Permutation Enumeration and Symmetric Functions

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April 2, 2022
Main goal: results on permutation enumeration related to symmetric functions
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Topics

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- class multiplication and characters
- commutators and characters
- border strips and descents
- alternating permutations and the Foulkes representation
- Lyndon symmetric functions
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- class multiplication and characters
- commutators and characters
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- alternating permutations and the Foulkes representation
- Lyndon symmetric functions
- generalized descent sets
The class multiplication theorem

**G**: finite group with conjugacy classes \( C_1, \ldots, C_t \)

Let \( i, j \in [t] = \{1, \ldots, t\} \).

\( \chi^1, \ldots, \chi^t \): the irreducible (complex) characters of \( G \)

\( d_r = \deg \chi^r \)

\( \chi^r_i \): \( \chi^r(v) \) for any \( v \in C_i \)
The class multiplication theorem

$G$: finite group with conjugacy classes $C_1, \ldots, C_t$

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$\chi^1, \ldots, \chi^t$: the irreducible (complex) characters of $G$

$d_r = \text{deg } \chi^r$

$\chi^r_i$: $\chi^r(v)$ for any $v \in C_i$

**Theorem.** Let $w \in C_k$. Then

$$\#\{(u, v) \in C_i \times C_j : uvw = \text{id}\} = \frac{|C_i| \cdot |C_j|}{|G|} \sum_{r=1}^{t} \frac{1}{d_r} \chi^r_i \chi^r_j \chi^r_k$$
Reformulation for $G = \mathfrak{S}_n$

$(x)$: the variables $x_1, x_2, \ldots$, and similarly $(y), (z)$

$H_\lambda$: product of hook lengths of $\lambda$ for $\lambda \vdash n$

**Theorem.**

$$\sum_{\lambda \vdash n} H_\lambda \ s_\lambda(x)s_\lambda(y)s_\lambda(z) = \frac{1}{n!} \sum_{uvw = \text{id} \text{ in } \mathfrak{S}_n} p_{\rho(u)}(x)p_{\rho(v)}(y)p_{\rho(w)}(z),$$

where $\rho(u)$ is the cycle type of $u$. 
Sample application

**Theorem.**

\[
\sum_{\lambda \vdash n} H_\lambda = \frac{1}{n!} \#\{(u, v, w) \in S^3_n : u^2 v^2 w^2 = 1\}.
\]
Sample application

Theorem.

\[ \sum_{\lambda \vdash n} H_\lambda = \frac{1}{n!} \# \{ (u, v, w) \in S_n^3 : u^2 v^2 w^2 = 1 \}. \]

Idea of proof. For \( w \in S_n \) let \( \text{sq}(w) = \# \{ u \in S_n : u^2 = w \} \). Let \( \varphi : \Lambda_{\mathbb{Q}} \to \mathbb{Q} \) be the linear transformation defined by \( \varphi(s_\lambda) = 1 \).

Well-known: \( p_\lambda = \sum_\mu \chi^\mu(\lambda)s_\mu \), so

\[
\varphi(p_\lambda) = \sum_\mu \chi^\mu(\lambda) = \text{sq}(w) \quad (\text{Frobenius}),
\]

where \( \rho(w) = \lambda \).
Proof (concluded)

\[ \varphi(s_\lambda) = 1, \quad \varphi(p_\lambda) = \text{sq}(w) \text{ where } \rho(w) = \lambda \]
ϕ(s_λ) = 1, ϕ(p_λ) = sq(w) where ρ(w) = λ

Apply ϕ separately to each set of variables in

\[ \sum_{\lambda \vdash n} H_\lambda \ s_\lambda(x)s_\lambda(y)s_\lambda(z) = \frac{1}{n!} \sum_{uvw=\text{id in } S_n} p_\rho(u)(x)p_\rho(v)(y)p_\rho(w)(z). \]  □
Commutators

