with sizes weakly decreasing. If \( \delta \in \mathfrak{S}_n \), then

\[
\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).
\]

**Example**

If \( \pi = 134/25/6/78 \vdash [8] \), then

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\lambda(134/25/6/78) = 3221
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and \( \delta = 14325876 \in \mathfrak{S}_8 \)

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\delta(134/25/6/78) = \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8)
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### Slash product

If

\[ S + n = \{s + n : s \in S\} \]

then for \( \pi \vdash [n] \) and \( \sigma = B_1/B_2/\cdots/B_\ell \vdash [m] \) the slash product is

\[ \pi \mid \sigma = \pi/(B_1 + n)/(B_2 + n)/\cdots/(B_\ell + n) \vdash [n + m]. \]

#### Example

If \( \pi = 134/25 \vdash [5] \) and \( \sigma = 1/23 \vdash [3] \) then

\[ \pi \mid \sigma = 134/25/6/78 \vdash [8]. \]
NCSym \textcolor{red}{(Rosas-Sagan 2004)}

NCSym is the algebra of symmetric functions in noncommuting variables $x_1, x_2, x_3, \ldots$

\[ NCSym = NCSym^0 \oplus NCSym^1 \oplus \cdots \subset \mathbb{Q} \ll x_1, x_2, x_3, \ldots \gg \]

where $NCSym^0 = \text{span}\{1\}$ and for $n > 0$

\[ NCSym^n = \text{span}\{m_{\pi} : \pi \vdash [n]\} \]
\[ = \text{span}\{p_{\pi} : \pi \vdash [n]\} \]
\[ = \text{span}\{e_{\pi} : \pi \vdash [n]\} \]
\[ = \text{span}\{h_{\pi} : \pi \vdash [n]\}. \]

\textbf{Note:} The $e_{\pi}$ defined by Wolf in 1936.
Monomial function in NCSym

The monomial symmetric function in NCSym for $\pi \vdash [n]$ is

$$m_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j = i_k$$

if and only if $j$ and $k$ are in the same block of $\pi$.

**Example**

$$m_{13/2} = x_1 x_2 x_1 + x_2 x_1 x_2 + x_1 x_3 x_1 + x_2 x_3 x_2 + \cdots$$
**Power sum function in NCSym**

The power sum symmetric function in NCSym for $\pi \vdash [n]$ is

$$p_{\pi} = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with $i_j = i_k$ if $j$ and $k$ are in the same block of $\pi$.

**Example**

$$p_{13/2} = x_1 x_2 x_1 + x_2 x_1 x_2 + \cdots + x_1^3 + x_2^3 + \cdots$$
Elementary function in NCSym

The elementary symmetric function in NCSym for $\pi \vdash [n]$ is

$$e_{\pi} = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j \neq i_k$$

if $j$ and $k$ are in the same block of $\pi$.

Example

$$e_{13/2} = x_1 x_1 x_2 + x_1 x_2 x_2 + x_2 x_2 x_1 + x_2 x_1 x_1 + \cdots + x_1 x_2 x_3 + \cdots$$
**Complete homogeneous function in NCSym**

The complete homogeneous symmetric function in NCSym for $\pi \vdash [n]$ is

$$h_{\pi} = \sum_{\sigma} \lambda(\sigma \wedge \pi)! m_\sigma$$

where $\sigma \wedge \pi$ is the maximal set partition where every block is a subblock of $\sigma$ and $\pi$.

**Example**

$$h_{13/2} = 2m_{123} + m_{12/3} + m_{1/23} + 2m_{13/2} + m_{1/2/3}$$

$123 \wedge 13/2 = 13/2 \wedge 13/2 = 13/2 \quad \lambda(13/2)! = 2$

$12/3 \wedge 13/2 = 1/23 \wedge 13/2 = 1/2/3 \wedge 13/2 = 1/2/3 \quad \lambda(1/2/3)! = 1$
**Permutations and products**

Fact: If $\pi \vdash [n]$, $\delta \in S_n$ and $\delta \circ m_\pi = m_{\delta(\pi)}$, then for $b = p, e, h$

$$\delta \circ b_\pi = b_{\delta(\pi)}.$$

Fact: If $\pi, \sigma$ are set partitions, then for $b = p, e, h$

$$b_\pi b_\sigma = b_{\pi|\sigma}.$$

**Example**

<table>
<thead>
<tr>
<th>Example</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$132 \circ m_{12/3} = m_{13/2}$</td>
<td>$p_{13/2} p_1 = p_{13/2</td>
</tr>
</tbody>
</table>
What's in a name?

