The Hecke algebra as a natural setting for the k-random-to-random shuffle family

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5/9/2026

CAscade Lectures In COmbinatorics

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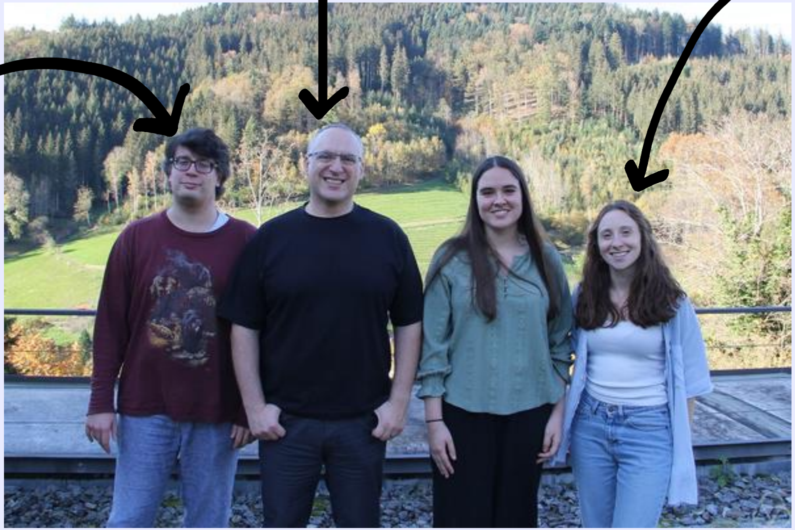

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(Drexel U)

Outline

I. Intro to shuffling families

"k-random-to-bottom" family

II. Motivation: a more mysterious family

"k-random-to-random" family

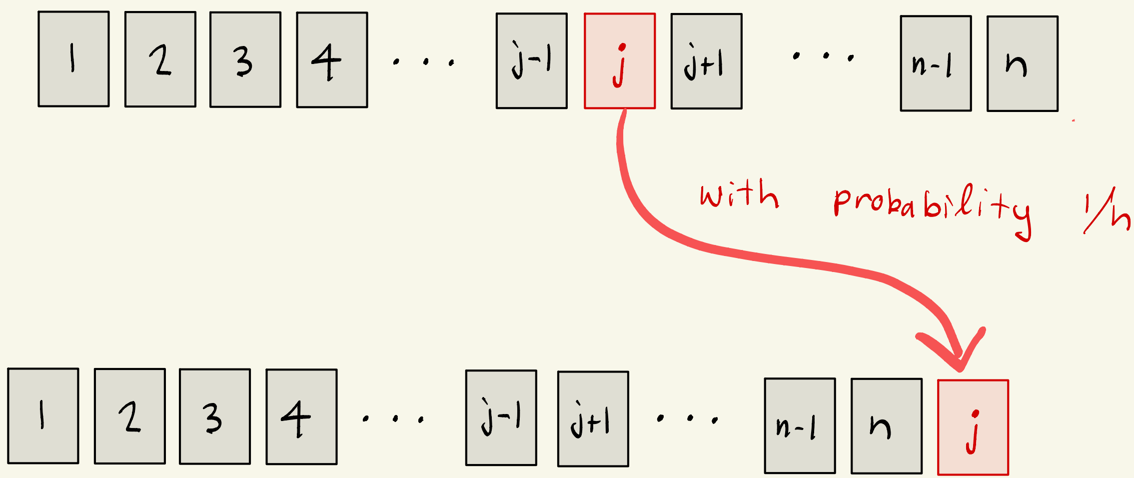
III. Our work: a surprisingly "natural" q-analogue:

The "(q, k) random-to-random" family

Slow ways to shuffle cards

A single random-to-bottom shuffle consists of:

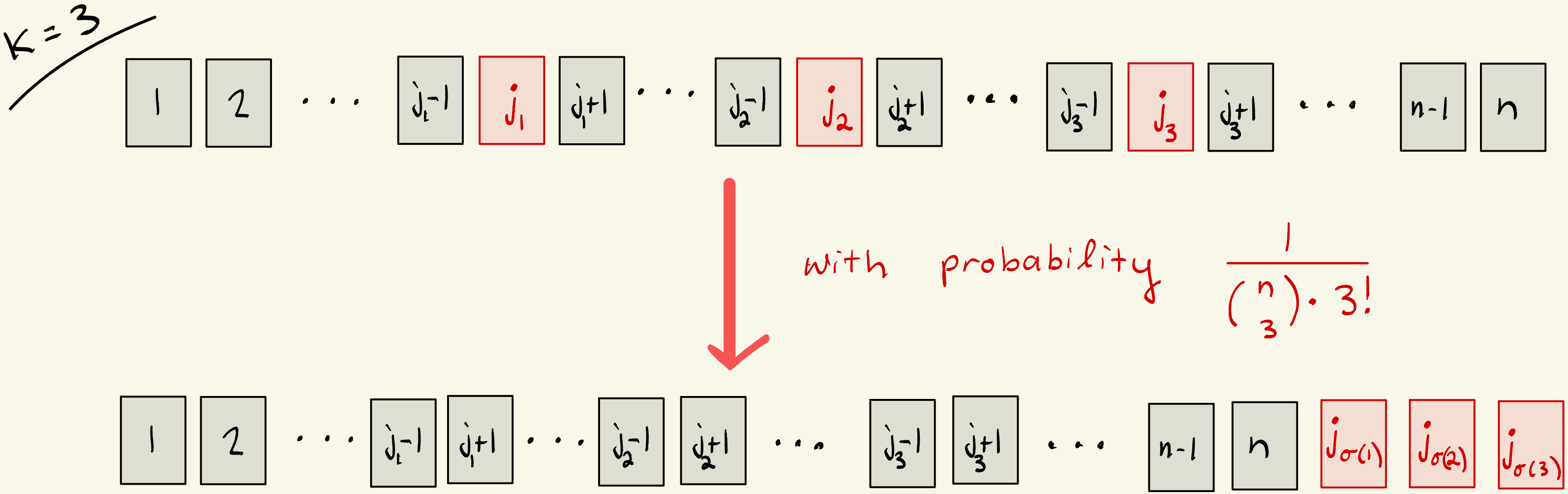
- picking a card uniformly at random
- moving it to the bottom of your deck



Slow ways to shuffle cards

The k-random-to-bottom shuffle consists of:

- Picking k cards at random
- Picking a random order for those cards
- Moving those cards to the bottom in that order



The Symmetric Group Algebra

- S_n := symmetric group on n letters
  ↳ Elements written in:
  - one-line notation 3124
  - cycle notation $(1,3,2)$.
  - or with simple transpositions $s_i = (i, i+1)$

- $\mathbb{C}[S_n]$:= symmetric group algebra
  ↳ As a vector space: $\operatorname{span}_{\mathbb{C}}\{u : u \in S_n\}$
  ↳ As a ring: $\left(\sum_{u \in S_n} c_u\, u\right)\left(\sum_{v \in S_n} d_v\, v\right) = \sum_{u,v} c_u d_v\, uv$

Shuffling Operators + the Symmetric Group Algebra

Ordered Decks of n Cards on $\{1, 2, \ldots, n\}$ $\longrightarrow$ Elements of S_n

$\boxed{2}\ \boxed{1}\ \boxed{3}$ $\qquad$ $2\ 1\ 3$

Shuffling Operators $\longrightarrow$ Elements of $\mathbb{C}[S_n]$

With probability p_π: permute the positions of your cards by π

$$\_\_\_ \cdot \sum_{\pi \in S_n} p_\pi \, \pi$$

Shuffle according to $\{p_\pi\}$, and then $\{q_\pi\}$

$$\_\_\_ \cdot \left(\sum_{\pi} p_\pi \, \pi \right) \cdot \left(\sum_{\pi} q_\pi \, \pi \right)$$

k-Random-to-Bottom in the Symmetric Group Algebra

$R2B_{n,k}$:= element in $\mathbb{C}[S_n]$ corresponding to k-random-to-bottom

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- $$R2B_{n,1} = \frac{1}{n} \sum_{w \in S_n :\, w(1) < w(2) < \cdots < w(n-1)} w$$

↳ Ex: $R2B_{3,1} = \frac{1}{3}\left[123 + 231 + 132\right]$

- $$R2B_{n,k} = \frac{1}{\binom{n}{k} \cdot k!} \sum_{w \in S_n :\, w(1) < w(2) < \cdots < w(n-k)} w$$

Shuffling Operators as Linear Transformations

Each $x \in \mathbb{C}[S_n]$ induces a linear transformation on $\mathbb{C}[S_n]$.

$$x : \mathbb{C}S_n \longrightarrow \mathbb{C}S_n$$
$$w \longmapsto w \cdot x \, .$$

Ex:

$$R2B_{3,1} \sim \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 0 & 0 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

$$123 \cdot R2B_{3,1} = \frac{1}{3}\left[123 + 231 + 132 \right]$$

We identify x with this transformation and study the eigenvalues of x.