$G$: finite group of order $g$

For $w \in G$, define

$$f(w) = \# \{(u, v) \in G \times G : w = uvu^{-1}v^{-1}\}.$$ 

$\text{Irr}(G)$: set of irreducible (complex) characters of $G$
Commutators

$G$: finite group of order $g$

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$$f(w) = \#\{(u, v) \in G \times G : w = uvu^{-1}v^{-1}\}.$$

$Irr(G)$: set of irreducible (complex) characters of $G$

**Theorem.** $f = \sum_{\chi \in \text{Irr}(G)} \frac{g}{\chi(1)^2} \chi$
Reformulation for $G = S_n$

Theorem. $\frac{1}{n!} \sum_{u,v \in S_n} p_\rho(uvu^{-1}v^{-1}) = \sum_{\lambda \vdash n} H_\lambda s_\lambda$  \hspace{1cm} (*)
Reformulation for $G = \mathfrak{S}_n$

Theorem.  \[
\frac{1}{n!} \sum_{u,v \in \mathfrak{S}_n} p_{\rho(uvu^{-1}v^{-1})} = \sum_{\lambda \vdash n} H_\lambda s_\lambda \quad (*)
\]

Sample application. For $w \in \mathfrak{S}_n$, let $\kappa(w)$ be the number of cycles of $w$. Then

$$
\frac{1}{n!} \sum_{u,v \in \mathfrak{S}_n} q^{\kappa(uvu^{-1}v^{-1})} = \sum_{\lambda \vdash n} \prod_{t \in \lambda} (q + c(t)),
$$

where $c(t)$ denotes the content of the square $t$. 
Reformulation for $G = \mathfrak{S}_n$

**Theorem.** \( \frac{1}{n!} \sum_{u,v \in \mathfrak{S}_n} p_{\rho(\text{uvu}^{-1}\text{v}^{-1})} = \sum_{\lambda \vdash n} H_\lambda s_\lambda \quad (\ast) \)

**Sample application.** For \( w \in \mathfrak{S}_n \), let \( \kappa(w) \) be the number of cycles of \( w \). Then

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\frac{1}{n!} \sum_{u,v \in \mathfrak{S}_n} q^{\kappa(\text{uvu}^{-1}\text{v}^{-1})} = \sum_{\lambda \vdash n} \prod_{t \in \lambda} (q + c(t)),
\]

where \( c(t) \) denotes the content of the square \( t \).

**Proof.** Let \( q \in \mathbb{P} \). Set \( x_1 = \cdots = x_q = 1 \), other \( x_i = 0 \) in (\ast). Note that \( p_{\rho(w)}(1^q) = q^{\kappa(w)} \) (since \( p_i(1^q) = q \)), etc. \( \square \)
Border strips (or ribbons)

\[ S = \{ b_1 < b_2 < \cdots < b_k \} \subseteq [n - 1] := \{1, 2, \ldots, n - 1\} \]

\( B_S \): the border strip with row lengths 
\( b_1, b_2 - b_1, b_3 - b_2, \ldots, n - b_k \).

\[ B_{\{3,4,6\}}, \ n=8 \]
Theorem (Foulkes). Let $S, T \subseteq [n - 1]$. Then

$$\langle s_{BS}, s_{BT} \rangle = \#\{w \in \mathfrak{S}_n : D(w) = S, D(w^{-1}) = T\},$$

where $D$ denotes descent set.
Theorems of Foulkes and Niven-de Bruijn

**Theorem (Foulkes).** Let $S, T \subseteq [n - 1]$. Then

$$\langle s_{BS}, s_{BT} \rangle = \# \{ w \in \mathcal{S}_n : D(w) = S, D(w^{-1}) = T \},$$

where $D$ denotes descent set.

$$\beta_n(S) := \# \{ w \in \mathcal{S}_n : D(w) = S \}$$

**Theorem (Niven, de Bruijn)** Fix $n$. Then $\beta_n(S)$ is maximized by $S = \{1, 3, 5, \ldots \}$ and $S = \{2, 4, 6, \ldots \}$. 
Gessel’s conjecture

Recall

\[ \langle s_{B_S}, s_{B_T} \rangle = \#\{w \in \mathfrak{S}_n : D(w) = S, D(w^{-1}) = T\} \].