Let the variables commute:

\[ \rho : \text{NCSym} \to \text{Sym} \]

Theorem (Rosas-Sagan 2004)

Let \( \pi \) be a set partition.

\[
\begin{align*}
\rho(m_\pi) &= \lambda(\pi)! m_{\lambda(\pi)} & \rho(p_\pi) &= p_{\lambda(\pi)} \\
\rho(e_\pi) &= \lambda(\pi)! e_{\lambda(\pi)} & \rho(h_\pi) &= \lambda(\pi)! h_{\lambda(\pi)}
\end{align*}
\]

Note: The images are classical monomial, power sum, elementary and complete homogeneous symmetric functions.

Question: Where are the Schur functions?
Partitions and diagrams

An integer partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ of $n$ is a list of positive integers whose sum is $n$: $3221 \vdash 8$.

The diagram $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ is the array of boxes with $\lambda_i$ boxes in row $i$ from the top.

\[
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \\
\square & & & \\
\end{array}
\]

3221
A semistandard Young tableau (SSYT) $T$ of shape $\lambda$ is a filling with $1, 2, 3, \ldots$ so rows weakly increase and columns increase.

\[
\begin{array}{cc}
1 & 1 \\
2 & \\
\end{array}
\]

Given an SSYT $T$ we have for commuting variables

\[x^T = x_1^{#1s} x_2^{#2s} x_3^{#3s} \ldots .\]

\[x_1^2 x_2\]
Schur functions

The Schur function in Sym is

\[ s_\lambda = \sum_{T \text{ SSYT of shape } \lambda} x^T. \]

**Example**

\[ s_{21} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3 + \cdots \]

1 1 1 1 1 2 2 2 2 2 3 3 3 1 2 1 3
2 2 3 3 3 3 3 3 3 2
Dotted Young tableaux

A dotted Young tableau (DYT) $\hat{T}$ of shape $\lambda$ is a filling with 1, 2, 3, \ldots so rows weakly increase and columns increase, and 1, 2, \ldots, $n$ dots appear exactly once.

\[
\begin{array}{cc}
\cdot & 1 \\
1 & 1 \\
\cdot & 2 \\
\end{array}
\]

Given a DYT $\hat{T}$ we have for noncommuting variables

\[x^\hat{T} = x_i \text{ in position } j \text{ iff } i \text{ has } j \text{ dots above it.}\]

\[x_1 x_2 x_1\]
The Rosas-Sagan Schur function in NCSym is

\[ S_\lambda^{RS} = \sum_{\dot{T} \text{ DYT of shape } \lambda} x^\dot{T}. \]

**Example**

\[ S_{21}^{RS} = 2x_1x_1x_2 + 2x_1x_2x_1 + 2x_2x_1x_1 + \cdots \]

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]
Theorem (Rosas-Sagan 2004)

Let $\lambda \vdash n$.

- The $S^R_\lambda$ are linearly independent.
- We have $\rho(S^R_\lambda) = n!s_\lambda$.

Note: However they are not a basis for NCSym because we only have one for each integer partition, not set partition.
**Rosas-Sagan Schur functions**

**Theorem (Rosas-Sagan 2004)**

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Rosas-Sagan 2004:

Is there a way to define functions ... for set partitions $\pi \vdash [n]$ having properties analogous to the ordinary Schur functions $s_\lambda$?
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Yes there is!
The three musketeers

ALEXANDRE DUMAS

The Three Musketeers
**Schur functions revisited**

**Definition**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the Jacobi-Trudi identity

$$s_\lambda = \det (h_{\lambda_i - i + j})_{1 \leq i, j \leq \ell}$$

where $h_0 = 1$ and $h_{-\text{ve}} = 0$.