Why eigenvalues?

Of interest to...

- **Probabalists:**

  Help determine mixing time + presence of a cutoff phenomenon

- **Combinatorialists:**

  For *very special* operators, the eigenvalues are nonnegative integers with "combinatorial" multiplicities.

Eigenvalues of k-random-to-bottom

Theorem (Phatarfod '91, Diaconis–Fill–Pitman '92, Bidigare–Hanlon–Rockmore '99, Brown '00)

- Eigenvalues of $\binom{n}{k} R2B_{n,k}$ are $\left\{ \binom{j}{k} : 0 \le j \le n \right\}$

non-negative integers.

- Multiplicity of $\binom{j}{k}$ is

$$\#\{\sigma \in S_n : \sigma \text{ has exactly } j \text{ fixed pts}\}$$

The random-to-bottom algebra

- Prop (Garsia, '02):

$$\left(R2B_{n,1}\right)^m = \sum_{k=0}^{m} S(m,k) \cdot R2B_{n,k} .$$

Stirling number of the second kind

- Corollary 1: The subalgebras of $\mathbb{C}[S_n]$ generated by $\{R2B_{n,1}\}$ and $\{R2B_{n,k} : 1 \le k \le n\}$ agree!

$$\langle R2B_{n,1} \rangle = \langle R2B_{n,k} : 0 \le k \le n \rangle .$$

- Corollary 2: The elements $\{R2B_{n,k} : 1 \le k \le n\}$ pairwise commute!

In summary...

Three nice properties of $\{R2B_{n,k}\}_k$:

① non-negative integer eigenvalues (after rescaling) ✓

② generate the same subalgebra as $\{R2B_n(1)\}$ ✓

③ Pairwise commute ✓

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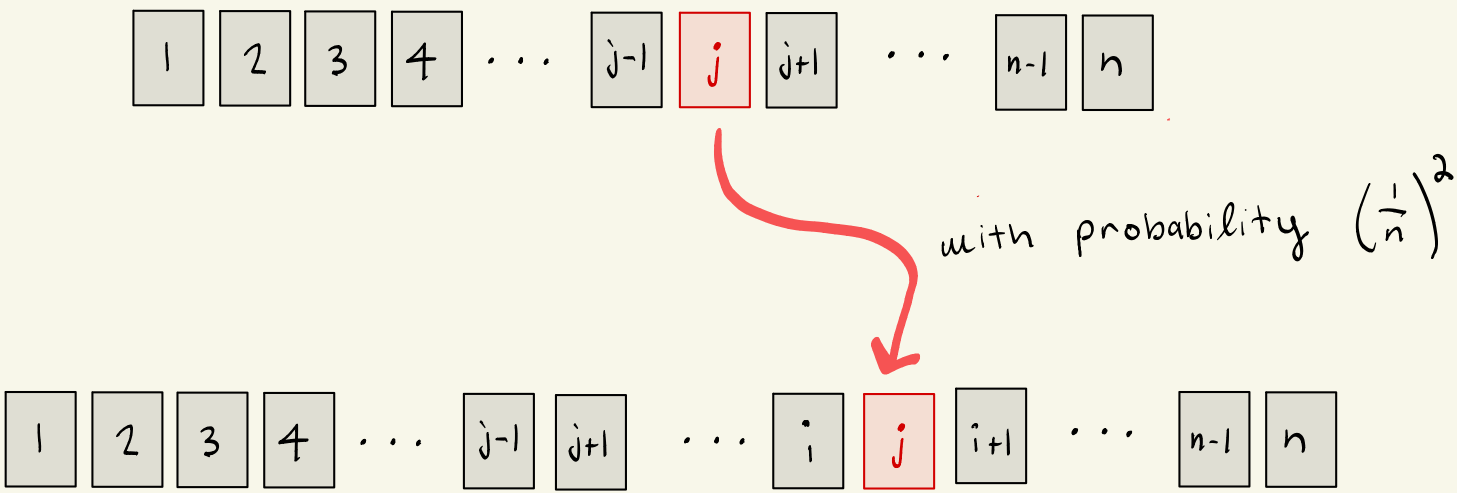
Random-to-random shuffling

A single random-to-random shuffle consists of:

- picking a card, uniformly at random
- moving it to some position, chosen uniformly at random



Random-to-random shuffling

- $\mathrm{R2R}_{n,1}$:= the element in $\mathbb{C}[S_n]$ associated to random-to-random

- Example:

$$\mathrm{R2R}_{3,1} = \frac{3}{9}\cdot(123) + \frac{2}{9}\cdot(132) + \frac{2}{9}\cdot(213) + \frac{1}{9}(231) + \frac{1}{9}(312)$$

$1\,2\,3$ or $1\,2\,3$

- Random-to-random can be factored as:

$$\mathrm{R2R}_{n,1} = \mathrm{R2B}_{n,1} \cdot \left(\mathrm{R2B}_{n,1}\right)^T$$

"bottom-to-random"

↳ $\mathrm{R2R}_{n,1}$ is diagonalizable.

