Clasped Web Bases from Hourglass Plabic Graphs

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Representation theory Background

- G denotes the **special linear group** SL_r and $\mathfrak{g}$ its Lie algebra $\mathfrak{sl}_r$, with the **standard representation** V of dimension r.
- The **fundamental representations** of G are

$$V(\omega_k) = \bigwedge^k V, \quad 1 \le k \le r-1.$$

- For $\underline{a} = (a_1, \dots, a_n) \in [r-1]^n$,

$$\mathrm{Inv}_G\left(\bigwedge^{\underline{a}} V\right) = \mathrm{Hom}_G\left(\bigotimes_{i=1}^{n} V(\omega_{a_i}), \mathbb{C}\right)$$

denotes the space of G-invariant maps from $\bigotimes_{i=1}^{n} V(\omega_{a_i})$ to $\mathbb{C}$.

Remark

$\dim \mathrm{Inv}_G(V^{\otimes n})$ = number of standard Young tableaux of shape $r \times \frac{n}{r}$ rectangle.

Web bases for invariant spaces

1. $r = 2$: The **Temperley-Lieb diagrams** of non-crossing matchings of n points drawn in a circle gives the first example of a web basis for $\mathrm{Inv}_G(\bigwedge^{\underline{a}} V)$.

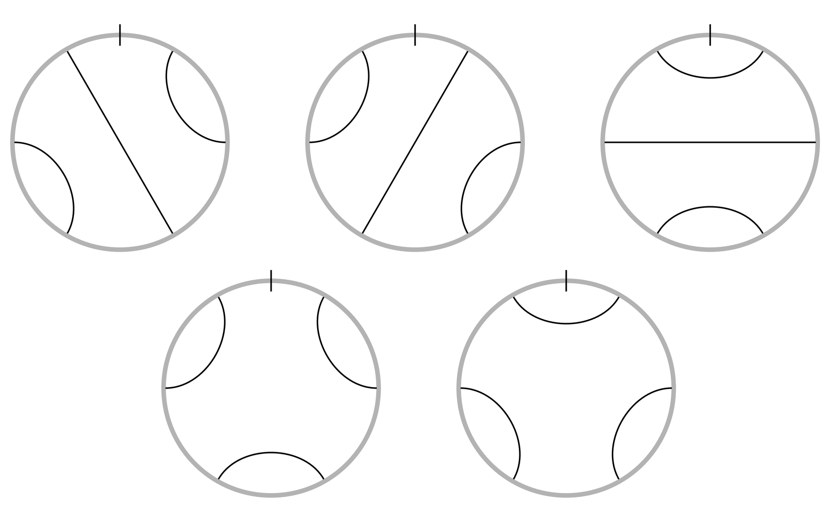


Figure: The $\mathfrak{sl}_2$ web basis for $V^{\otimes 6}$

2. $r = 3$: Kuperberg introduced the **non-elliptic web basis**, which can be interpreted as 3-regular plabic graphs.
3. $r = 4$: Gaetz–Pechenik–Pfannerer–Striker–Swanson introduce **hourglass plabic graphs** to extend Kuperberg's web basis to $\mathfrak{sl}_4$.

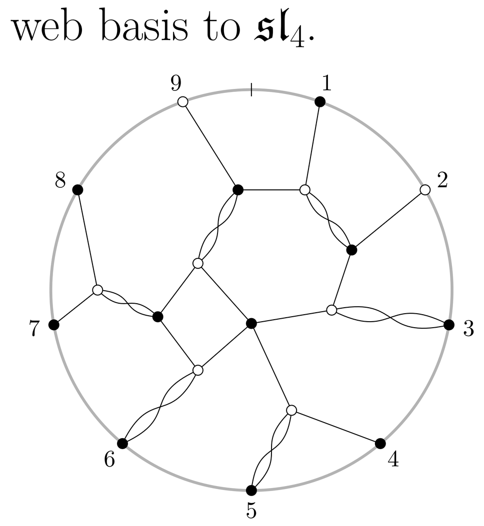


Figure: An $\mathfrak{sl}_4$ web. Boundary vertices $1, 4, 7, 8$ correspond to $V(\omega_1) = V$, vertices $3, 5, 6$ correspond to $V(\omega_2) = \bigwedge^2 V$, and vertices $2, 9$ correspond to $V(\omega_3) = V^*$.

Applications of web bases

Quantum enrichments of web bases have found applications in a plethora of fields of mathematics, including:

- Quantum link invariants,
- Skein algebras and skein modules,
- Cluster algebras and canonical bases,
- Cyclic sieving phenomena.

Web basis facts

- A web W defines a **web invariant** $[W] \in \mathrm{Inv}_G(\bigwedge^{\underline{a}} V)$. An explicit expression for $[W]$ can be given in terms of **proper colorings** of W.
- Many algebraic operations on invariant spaces correspond to combinatorial operations on webs:

Tensor product	$\longleftrightarrow$	Superposition
Contraction	$\longleftrightarrow$	Stitch
Cyclic permutation	$\longleftrightarrow$	Rotation

Other irreducible representations

Question: What about other irreducible representations?