**Example**

$$s_{21} = \det \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = h_2 h_1 - h_3 h_0 = h_{21} - h_3$$
**Schur functions revisited**

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Example

$$s_{21} = \det \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = h_2 h_1 - h_3 h_0 = h_{21} - h_3$$
The noncommutative Leibniz formula for \( A = (a_{ij})_{1 \leq i, j \leq n} \) with noncommuting entries \( a_{ij} \) is

\[
\det(A) = \sum_{\varepsilon \in S_n} \text{sgn}(\varepsilon) a_{1\varepsilon(1)} a_{2\varepsilon(2)} \cdots a_{n\varepsilon(n)}
\]

- product of the entries is taken top row to the bottom row
- \( \text{sgn}(\varepsilon) \) is the sign of permutation \( \varepsilon \).
“... AND ONE FOR ALL,” — DUMAS, T3M

**Definition**

Let \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell \). Then the source Schur function

\[
s[\lambda] = \det \left( \frac{1}{(\lambda_i - i + j)!} h[\lambda_i - i + j] \right)_{1 \leq i, j \leq \ell}
\]

where \( h[0] = h_\emptyset = 1 \) and \( h_{-ve} = 0 \).

**Example**

\[
s_{[21]} = \det \left( \begin{array}{cc} \frac{1}{2!} h_{12} & \frac{1}{3!} h_{123} \\ \frac{1}{0!} h_\emptyset & \frac{1}{1!} h_1 \end{array} \right) = \frac{1}{2!} h_{12} \frac{1}{1!} h_1 - \frac{1}{3!} h_{123} \frac{1}{0!} h_\emptyset
\]

\[
= \frac{1}{2} h_{12} |1 \rangle - \frac{1}{6} h_{123} = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}.
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\[
= \frac{1}{2} h_{121} - \frac{1}{6} h_{123} = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}.
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$$s_{[21]} = \det \left( \frac{1}{2!} h_{12} \begin{array}{cc} \frac{1}{0!} h_\emptyset & \frac{1}{3!} h_{123} \\ \frac{1}{1!} h_1 & \end{array} \right) = \frac{1}{2!} h_{12} \frac{1}{1!} h_1 - \frac{1}{3!} h_{123} \frac{1}{0!} h_\emptyset$$

$$= \frac{1}{2} h_{12|1} - \frac{1}{6} h_{123} = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}.$$
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**Definition**

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$$= \frac{1}{2} h_{12} |1 - \frac{1}{6} h_{123} = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}.$$
We now create a set partition $\pi \vdash [n]$ from an integer partition $\lambda \vdash n$ where

$$\lambda(\pi) = \lambda$$

using tableaux $T_\pi$ of shape $\lambda$ such that

- every element $1, 2, \ldots, n$ appears exactly once
- rows increase left to right
- first column of rows of same length increase top to bottom.

rows $\longleftrightarrow$ blocks

**Example**

$$T_\pi = \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 \\
7 & 8 \\
6
\end{array} \quad \longleftrightarrow \quad \pi = 134/25/78/6$$
We now create a permutation $\delta_\pi \in \mathfrak{S}_n$ using $T_\pi$.

read by row $\longleftrightarrow$ one line notation

**Example**

$$T_\pi = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 \\ 7 & 8 \\ 6 \end{bmatrix} \quad \longleftrightarrow \quad \delta_\pi = 13425786$$
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Let \( \pi = B_1/B_2/\cdots/B_\ell \). Then the Schur function in NCSym is

\[
s_\pi = \delta_\pi \circ s[\lambda(\pi)] = \delta_\pi \circ \det \left( \frac{1}{(\lambda_i - i + j)!} h[\lambda_i - i + j] \right)_{1 \leq i,j \leq \ell}.
\]

**Example**

If \( \pi = 13/2 \) then \( T_\pi = \begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \) and \( \delta_\pi = 132 \).

\[
s_{13/2} = 132 \circ s[21] = 132 \circ \left( \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123} \right) = \frac{1}{2} h_{13/2} - \frac{1}{6} h_{123}
\]
A refined basis