Eigenvalues of random-to-random

1993: Diaconis — Saloff-Coste introduce random-to-random

2002: Uyemura-Reyes conjectures non-negative integer eigenvalues

2003: Hanlon—Hersh connect conjecture to the Complex of injective words

2014: Reiner-Saliola-Welker conjecture k-random-to-random has non-negative integer eigenvalues.

2018: Dieker-Saliola prove Uyemura-Reyes's conjecture.

↳ 2017: Bernstein-Nestoridi use to compute mixing time

2019: Lafrenière proves Reiner-Saliola-Welker's conjecture

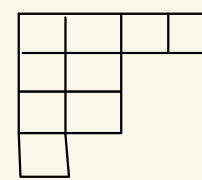
Tableaux Combinatorics

The spectrum of $R2R_{n,1}(q)$ is expressed in terms of tableaux combinatorics:

- A partition of n $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ is a weakly decreasing sequence of positive integers summing to n.

- Partitions are represented by Young diagrams of shape λ.

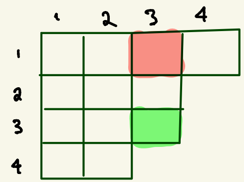
<u>Example</u> $\lambda = (4, 2, 2, 1) \vdash 9$ has Young diagram



Content, horizontal strips

- The content of a box in a Young diagram Y is

$$\mathrm{cont}(\mathrm{box}) = \mathrm{col}(\mathrm{box}) - \mathrm{row}(\mathrm{box}).$$

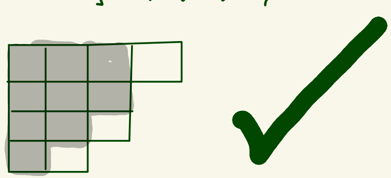


$\mathrm{cont}(\ ) = 3 - 1 = 2$

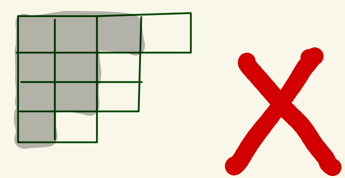
$\mathrm{cont}(\ ) = 3 - 3 = 0$

- For partitions $\mu \subseteq \lambda$, the skew partition $\lambda\backslash\mu$ is a horizontal strip if $\lambda\backslash\mu$ has at most one box per column.

$(4,3,3,2)\backslash(3,3,2,1)$



$(4,3,3,2)\backslash(3,2,2,1)$



Standard Young Tableaux

- A Standard Young tableau (SYT) is a filling of a Young diagram of size n with $\{1, 2, \ldots, n\}$ so that the entries increase along rows and columns.

- Let $SYT(\lambda) :=$ standard Young tableaux of shape λ.

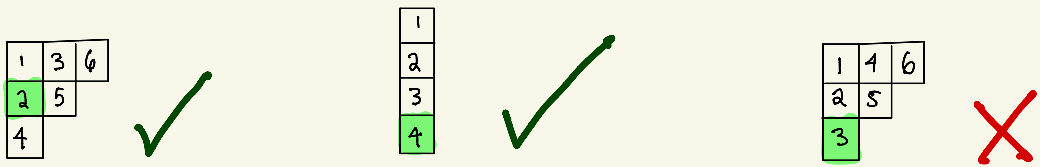
Example

$$SYT(2,2,1) = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \multicolumn{1}{|c|}{5} \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \multicolumn{1}{|c|}{5} \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1}\end{array} \right\}.$$

Desarrangement Tableaux

- An integer i is a descent of an SYT T if $i+1$ appears in a row below i.

- A desarrangement tableau is an SYT T for which the minimum non-descent is even.



- Let $d^{\mu} := \#\{\text{Desarrangement } T : \text{shape}(T) = \mu\}$.