**Conjecture.** Fix \( n \). Then \( \langle s_{B_S}, s_{B_T} \rangle \) is maximized by

\( S = T = \{1, 3, 5, \ldots \} \) and \( S = T = \{2, 4, 6, \ldots \} \).
Gessel’s conjecture

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**Theorem.** The maximum of value of \( \langle s_{BS}, s_{BT} \rangle \) is achieved by some \( S = T \).
Gessel’s conjecture

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$$\langle s_{B_S}, s_{B_T} \rangle = \# \{w \in \mathfrak{S}_n : D(w) = S, D(w^{-1}) = T \}.$$

**Conjecture.** Fix $n$. Then $\langle s_{B_S}, s_{B_T} \rangle$ is maximized by $S = T = \{1, 3, 5, \ldots \}$ and $S = T = \{2, 4, 6, \ldots \}$.

**Theorem.** The maximum of value of $\langle s_{B_S}, s_{B_T} \rangle$ is achieved by some $S = T$.

**Proof.** $\langle s_{B_S} - s_{B_T}, s_{B_S} - s_{B_T} \rangle \geq 0$

$$\Rightarrow \langle s_{B_S}, s_{B_S} \rangle + \langle s_{B_T}, s_{B_T} \rangle \geq 2 \langle s_{B_S}, s_{B_T} \rangle,$$

so either $\langle s_{B_S}, s_{B_S} \rangle \geq \langle s_{B_S}, s_{B_T} \rangle$ or $\langle s_{B_T}, s_{B_T} \rangle \geq \langle s_{B_S}, s_{B_T} \rangle$. \qed
Alternating permutations

\[ w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n \text{ is alternating if} \]

\[ a_1 > a_2 < a_3 > a_4 < \cdots a_n. \]

\( E_n \): number of alternating \( w \in \mathfrak{S}_n \) (Euler number)
Alternating permutations

\[ w = a_1 a_2 \cdots a_n \in S_n \text{ is alternating if} \]
\[ a_1 > a_2 < a_3 > a_4 < \cdots a_n. \]

**Eₙ**: number of alternating \( w \in S_n \) (**Euler number**)

**Theorem** (D. André, 1879)

\[
\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x
\]
Let $R_n$ be the **ribbon staircase**: the border strip with row lengths $(1, 2, 2, \ldots , 2, 2, 1)$ ($n$ even) or $(1, 2, 2, \ldots , 2, 2)$ ($n$ odd).
Another theorem of Foulkes

$\chi^{R_n}$: the (reducible) character of $\mathfrak{S}_n$ corresponding to $R_n$, i.e., $\text{ch}(\chi^{R_n}) = s_{R_n}$. Equivalently,

$$s_{R_n} = \sum_{\mu \vdash n} z_{\mu}^{-1} \chi^{R_n}(\mu) p_{\mu}.$$
Another theorem of Foulkes

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$$s_{R_n} = \sum_{\mu \vdash n} z_{\mu}^{-1} \chi^{R_n}(\mu)p_{\mu}.$$ 

**Theorem (Foulkes).** Let $\mu \vdash n = 2k + 1$. Then

$$\chi^{R_n}(\mu) = \begin{cases} 
0, & \text{if } \mu \text{ has an even part} \\
(-1)^{k+r} E_{2r+1}, & \text{if } \mu \text{ has } 2r + 1 \text{ odd parts and no even parts.}
\end{cases}$$

Similar result for $n = 2k$. 

Sample application

\[ L(t) = \frac{1}{2} \log \frac{1 + t}{1 - t} = t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots \]

\[ f(n) = \# \{ w \in S_n : w \text{ and } w^{-1} \text{ are alternating} \} \]
Sample application

\[ L(t) = \frac{1}{2} \log \frac{1 + t}{1 - t} = t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots \]

\[ f(n) = \#\{w \in \mathcal{S}_n : w \text{ and } w^{-1} \text{ are alternating}\} \]

**Theorem.** \[ \sum_{k \geq 0} f(2k + 1) t^{2k+1} = \sum_{r \geq 0} E_{2r+1}^2 \frac{L(t)^{2r+1}}{(2r + 1)!}. \]

*Similar result for f(2k).*
Idea of proof.