**Theorem (Aliniaeifard-Li-vW 2022)**

\[ NCSym^n = \text{span}\{s_\pi : \pi \vdash [n]\} \quad \rho(s_\pi) = s_{\lambda(\pi)} \]

\[ S_\lambda^{RS} = \sum_{\delta \in S_n} \delta \circ s_{\lambda} \]
A refined basis

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\[ S_\lambda^{\text{RS}} = \sum_{\delta \in S_n} \delta \circ s_{[\lambda]} \]

Note: \( \delta \circ s_\pi \neq s_\delta(\pi) \) in general.
A refined basis

**Theorem (Aliniaeifard-Li-vW 2022)**

\[
\text{NCSym}^n = \text{span}\{s_\pi : \pi \vdash [n]\} \quad \rho(s_\pi) = s_{\lambda(\pi)}
\]

\[
S^R_S = \sum_{\delta \in \mathcal{G}_n} \delta \circ s_{[\lambda]}
\]

**Note:** \(\delta \circ s_\pi \neq s_{\delta(\pi)}\) in general.

**Theorem (Aliniaeifard-Li-vW 2022)**

*For \(n \geq 5\) we have \(n!\) different bases:*

\[
\text{NCSym}^n = \text{span}\{\delta \circ s_\pi : \pi \vdash [n]\} \quad \rho(\delta \circ s_\pi) = s_{\lambda(\pi)}
\]
**Young Tableaux**

A Young tableau (YT) \( t \) of shape \( \lambda \) is a filling with 1, 2, 3, \ldots so each number appears exactly **once**.

We now create a permutation \( \delta_t \in S_n \) using \( t \).

read by row \( \leftrightarrow \) one line notation

Then

\[
s_t = \delta_t \circ s_{[\lambda]}.
\]

**Example**

\[
t = \begin{array}{cc}
2 & 1 \\
3 & \\
one
\end{array}
\quad \delta_t = 213
\]

\[
s_t = 213 \circ s_{[21]} = 213 \circ \left( \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123} \right) = \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123}
\]
The row-equivalence class \([t]\) of \(t\) consists of \(\tilde{t}\)
- same shape as \(t\)
- same set of row entries as \(t\).

**Example**

\[
t = \begin{array}{cc}
2 & 1 \\
3 & \end{array} \quad \quad \quad \quad \quad [t] = \left\{ \begin{array}{cc}
2 & 1 \\
3 & \end{array}, \begin{array}{cc}
1 & 2 \\
3 & \end{array} \right\}
\]

\[
\delta_t = 213 \quad \delta_{\tilde{t}} = 123
\]
Young tableaux

The row-equivalence class $[t]$ of $t$ consists of $\tilde{t}$
- same shape as $t$
- same set of row entries as $t$.

Example

$$t = \begin{array}{c}
2 \\
1 \\
3 \\
\end{array} \quad [t] = \left\{ \begin{array}{c}
2 \\
1 \\
3 \\
\end{array}, \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \right\}$$

$$\delta_t = 213 \quad \delta_{\tilde{t}} = 123$$
A tabloid basis

The tabloid Schur function in NCSym is

$$s[t] = \sum_{\tilde{t}} s_{\tilde{t}}$$

where $\tilde{t}$ and $t$ are row equivalent.

**Example**

\[
\begin{bmatrix}
2 & 1 \\
3 \\
\end{bmatrix}
\]

\[
s\begin{bmatrix}
2 & 1 \\
3 \\
\end{bmatrix}
= 213 \circ s_{[21]} + 123 \circ s_{[21]}
\]

\[
= \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123} + \frac{1}{2} h_{12/3} - \frac{1}{6} h_{123} = h_{12/3} - \frac{1}{3} h_{123}
\]
A tabloid basis

The tabloid Schur function in NCSym is

\[ s_{[t]} = \sum \tilde{t} s_{\tilde{t}} \]

where \( \tilde{t} \) and \( t \) are row equivalent.