- Any dominant integral weight $\lambda \in \Lambda^+$ can be written uniquely as $\lambda = \sum_{i=1}^{r-1} m_i \omega_i$, giving a (unique up to scaling) embedding

$$V(\lambda) \to \bigotimes_{i=1}^{r-1} V(\omega_i)^{\otimes m_i}.$$

- For any sequence $\lambda_1, \dots, \lambda_n$ of weights, we get a projection

$$\pi : \mathrm{Inv}_G\left(\bigwedge^{\underline{a}} V\right) \to \mathrm{Inv}_G\left(\bigotimes_{i=1}^{m} V(\lambda_i)\right).$$

Clasped webs

- The projection can be pictured diagrammatically as clasping of sets of consecutive boundary vertices of the web.

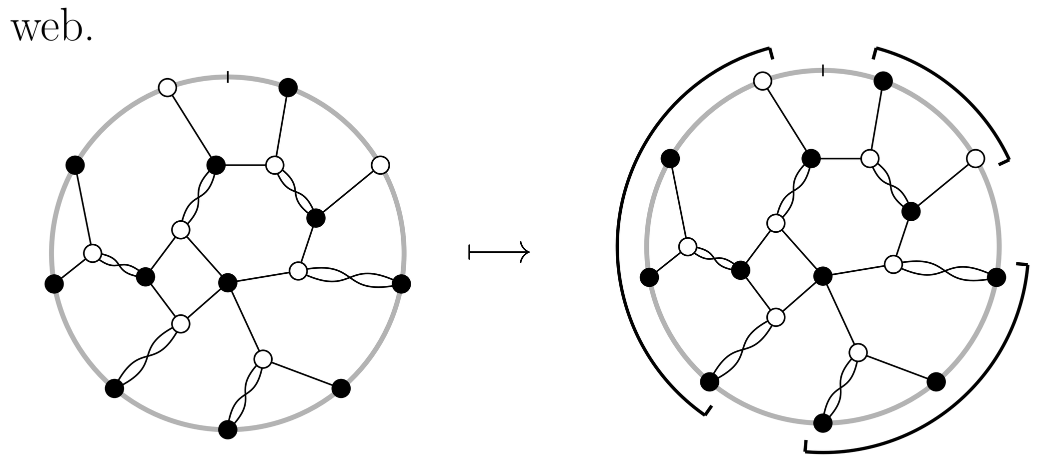


Figure: Clasping projection to $\mathrm{Inv}_G(V(\omega_1+\omega_3) \otimes V(\omega_1+2\omega_2) \otimes V(2\omega_1+\omega_2+\omega_3))$

- Kuperberg introduced the notion of **non-convexity** for clasped webs in terms of **cut paths** to describe a basis for $\mathrm{Inv}_G\left(\bigotimes_{i=1}^{m} V(\lambda_i)\right)$.

Theorem (Kuperberg 1996)

Let $r = 2$ or 3. Then, $\pi([W]) \neq 0$ iff W is non-convex. Moreover, the web invariants of non-convex clasped webs is a basis for $\mathrm{Inv}_G\left(\bigotimes_{i=1}^{m} V(\lambda_i)\right)$.

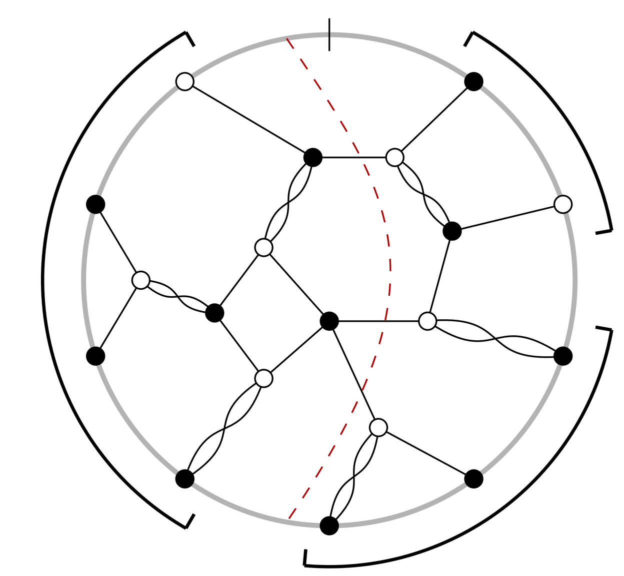


Figure: Cut path in a clasped web for the leftmost clasp

Trips in hourglass plabic graphs

Starting from each boundary vertex, there are 3 types of **trip strands** – namely trip_i for $i \in \{1, 2, 3\}$ given by the rules of the road:

- take the i^{th} right at a black vertex, and
- take the i^{th} left at a white vertex.

Note: At a boundary vertex of type $V(\omega_2)$, there are two non-trivial choices for trip_2, but only one choice each for trip_1 and trip_3 that are non-trivial.