Let $\text{OP}(n)$ be the set of partitions of $n$ with odd parts. Then for $n = 2k + 1$,

$$f(n) = \langle s_{R_n}, s_{R_n} \rangle = \langle \sum_{\mu \vdash n} z_\mu^{-1} \chi^{R_n}(\mu)p_\mu, \sum_{\mu \vdash n} z_\mu^{-1} \chi^{R_n}(\mu)p_\mu \rangle = \sum_{\mu \vdash n} z_\mu^{-1} \left(\chi^{R_n}(\mu)\right)^2.$$
Idea of proof.

Let \( \text{OP}(n) \) be the set of partitions of \( n \) with odd parts. Then for \( n = 2k + 1 \),

\[
f(n) = \left\langle s_{R_n}, s_{R_n} \right\rangle \\
= \left\langle \sum_{\mu \vdash n} z_{\mu}^{-1} \chi_{R_n}(\mu)p_{\mu}, \sum_{\mu \vdash n} z_{\mu}^{-1} \chi_{R_n}(\mu)p_{\mu} \right\rangle \\
= \sum_{\mu \vdash n} z_{\mu}^{-1} \left( \chi_{R_n}(\mu) \right)^2.
\]

Use Foulkes’ theorem on value of \( \chi_{R_n}(\mu) \) to get

\[
f(n) = \sum_{\mu \in \text{OP}(n)} z_{\mu}^{-1} E_{2r+1}^2.
\]

Now use elementary generating function manipulators. \( \square \)
Lyndon symmetric functions

For $\lambda \vdash n$, let

$$K_\lambda = \{ w \in S_n : \rho(w) = \lambda \},$$

a conjugacy class in $S_n$.

For $S \subseteq [n-1]$, define

$$F_S = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n, \atop i_j < i_{j+1} \text{ if } j \in S} x_{i_1} x_{i_2} \cdots x_{i_n},$$

known as \textit{(Gessel’s) fundamental quasisymmetric function}. 
Lyndon symmetric functions

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$$F_S = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \atop i_j < i_{j+1} \text{ if } j \in S} x_{i_1} x_{i_2} \cdots x_{i_n},$$

known as (Gessel’s) fundamental quasisymmetric function.

Define the Lyndon symmetric function

$$L_{\lambda} = \sum_{w \in K_{\lambda}} F_{D(w)},$$

a generating function for the number of permutations of cycle type $\lambda$ by descent set.
Example.  $n = 3, \lambda = (2, 1)$

<table>
<thead>
<tr>
<th>$w$</th>
<th>$D(w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>213</td>
<td>1</td>
</tr>
<tr>
<td>132</td>
<td>2</td>
</tr>
<tr>
<td>321</td>
<td>1,2</td>
</tr>
</tbody>
</table>

$L_{(2,1)} = F_1 + F_2 + F_{1,2} = s_{2,1} + s_{1,1,1}$
Theorem. $L_\lambda$ is a symmetric function given by

$$L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_{d}^{n/d}$$

$$L_{\langle n^k \rangle} = h_k[L_n] \quad \text{(plethysm)}$$

$$L_{\langle 1^{k_1} 2^{k_2} \ldots \rangle} = L_{\langle 1^{k_1} \rangle} L_{\langle 2^{k_2} \rangle} \ldots .$$
Theorem. $L_\lambda$ is a symmetric function given by

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$$L_{\langle n^k \rangle} = h_k[L_n] \text{ (plethysm)}$$

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$\blacktriangleright$ $L_\lambda$ is Schur positive.
Theorem. $L_\lambda$ is a symmetric function given by

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L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_n^{n/d}
\]

\[
L_{\langle n^k \rangle} = h_k[L_n] \text{ (plethysm)}
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\]

- $L_\lambda$ is Schur positive.
- $\sum_{\lambda \vdash n} L_\lambda = p_1^n$
Theorem. $L_\lambda$ is a symmetric function given by

$$L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}$$

$$L_{\langle n^k \rangle} = h_k[L_n] \text{ (plethysm)}$$

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- $L_\lambda$ is Schur positive.
- $\sum_{\lambda \vdash n} L_\lambda = p_1^n$
- Let $d(n)$ be the codimension of the span of the $L_\lambda$'s, $\lambda \vdash n$, in $\Lambda^n_Q$. **Open:** what is $d(n)$?