**Theorem (Aliniaeifard-Li-vW 2022)**

We have basis

\[ \text{NCSym}^n = \text{span}\{ s_{[t]} : t \leftrightarrow \pi, \pi \vdash [n] \}. \]

If \( t \) has shape \( \lambda \) then

\[ \rho(s_{[t]}) = \lambda! s_\lambda. \]
**Specht modules**

Given Young tableau $t$

$$e_t = \left( \sum_{\delta} \text{sgn}(\delta) \delta \right) \circ [t]$$

$$e_t = \left( \sum_{\delta} \text{sgn}(\delta) \delta \right) \circ s[t]$$

where the sums are over all column-stabilizers of $t$ that permute elements within each column.

Given partition $\lambda$ we have *Specht module*; submodule of NCSym.

$$S^\lambda = \text{span}\{e_t : t \text{ has shape } \lambda\}$$

**Theorem (Aliniaeifard-Li-vW 2022)**

$$S^\lambda \cong S^\lambda$$

$$e_t \mapsto e_t$$
For $\lambda \vdash n, \mu \vdash m$ the skew diagram $\lambda/\mu$ is the array of $n - m$ boxes contained in $\lambda$ but not in $\mu$.

A skew shape $\lambda/\mu$ is a ribbon if it is connected with no $2 \times 2$ square.

A ribbon can be denoted by row lengths $\alpha$: 