Eigenvalues of $R2R_{n,1}$

Theorem (Dieker – Saliola '18)

For $\lambda \vdash n$, the eigenvalues of $R2R_{n,1}$ are:

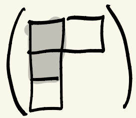
- in $\mathbb{Z}_{\geq 0}$.
- indexed by horizontal strips: $\{\mathrm{Eig}_1(\lambda/\mu)\}$

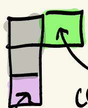
↳ $\mathrm{Eig}_1(\lambda/\mu)$ has multiplicity $d^{\mu} \cdot \#\,SYT(\lambda)$

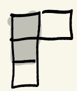
↳ $$\mathrm{Eig}_1(\lambda/\mu) = \sum_{\text{box in } \lambda\setminus\mu} \mathrm{cont}(\text{box}) + \sum_{i=|\mu|+1}^{n} i$$

Example eigenvalue computation

$$\mathrm{Eig}_1(\lambda/\mu) = \sum_{\text{box in } \lambda\backslash\mu} \mathrm{cont}(\text{box}) + \sum_{i=|\mu|+1}^{n} i$$

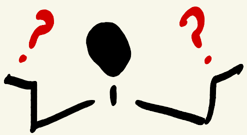
Computing Eig_1 

- 

  Eig_1 () $= (1 + -2) + (3+4) = 6$

- Multiplicity in $\mathbb{C}[S_n]$ $= \underbrace{\#\left\{ T \in \mathrm{SYT}(\right.}$  $\left.\underbrace{): T \text{ is desarrangement} \right\}}_{1} \cdot \underbrace{\#\,\mathrm{SYT}(}$  $\underbrace{)}_{3}$

  $= \boxed{3}$



Why are these objects showing up?

- Partitions + SYT
- Desarrangement tableaux

was clear

- Horizontal strips

there seemed to be more to the story

- Content

seemed like magic

Why partitions / SYT?

To understand spectra of $R2R_{n,1}$ on $\mathbb{C}[S_n]$, it's enough to understand its spectra on each irreducible right S_n-module.

$$\{S^\lambda : \lambda \vdash n\}.$$

λ a partition of n

$$\mathbb{C}[S_n] \circlearrowleft R2R_{n,1} \quad \cong \quad \bigoplus_{\lambda \vdash n} (S^\lambda)^* \otimes S^\lambda \circlearrowleft R2R_{n,1}$$

Eigenvalue ε with multiplicity $\underbrace{\dim(S^\lambda)^*}_{\#\,\mathrm{SYT}(\lambda)} \cdot m$ $\longleftrightarrow$ Eigenvalue ε with multiplicity m

Why desarrangement tableaux + horizontal strips?

- **Note** $\ker R2R_{n,1} = \ker R2B_{n,1} \cdot (R2B_{n,1})^T = \ker R2B_{n,1}$

- **Recall** $\dim \ker R2B_{n,1} = \#\underbrace{\text{permutations in } S_n \text{ with no fixed pts}}_{\text{"derangements"}}$

Thm (Désarménien, '84): In S_n, there is a bijection

$$\text{Derangements} \longleftrightarrow \text{Desarrangements}$$

$$\text{Desarrangements} = \{w \in S_n \mid \text{first non-descent of } w \text{ is even}\}$$

Why desarrangement tableaux + horizontal strips?

Thm (Reiner–Wachs, see Reiner–Saliola Welker '14)

As left S_n-rep'ns,

$$\ker R2B_n \cong \bigoplus_{\substack{\text{desarrangement} \\ \text{tableaux } T}} \left(S^{\text{shape}(T)}\right)^*.$$

The eigenvector in S^λ associated to $\mathrm{Eig}_1(\lambda/\mu)$ is obtained by "lifting" an element of S^μ in $\ker R2R_{|\mu|}$.

Think in the sense of

$$S^\mu \otimes S^{(|\lambda| - |\mu|)} \Big\uparrow_{S_{|\mu|} \times S_{n-|\mu|}}^{S_n}$$

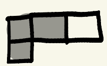
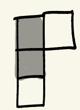
Indexing sets + "accidental" eigenvalue friends

$$\mathrm{Eig}_1(\lambda/\mu) = \left(\sum_{\mathrm{box} \in \lambda/\mu} \mathrm{cont}(\mathrm{box}) \right) + \sum_{k=|\mu|+1}^{|\lambda|} k$$

The eigenvalue formula implies

- $\mathrm{contents}(\lambda/\mu) = \mathrm{contents}(\alpha/\beta) \Longrightarrow \mathrm{Eig}_1(\lambda/\mu) = \mathrm{Eig}_1(\alpha/\beta)$.

However, the converse does <u>not</u> hold!

$\mathrm{Eig}_1\Big($  $\Big) = 6 = \mathrm{Eig}_1\Big($  $\Big)$

What's the "right" indexing set for the eigenvalues?