<table>
<thead>
<tr>
<th>$n$</th>
<th>1–3</th>
<th>4–6</th>
<th>7</th>
<th>8</th>
<th>9–11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d(n)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>12</td>
<td>15</td>
</tr>
</tbody>
</table>
A consequence of Gessel-Reutenauer

**Theorem (Gessel-Reutenauer).** Let $\lambda \vdash n$ and $S \subset [n - 1]$. Then

$$\langle L_\lambda, s_{B_S} \rangle = \# \{ w \in \mathfrak{S}_n : \rho(w) = \lambda, \ D(w) = S \}.$$
Sample application

\[ f(n) = \#\{ w \in \mathcal{S}_{2n} : \rho(w) = (2, 2, \ldots, 2), \ D(w) = \{1, 3, 5, \ldots\} \} \]

Thus \( f(n) = \langle L_{\langle 2^n \rangle}, s_{R_{2n}} \rangle \). Using

\[ L_{\langle 2^n \rangle} = h_n \left[ \frac{1}{2} (p_1^2 - p_2) \right] \]

and Foulkes’ theorem on \( s_{R_{2n}} \), we obtain (after quite a bit of manipulatorics):
Sample application

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Thus \( f(n) = \langle L_{\langle 2^n \rangle}, s_{R_{2^n}} \rangle \). Using

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and Foulkes' theorem on \( s_{R_{2^n}} \), we obtain (after quite a bit of manipulatorics):

**Theorem.** Let \( E \) be an indeterminate. Let \( \Omega \) be the linear operator sending \( E^k \) to the Euler number \( E_k \). Then

\[
\sum_{n \geq 0} f(n) t^n = \Omega \left( \frac{1 + t}{1 - t} \right)^{(E^2 + 1)/4}.
\]
Computation of $\Omega \left( \frac{1+t}{1-t} \right)^{(E^2+1)/4}$

\[
\Omega \left( \frac{1+t}{1-t} \right)^{\frac{E^2+1}{4}} = \Omega \left( 1 + \frac{1}{2}(E^2 + 1)t + \frac{1}{8}(E^4 + 2E^2 + 1)t^2 + \cdots \right)
\]
\[
= 1 + \frac{1}{2}(E_2 + 1)t + \frac{1}{8}(E^4 + 2E_2 + 1)t^2 + \cdots
\]
\[
= 1 + \frac{1}{2}(1 + 1)t + \frac{1}{8}(5 + 2 \cdot 1 + 1)t^2 + \cdots
\]
\[
= 1 + t + t^2 + \cdots
\]
Computation of $\Omega \left( \frac{1+t}{1-t} \right)^{(E^2+1)/4}$

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= 1 + \frac{1}{2}(E_2 + 1)t + \frac{1}{8}(E_4 + 2E_2 + 1)t^2 + \cdots \\
= 1 + \frac{1}{2}(1 + 1)t + \frac{1}{8}(5 + 2 \cdot 1 + 1)t^2 + \cdots \\
= 1 + t + t^2 + \cdots
\]

E.g., the unique $w \in S_4$ that is alternating and has cycle type $(2, 2)$ is 2143.
Descent set enumeration in the alternating group

$\mathcal{A}_n$: alternating group of degree $n$

$\gamma_n(S) = \# \{ w \in \mathcal{A}_n : D(w) = S \}$
Descent set enumeration in the alternating group

$\mathcal{A}_n$: alternating group of degree $n$

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Recall

$\langle L_\lambda, s_{B_S} \rangle = \# \{ w \in \mathfrak{S}_n : \rho(w) = \lambda, \ D(w) = S \}$. 