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5332/221
```
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Skew diagrams

For $\lambda \vdash n, \mu \vdash m$ the skew diagram $\lambda/\mu$ is the array of $n - m$ boxes contained in $\lambda$ but not in $\mu$.

A skew shape $\lambda/\mu$ is a ribbon if it is connected with no $2 \times 2$ square.

A ribbon can be denoted by row lengths $\alpha$: 3122.
skew Schur functions

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the skew Schur function in $\text{Sym}$ is

$$s_{\lambda/\mu} = \det \left( h_{\lambda_i - \mu_j - i + j} \right)_{1 \leq i, j \leq \ell}$$

$$= \sum_{T \text{ SSYT of shape } \lambda/\mu} x^T$$

$$= \sum_{\nu} c_{\mu\nu}^\lambda s_\nu$$
Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the skew Schur function in Sym is

$$s_{\lambda/\mu} = \det (h_{\lambda_i - \mu_j - i + j})_{1 \leq i,j, i,j \leq \ell}$$

$$= \sum_{T \text{ SSYT of shape } \lambda/\mu} x^T$$

$$= \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}$$

and the $c_{\mu\nu}^{\lambda}$ are the Littlewood-Richardson coefficients.
skew Schur functions

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the skew Schur function in $\text{Sym}$ is

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$$= \sum_{\nu} c_{\mu\nu}^\lambda s_\nu$$

and the $c_{\mu\nu}^\lambda$ are the Littlewood-Richardson coefficients.

Let $\lambda/\mu$ be a ribbon $\alpha$. Then the ribbon Schur function in $\text{Sym}$ is

$$r_\alpha = s_{\lambda/\mu}.$$
**Skew Schur functions in noncommuting variables**

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell$. Then the source skew Schur function

$$s[\lambda/\mu] = \det \left( \frac{1}{(\lambda_i - \mu_j - i + j)!} h_{[\lambda_i - \mu_j - i + j]} \right)_{1 \leq i,j \leq \ell}$$

and the skew Schur function in NCSym is

$$s(\delta,\lambda/\mu) = \delta \circ s[\lambda/\mu] = \delta \circ \det \left( \frac{1}{(\lambda_i - \mu_j - i + j)!} h_{[\lambda_i - \mu_j - i + j]} \right)_{1 \leq i,j \leq \ell}.$$ 

**Theorem (Aliniaeifard-Li-vW 2022)**

$$\rho(s(\delta,\lambda/\mu)) = s_{\lambda/\mu}$$
Skew Schur functions in noncommuting variables

\[ s_{[2/1]} = \det \left( \begin{array}{cc} \frac{1}{1!} h_1 & \frac{1}{3!} h_{123} \\ \frac{1}{0!} h_{\emptyset} & \frac{1}{2!} h_{12} \end{array} \right) = \frac{1}{1!} h_1 \frac{1}{2!} h_{12} - \frac{1}{3!} h_{123} \frac{1}{0!} h_{\emptyset} \]

\[ = \frac{1}{2} h_{1|12} - \frac{1}{6} h_{123} = \frac{1}{2} h_{1/23} - \frac{1}{6} h_{123} \]

\[ s_{(132, 2/1)} = 132^\circ s_{[2/1]} = 132^\circ \left( \frac{1}{2} h_{1/23} - \frac{1}{6} h_{123} \right) = \frac{1}{2} h_{1/23} - \frac{1}{6} h_{123} \]
ROSAS-SAGAN SKEW SCHUR FUNCTIONS

The Rosas-Sagan skew Schur function in NCSym is

\[ S_{\lambda/\mu}^{RS} = \sum_{\hat{T}} \chi^{\hat{T}}. \]

\( \hat{T} \) DYT of shape \( \lambda/\mu \)

THEOREM (ALINIAEIFARD-LI-vW 2022)

Let \( \lambda \vdash n \) and \( \mu \vdash m \).

\[ \rho(S_{\lambda/\mu}^{RS}) = (n - m)! s_{\lambda/\mu} \]

\[ S_{\lambda/\mu}^{RS} = \sum_{\delta \in \mathcal{G}(n-m)} s_{(\delta, \lambda/\mu)} \]

\[ = \sum_{\nu \vdash (n-m)} c_{\mu \nu}^{\lambda} S_{\nu}^{RS} \]
The Rosas-Sagan skew Schur function in NCSym is

$$S_{\lambda/\mu}^{RS} = \sum_{\hat{T} \text{ DYT of shape } \lambda/\mu} \hat{T}.$$ 

**Theorem (Aliniaeifard-Li-vW 2022)**

Let $\lambda \vdash n$ and $\mu \vdash m$.

$$\rho(S_{\lambda/\mu}^{RS}) = (n - m)! s_{\lambda/\mu}$$

$$S_{\lambda/\mu}^{RS} = \sum_{\delta \in \mathcal{S}(n-m)} s_{(\delta, \lambda/\mu)}$$

$$= \sum_{\nu \vdash (n-m)} c_{\mu \nu}^{\lambda} S_{\nu}^{RS}$$

and the $c_{\mu \nu}^{\lambda}$ are the Littlewood-Richardson coefficients.
Noncommutative symmetric functions

Take the noncommutative symmetric functions (Gelfand-Krob-Lascoux-Leclerc-Thibon) with map $\mathcal{J}$ (Bergeron-Reutenauer-Rosas-Zabrocki).

\[ \mathcal{J} : \sum_{j_1 \leq j_2 \leq \cdots \leq j_n} x_{j_1} x_{j_2} \cdots x_{j_n} \mapsto \frac{1}{n!} h[n] \in \text{NCSym} \]

**Theorem (Aliniaeifard-Li-vW 2022)**

For the immaculate function $\mathcal{G}_\lambda$ of Berg-Bergeron-Saliola-Serrano-Zabrocki

\[ \mathcal{J}(\mathcal{G}_\lambda) = s[\lambda] \]

For the noncommutative ribbon Schur function $r_\alpha$ of Gelfand et al.

\[ \mathcal{J}(r_\alpha) = r[\alpha] \]

the ribbon source Schur function: $r[\alpha] = s[\lambda/\mu]$. 
Further avenues

- Find **product** rules

\[ s_\pi s_\sigma = \sum \tau c_{\pi \sigma}^{\tau} s_\tau \]

\[ S_\lambda^{RS} S_\mu^{RS} = \sum \nu c_{\lambda \mu}^{\nu} S_\nu^{RS} \]

- Find a **coproduct** rule

\[ \Delta(s_\pi) = \sum_{\sigma, \tau} d_{\sigma \tau}^{\pi} s_\sigma \otimes s_\tau \]

- **Generalize** \( s_\pi \), for example to MacMahon symmetric functions.
- **Find the dual basis** to Schur functions in NCSym.
- **Relationship** to \( x \)-basis of Bergeron-Hohlweg-Rosas-Zabrocki.
“Never fear quarrels, but seek hazardous adventures”
– Alexandre Dumas, The Three Musketeers

Schur functions in noncommuting variables
“Never fear quarrels, but seek hazardous adventures”
– Alexandre Dumas, The Three Musketeers

Schur functions in noncommuting variables

Thank you very much,
and a big thank you to the organizers for a wonderful conference!