The k-random-to-random family

- More generally, we can consider k-random-to-random,

$$R2R_{n,k} := R2B_{n,k} \circ (R2B_{n,k})^T.$$

Theorem (Reiner-Saliola-Welker '14, Lafrenière '19)

The operators $\{R2R_{n,k}\}$ are simultaneously diagonalizable:

- they pairwise commute!
- their eigenvalues are all indexed by horiz. strips: $Eig_k(\lambda/\mu)$, with same multiplicities

The k-random-to-random family

Theorem (Lafrenière '19)

The eigenvalues of $R2R_{n,k}$ are in $\mathbb{Z}_{\geq 0}$.

↳ proved with a recursive formula for $Eig_k(\lambda/\mu)$

Unlike the $\{R2B_{n,k}\}$ family:

$$\langle R2R_{n,1} \rangle \neq \langle R2R_{n,k} : 1 \leq k \leq n \rangle$$

:(

In summary...

The k-random-to-random family:

- Has eigenvalues in $\mathbb{Z}_{\geq 0}$ with interesting combinatorics! ✓
- pairwise commute! ✓

but the initial study left some mysteries:

① Why is content showing up?

② What is the "right" eigenvalue indexing set?

③ How far away from $\langle R2R_{n,k} \rangle_k$ is $\langle R2R_{n,1} \rangle$?

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q-Analogues

More general objects for which plugging in $q=1$ recovers classical objects

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- $[n]_q := \dfrac{1-q^n}{1-q} = \begin{cases} 1+q+q^2+\dots+q^{n-1} & n \geq 0 \\ -q^{-1}-q^{-2}-\dots-q^{n} & n<0 \end{cases} \xrightarrow{q=1} n$

- $[n]!_q := [n]_q \cdot [n-1]_q \cdots [2]_q\,[1]_q \xrightarrow{q=1} n!$

- $\begin{bmatrix} n \\ k \end{bmatrix}_q := \dfrac{[n]!_q}{[n-k]!_q \cdot [k]!_q} \xrightarrow{q=1} \dbinom{n}{k}$

- Hecke algebra $H_n(q) \xrightarrow{q=1}$ symmetric group algebra $\mathbb{C}[S_n]$

The Hecke Algebra

For $q \in \mathbb{C}$, the (type A, Iwahori) Hecke algebra $\mathcal{H}_n(q)$ is:

- the $\mathbb{C}$-vector space with basis $\{T_w : w \in S_n\}$,
- with multiplication defined for a reduced word $s_{i_1} s_{i_2} \cdots s_{i_k}$ by:

$$T_w \, T_{s_{i_1} s_{i_2} \cdots s_{i_k}} := \left(\left(\left(T_w \cdot T_{s_{i_1}}\right) \cdot T_{s_{i_2}}\right) \cdots T_{s_{i_k}}\right),$$

where

$$T_w T_{s_i} = \begin{cases} T_{w \cdot s_i} & \text{if } \ell(ws_i) = \ell(w) + 1 \\ q T_{ws_i} + (q-1) T_w & \text{otherwise} \end{cases}$$

Aside: $\mathcal{H}_n(q)$ + probability

- Diaconis – Ram ('00):
  - For $q \in \mathbb{R}_{\geq 1}$, we can think of $q^{-1} \in (0,1]$ as a probability.
  - Setting $\widetilde{T}_w := q^{-\ell(w)} \cdot T_w$, our multiplication formula becomes:

$$\widetilde{T}_w \cdot \widetilde{T}_{s_i} = \begin{cases} \widetilde{T}_{w \cdot s_i} & \ell(w \cdot s_i) = \ell(w) + 1, \\ q^{-1}\widetilde{T}_{w \cdot s_i} + (1-q^{-1})\widetilde{T}_w & \text{otherwise} \end{cases}$$

  - Each $\widetilde{T}_{s_i}$ induces a Markov chain on permutations

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- Also: See recent Ayyer – Brauner – De Gier – Schilling ('26) for unequal probabilities generalization of $R2B_{n,1}(q)$

The q-deformed Operators

- **(q,k)-random-to-bottom**

$$R2B_{n,k}(q) := \frac{1}{\begin{bmatrix} n \\ k \end{bmatrix}_q \cdot [k]!_q} \sum_{\substack{w \in S_n : \\ w(1) < w(2) < \cdots < w(n-k)}} T_w$$

linear map $T_w \longmapsto T_{w^{-1}}$

- **(q,k)-random-to-random**

$$R2R_{n,k}(q) := \frac{1}{[k]!_q} R2B_{n,k}(q) \cdot \left(R2B_{n,k}(q) \right)^{*}$$