Recall the notation: let $\rho(w) = \lambda$. Then $\varepsilon_\lambda = \text{sgn}(w)$. Hence:
Descent set enumeration in the alternating group

\( \mathfrak{A}_n \): alternating group of degree \( n \)

\( \gamma_n(S) = \# \{ w \in \mathfrak{A}_n : D(w) = S \} \)

Recall

\[ \langle L_\lambda, s_{B_S} \rangle = \# \{ w \in \mathfrak{S}_n : \rho(w) = \lambda, \ D(w) = S \}. \]

Recall the notation: let \( \rho(w) = \lambda \). Then \( \varepsilon_\lambda = \text{sgn}(w) \). Hence:

**Theorem.** \( \gamma_n(S) = \left\langle \sum_{\lambda \vdash n, \varepsilon_\lambda = 1} L_\lambda, s_{B_S} \right\rangle \)
A formula for $\sum_{\lambda \vdash n} L_\lambda$ for $\varepsilon_\lambda = 1$

Theorem.

$$\sum_{\lambda \vdash n} L_\lambda = \begin{cases} 
\frac{1}{2} \left( p_1^n + p_2^{n/2} \right), & \text{if } n \text{ is even} \\
\frac{1}{2} \left( p_1^n + p_1 p_2^{(n-1)/2} \right), & \text{if } n \text{ is odd.}
\end{cases}$$
A formula for \( \sum_{\lambda \vdash n} L_\lambda \)

**Theorem.**

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\]

Proof is a computation based on the Gessel-Reutenauer formula

\[
L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}
\]

\[
L_{\langle n^k \rangle} = h_k[L_n] \text{ (plethysm)}
\]

\[
L_{\langle 1^{k_1} 2^{k_2} \ldots \rangle} = L_{\langle 1^{k_1} \rangle} L_{\langle 2^{k_2} \rangle} \cdots
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A formula for $\sum_{\lambda \vdash n, \varepsilon_\lambda = 1} L_\lambda$

**Theorem.**

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Proof is a computation based on the Gessel-Reutenauer formula

$$L_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}$$

$$L_{\langle n^k \rangle} = h_k[L_n] \quad \text{(plethysm)}$$

$$L_{\langle 1^{k_1} 2^{k_2} \ldots \rangle} = L_{\langle 1^{k_1} \rangle} L_{\langle 2^{k_2} \rangle} \cdots$$

Is there a more conceptual proof?
Let $B_S$ be a border strip of even size $2m$. Tile it uniquely with $m$ dominos. Shrink each domino to a square to get $B_{S/2}$.

$S = \{3, 4, 5, 6, 9\}$  \hspace{1cm}  S/2 = \{2, 3\}$
A formula for $\gamma_n(S)$, $n$ even

$B_S$: a border strip of size $n = 2m$

$v(B_S)$: number of vertical dominos in the unique tiling of $B_S$ by $m$ dominos

Recall: $\beta_n(S) = \#\{w \in \mathcal{G}_n : D(w) = S\}$

$$\gamma_n(S) = \#\{w \in \mathcal{A}_n : D(w) = S\}$$
A formula for $\gamma_n(S)$, $n$ even

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More complicated formula when $n$ is odd.
**Sketch of proof.**

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\[
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\]

**Proof (sketch).**

\[\gamma_n(S) = \left\langle s_{B_S}, \sum_{\lambda \vdash n \atop \varepsilon_\lambda = 1} L_\lambda \right\rangle = \left\langle s_{B_S}, \frac{1}{2} (p_1^n + p_2^m) \right\rangle = \frac{1}{2} \left( \beta_n(S) + \left\langle s_{B_S}, p_2^m \right\rangle \right).\]
**Sketch of proof.**

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\[
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\]

\[
= \frac{1}{2} (\beta_n(S) + \left\langle s_{B_S}, p_2^m \right\rangle).
\]

Evaluate \( \left\langle s_{B_S}, p_2^m \right\rangle \) by the Murnaghan-Nakayama rule.
Completion of proof.