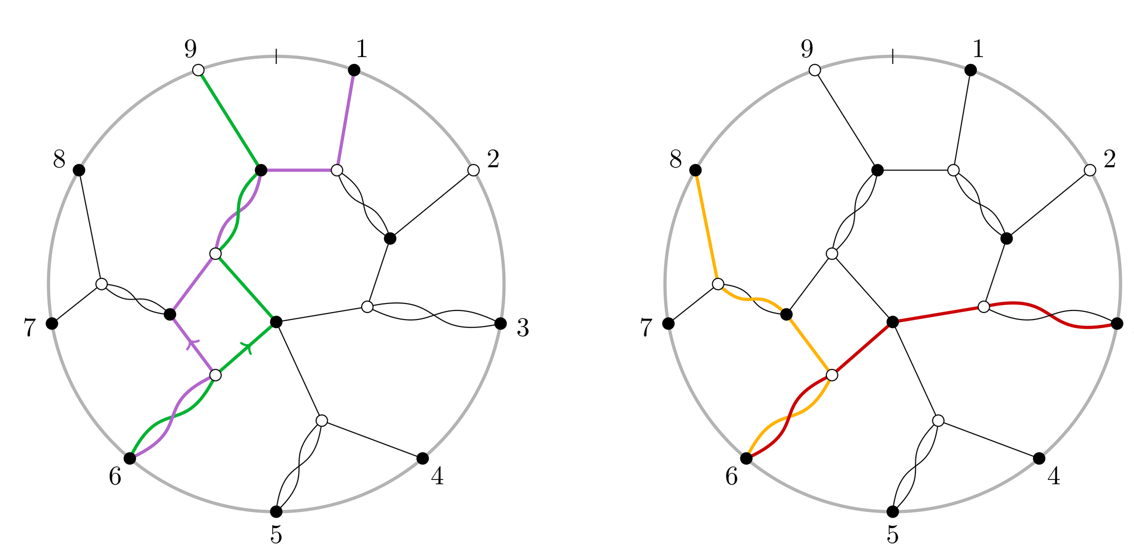


Figure: trip_1- (in purple), trip_2- (in orange/red), and trip_3- in (green) strands starting at vertex 6.

Remark

- Trips in hourglass plabic graphs generalize Postnikov's trips in plabic graphs, related to positroid varieties.
- Trips in webs are related to **promotion** permutations of SYT.

Main Theorems

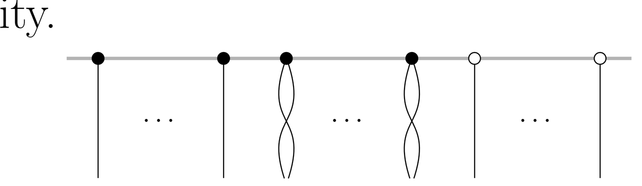
We generalize Kuperberg's notion of non-convexity to clasped hourglass plabic graphs, and give a **combinatorial** characterization of non-convexity.

Theorem (E.–Gaetz 2025)

Let $r = 2, 3$ or 4. A basis for $\mathrm{Inv}_G\left(\bigotimes_{i=1}^{m} V(\lambda_i)\right)$ is given by the web invariants of non-convex clasped webs. Moreover, the following are equivalent:

1. W is non-convex,
2. $\pi([W]) \neq 0$,
3. W has no non-trivial trips that start and end in the same clasp.

In the special case where the boundary vertices within each clasp are **sorted**, we have a **local characterization** of non-convexity.



Theorem (E.–Gaetz 2025)

A **sorted** clasped $\mathfrak{sl}_4$-web is non-convex iff it does not contain one of the following **bad configurations** within a clasp

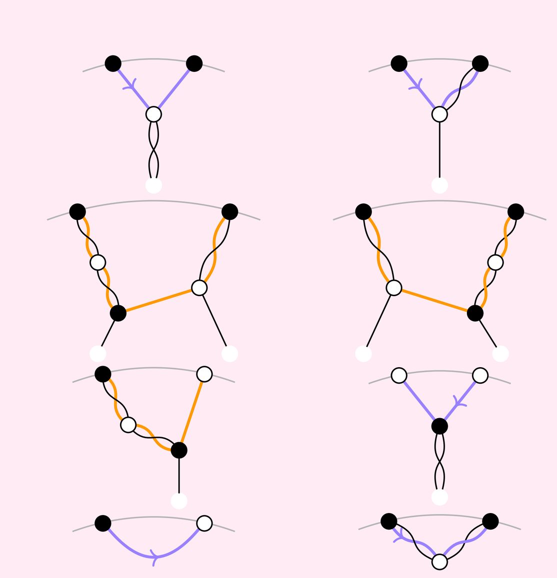


Figure: List of bad configurations, with returning trips highlighted.

Further directions

- **Higher rank**, i.e., $r \ge 5$.
- Other Lie types, i.e., **orthogonal** and **symplectic**.
- Lie **superalgebras**.

References

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