Our General Approach

For eigenvalues, we show we can reduce to the case that $\mathcal{H}_n(q)$ is semisimple, in which case the representation theory is very similar to $\mathbb{C}[S_n]$:

$$\mathcal{H}_n(q) \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^* \otimes S^\lambda$$

dim = # SYT (λ)

- There is a chain of inclusions

$$\mathcal{H}_1(q) \subseteq \mathcal{H}_2(q) \subseteq \cdots \subseteq \mathcal{H}_{n-1}(q) \subseteq \mathcal{H}_n(q) \subseteq \cdots$$

$$T_{w_1 w_2 \cdots w_{n-1}} \hookrightarrow T_{w_1 w_2 \cdots w_{n-1} n}$$

- We prove our results recursively via this structure.

Addressing the "content" mystery

- The (additive) Jucys - Murphy elements of $\mathcal{H}_n(q)$ are

$$J_i := \sum_{i=1}^{k-1} q^{i-k} T_{(i,k)} \quad \text{for} \quad 1 \leq i \leq n.$$

- For $\lambda \vdash n$, Young's seminormal basis $\{P_T : T \in SYT(\lambda)\}$ of S^λ forms a simultaneous eigenbasis for $\{J_i\}$, with

$$P_T \cdot J_i = \left[\operatorname{cont}(\operatorname{box}_T(i))\right]_q \cdot P_T$$

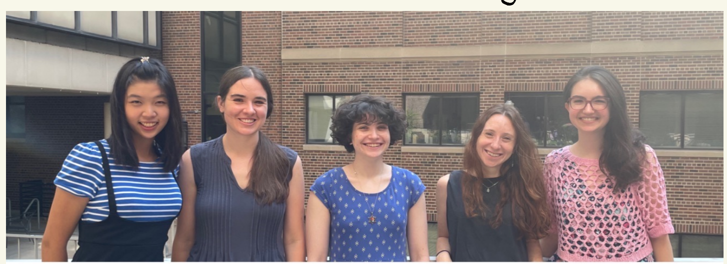
Ex: $$P_{\begin{smallmatrix}1&4\\2&5\\3&\end{smallmatrix}} \cdot J_3 = -[2]_q \, P_{\begin{smallmatrix}1&4\\2&5\\3&\end{smallmatrix}}$$

Addressing the "content" mystery

A fundamental recursion:

$$\boxed{\begin{aligned}&\mathrm{BaR}_{n,1}(q)\cdot \mathrm{RaR}_{n,k}(q) = \\ &\quad\left(q^{k}\,\mathrm{RaR}_{n-1,k}(q) + \left([n+1-k]_q + q^{n+1-k} J_n\right)\mathrm{RaR}_{n,k-1}(q)\right)\mathrm{BaR}_{n,1}(q)\end{aligned}}$$

↳ $k=1$: (Axelrod-Freed — Brauner — Chiang — C. — Lang, '24)



↳ General k: (Brauner — C. — Grinberg — Saliola, '25)

Eigenvalues of $R\partial R_{n,k}(q)$

Theorem (Brauner – C. – Grinberg – Saliola, '26+ (general k)
Axelrod-Freed – Brauner – Chiang – C. – Lang '24 ($k=1$))

The eigenvalues of $R\partial R_{n,k}(q)$ are

- in $\mathbb{Z}_{\geq 0}[q]$
- indexed by horizontal strips $\mathrm{Eig}_k^{(q)}(\lambda/\mu)$, with multiplicities $d^\mu \cdot \# SYT(\lambda)$

+ a closed formula $\forall k$!: $\mathrm{Eig}_k^{(q)}(\lambda/\mu)$

$$\| $$

$$q^{(n-k+1)k} \sum_{r=0}^{k} (-1)^{k-r} \cdot e_r\left(\left\{[\mathrm{cont}(\mathrm{box})]_q : \mathrm{box} \in \lambda/\mu\right\}\right) \cdot h_{k-r}\left([-(|\mu|+1)]_q, \ldots, [-(n-k+1)]_q\right).$$

elementary symmetric fns

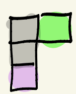
homogeneous symmetric fns

Example

$$\mathrm{Eig}_k^{(q)}(\lambda/\mu) = q^{(n-k+1)k} \sum_{r=0}^{k} (-1)^{k-r} \cdot e_r\left(\left\{[\mathrm{cont}(\mathrm{box})]_q : \mathrm{box} \in \lambda/\mu\right\}\right) \cdot h_{k-r}\left([-(|\mu|+1)]_q, \dots, [-(n-k+1)]_q\right).$$