$\langle s_{B_S}, p_2^m \rangle$ is the number of border-strip tableau of type $2^m$. There is a unique tiling by dominos. A border strip tableaux is an ordering of these dominos so that removing them in that order from the lower right boundary always leaves a skew shape. This corresponds to a (reverse) standard Young tableau of shape $B_{S/2}$, of which there are $\beta_m(S/2)$. The sign is $(-1)^{v(B_S)}$. □
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$S = \{3, 4, 5, 6, 9\} \quad S/2 = \{2, 3\}$
Generalized descent sets

\[ \mathbf{X} \subseteq \{(i,j) : 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\} \]
Generalized descent sets

\[ X \subseteq \{(i,j) : 1 \leq i \leq n, \ 1 \leq j \leq n, \ i \neq j\} \]

**X-descent** of \( w = a_1 \cdots a_n \in S_n \): an index \( 1 \leq i \leq n - 1 \) for which \( (a_i, a_{i+1}) \in X \)
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**X-descent set** \( X\text{Des}(w) \): set of X-descents

**Example.** (a) \( X = \{(i,j) : n - 1 \geq i > j \geq 1 \} \): \( X\text{Des} = \text{Des} \)

(b) \( X = \{(i,j) \in [n] \times [n] : i \neq j \} \): \( X\text{Des}(w) = [n - 1] \)
A generating function for the XDescent set

\[ U_X = \sum_{w \in S_n} F_{XDes}(w) \]
A generating function for the XDescent set

\[ U_X = \sum_{w \in \mathcal{S}_n} F_{X\text{Des}}(w) \]

**Example.** \( X = \{(1, 3), (2, 1), (3, 1), (3, 2)\} \)

<table>
<thead>
<tr>
<th>( w )</th>
<th>( X\text{Des}(w) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>132</td>
<td>( {1, 2} )</td>
</tr>
<tr>
<td>213</td>
<td>( {1, 2} )</td>
</tr>
<tr>
<td>231</td>
<td>( {2} )</td>
</tr>
<tr>
<td>312</td>
<td>( {1} )</td>
</tr>
<tr>
<td>321</td>
<td>( {1, 2} )</td>
</tr>
</tbody>
</table>

\[ U_X = F_{\emptyset} + F_1 + F_2 + 3F_{1,2} = p_1^3 - p_2p_1 + p_3 = s_3 + s_{21} + 2s_{111} \]
Two theorems

**Theorem** (easy). $U_X$ is a $p$-integral symmetric function.
Two theorems

Theorem (easy). $U_X$ is a $p$-integral symmetric function.

**record set** $\text{rec}(w)$ for $w = a_1 \cdots a_n \in S_n$:
$\text{rec}(w) = \{0 \leq i \leq n - 1 : a_i > a_j \text{ for all } j < i\}$. Thus always $0 \in \text{rec}(w)$.

**record partition** $\text{rp}(w)$: if $\text{rec}(w) = \{r_0, \ldots, r_j\}_<$, then $\text{rp}(w)$ is the numbers $r_1 - r_0, r_2 - r_1, \ldots, n - r_j$ arranged in decreasing order.
Two theorems

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**Theorem** (conjectured by RS, proved by I. Gessel) Let $X$ have the property that if $(i, j) \in X$ then $i > j$. Then

$$U_X = \sum_{\substack{w \in \mathcal{S}_n \\ X\text{Des}(w) = \emptyset}} p_{\text{rp}(w)}.$$  

In particular, $U_X$ is $p$-positive.
Connection with chromatic symmetric functions

\( P \): partial ordering of \([n]\)

\( Y_P = \{(i, j) : i >_P j\} \)

\( \text{inc}(P) \): incomparability graph of \( P \), i.e., vertex set \([n]\), edges \( ij \) if \( i \parallel j \) in \( P \)

\( X_G \): chromatic symmetric function of the graph \( G \)
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\textbf{Theorem.} \( U_{Y_P} = X_{\text{inc}(P)} \)
Reverse succession-free permutations