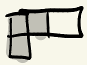
$$\Big\downarrow{\scriptstyle k=1}$$

$$q^n\left(\sum_{\substack{\text{box in}\\ \lambda/\mu}} [\mathrm{cont}(\lambda/\mu)]_q - \sum_{i=|\mu|+1}^{n} [-i]_q\right) \xrightarrow{q=1} \sum_{\substack{\text{box} \in \\ \lambda/\mu}} \mathrm{cont}(\mathrm{box}) + \sum_{i=|\mu|+1}^{n} i$$

$\mathrm{Eig}_1^{(q)}\Big($  $\Big)$ $= q^4\left[\left([1]_q + [-2]_q\right) - [-3]_q - [-4]_q\right]$

$$= 1 + 2q + q^2 + q^3 + q^4$$

$\mathcal{H}_n(q)$ "separates" accidental eigenvalue friends

λ/μ	$\mathrm{Eig}_1(\lambda/\mu)$	$\mathrm{Eig}_1^{(q)}(\lambda/\mu)$
	6	$1+2q+q^2+q^3+q^4 = [5]_q + q$
	6	$1+q+q^2+q^3+q^4+q^5 = [6]_q$

same

different

Thm (Brauner – C. – Grinberg – Saliola, '26+)

Let q be non-integral over $\mathbb{Z}$. Then,

$$\mathrm{Eig}_1^{(q)}(\lambda/\mu) = \mathrm{Eig}^{(q)}(\alpha/\beta) \iff \mathrm{contents}(\lambda/\mu) = \mathrm{contents}(\alpha/\beta)$$

The random-to-random algebra

Thm (Brauner – C. – Grinberg – Saliola, '26 +)

If q is non-integral over $\mathbb{Z}$, then the subalgebras

$$\langle R2R_{n,1}(q) \rangle = \langle R2R_{n,k}(q) : 0 \leq k \leq n \rangle .$$

(!) Recall: Not true when $q=1$

Note: $\{R2R_{n,k}(q)\}_k$ still pairwise commute for *all* q

(B-C-G-S '26+, via fundamental recursion)

The random-to-random algebra

Let $\mathcal{R} := \langle R2R_{n,1}(q) \rangle = \langle R2R_{n,k}(q) \rangle_k$

for "generic" q

Questions:

① $\dim \mathcal{R}$ = # possible content sets of horizontal strips.

Better formula for dimensions?

② How to expand $R2R_{n,k}(q)$ in terms of $\{(R2R_{n,1}(q))^i\}_i$?

↳ for R2B, in terms of Stirling #'s of first kind!

In summary...

	$A = R2B$	$A = R2R$	$A = R2R(q)$
$\{A_n(k)\}_k$ Pairwise Commute?	✓	✓	✓
$\langle A_n(1)\rangle = \langle A_n(k)\rangle_k$	✓	✗	✓
$A_n(k)$ has non-negative integer eigenvalues	✓	✓	✓ (in $\mathbb{Z}_{\geq 0}[q]$)

Preview for Oregonians

- Reiner - Saliola - Welker ('14) considered a symmetrized shuffling operator $\nu_\lambda \in \mathbb{C}[S_n]$ for each partition $\lambda \vdash n$. They conjecture:

CONJECTURE 4.4. *Let λ and γ be distinct partitions of n, both different from (1^n) and (n). The operators ν_λ and ν_γ commute if and only if they* both *belong to* $\{\nu_{(k,1^{n-k})} : 1 < k < n\}$ *or* $\{\nu_{(2^k,1^{n-2k})} : 0 < k \leq \lfloor \frac{n}{2} \rfloor\}$. *Furthermore, ν_λ has integer eigenvalues if and only if ν_λ belongs to one of these two families.*

k-random-to-random family

"Dyadic" shuffling family

- Hear more about our progress on dyadic shuffles next week!

May 14	Sarah Brauner *Brown University*	▼ Dyadic card shuffling There are many ways to shuffle a deck of cards. In this talk, I will discuss a strange one introduced by Reiner-Saliola-Welker in 2014 called dyadic shuffling. Many mysteries about this shuffling process have endured, especially related to its eigenvalues. I will present recent progress understanding these eigenvalues using the representation theory of the symmetric group, thereby proving several conjectures from Nadia Lafrenière's thesis and partially resolving a question by Reiner, Saliola, and Welker. Joint work with Patty Commins, Darij Grinberg, Trevor Karn, Nadia Lafrenière and Franco Saliola

Thank you for listening!