Let \( X = \{(2, 1), (3, 2), \ldots , (n, n-1)\} \).

\[ f_n = \#\{w \in \mathfrak{S}_n : \text{XDes}(w) = \emptyset\} \quad (\text{rs-free permutations}) \]
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Known result:

\[
\sum_{n \geq 0} f_n \frac{x^n}{n!} = \frac{e^{-x}}{(1 - x)^2} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{11x^4}{4!} + \cdots
\]
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$$

**Theorem.** $U_X = \sum_{i=1}^{n} f_i s_{i, 1^{n-i}}$

(Generating function for $w \in \mathfrak{S}_n$ by positions of reverse successions)
Reverse succession-free permutations

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(generating function for \( w \in \mathfrak{S}_n \) by positions of reverse successions)

**Example.** \( n = 4: \) \( U_X = 11s_4 + 3s_{31} + s_{211} + s_{1111} \)
Key fact for proof

\[ s_{i,1}^{n-i} = \sum_{S \in \binom{[n-1]}{n-i}} F_S. \]

In particular, no \( F_S \) appears in two different \( s_{i,1}^{n-i} \)'s.
Theorem. $U_X = \sum_{i=1}^{n} f_i s_{i,1}^{n-i}$
Further details

**Theorem.** \( U_X = \sum_{i=1}^{n} f_i s_{i,1}^{n-i} \)

**Proof.** For \( S \subseteq [n-1] \), take coefficient of \( F_S \) on both sides.
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Proof. For \( S \subseteq [n - 1] \), take coefficient of \( F_S \) on both sides.

Left-hand side: \( \#\{w \in \mathfrak{S}_n : X\text{Des}(w) = S\} \)
Theorem. \( U_X = \sum_{i=1}^{n} f_i s_{i,1^{n-i}} \)

Proof. For \( S \subseteq [n-1] \), take coefficient of \( F_S \) on both sides.

Left-hand side: \( \# \{ w \in \mathfrak{S}_n : X\text{Des}(w) = S \} \)

Right-hand side: Since

\[
s_{i,1^{n-i}} = \sum_{S \in \binom{[n-1]}{n-i}} F_S,
\]

we need to show: \( f_i = \# \{ w \in \mathfrak{S}_n : X\text{Des}(w) = S \} \) if \( \#S = n - i \).
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Will define a bijection

$$\{w \in \mathfrak{S}_n : XDes(w) = S\} \to \{u \in \mathfrak{S}_i : XDes(u) = \emptyset\}.$$
Conclusion of proof

To show: \( f_i = \#\{ w \in \mathfrak{S}_n : \text{XDes}(w) = S \} \) if \( \#S = n - i \).

Will define a bijection

\[
\{ w \in \mathfrak{S}_n : \text{XDes}(w) = S \} \rightarrow \{ u \in \mathfrak{S}_i : \text{XDes}(u) = \emptyset \}.
\]

Example. \( w = 3247651 \), so \( S = \{1, 4, 5\} \), \( n = 7 \), \( i = 4 \). Factor \( w \):

\[
w = 32 \cdot 4 \cdot 765 \cdot 1.
\]

Let \( 1 \rightarrow 1 \), \( 32 \rightarrow 2 \), \( 4 \rightarrow 3 \), \( 765 \rightarrow 4 \). get

\[
w \rightarrow 2341 = u. \quad \square
\]
A $q$-analogue for $X = \{(2, 1), (3, 2), \ldots, (n, n-1)\}$

Let $U_X(q) = \sum_{w \in \mathfrak{S}_n} q^{\text{des}(w^{-1})} F_{X\text{Des}(w)}$, where des denotes the number of (ordinary) descents.

$U_X(q)$ is the generating function for $w \in \mathfrak{S}_n$ by positions of reverse successions and by des($w^{-1}$).

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**Theorem.** $U_X(q) = \sum_{i=1}^{n} q^{n-i} f_i(q) s_{i,1}^{n-i}$
The final slide