Schubert polynomial formulas via degeneration

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"Proof:" Assume we have two pairs of lines that each intersect at a point. One line connects the intersection points, while the other line is the intersection of the planes defined by each pair of lines. Now apply:

Principle of the conservation of number (Schubert 1879)
The number of solutions remains unchanged when the parameters (here lines) are varied, provided the solution remains finite.

Schubert Calculus

Using his principle, Schubert 'solved' many enumerative geometry problems, e.g., a generic surface in $\mathbb{P}^3$ of degree $n > 2$ has

$$\frac{1}{12}n(n-4)(n-5)(n-6)(n-7)(n^3+6n^2+7n-30)$$

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Hilbert's 15th problem: Give a rigorous foundation to Schubert's method.

This problem has been solved for Schubert calculus and related techniques by realizing enumerative geometry questions as cohomology computations.

The Flag Variety

In the modern formulation, Schubert's computations live inside the cohomology ring of the *flag variety*

$$Fl(n) = \{(0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) : \dim(V_i) = i\}.$$

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Example: Flags in X_{35142} can be represented by matrices of the form

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & * & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 \\ * & 1 & 0 & 0 & 0 \end{bmatrix}$$

More generally,

$$\overline{X}_w = \{\text{flag}(A) : A \in GL(n), rk(A_{[i][j]}) \leq rk(w_{[i][j]})\}$$

Positivity

Generic intersections correspond to the cohomology cup product:

$$\overline{X}_u \cup \overline{X}_v = [\overline{X}_u \cap_{\mathrm{transverse}} \overline{X}_v] = \sum_{w \in S_n} c_{uv}^w [\overline{X}_w].$$

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Problem

Find a combinatorial formula for c_{uv}^w.

Note: Only very special cases are known.

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- Defined inductively via operators with $\mathfrak{S}_{n\ldots21} = x_1^{n-1} x_2^{n-2} \ldots x_{n-1}$.

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2. *For* $w \in S_n$, *let* a *be the first index so* $w(a) < n + 1 - a$. *Then*
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$$x_a \mathfrak{S}_w = \sum_{u \in \Psi(w)} \mathfrak{S}_u.$$

Example: For $w = 35142$, applying transition and cotransition gives

$$\mathfrak{S}_{35142} = x_4 \cdot \mathfrak{S}_{35124} + \mathfrak{S}_{35214} \quad \text{and} \quad x_1 \cdot \mathfrak{S}_{35142} = \mathfrak{S}_{45132} + \mathfrak{S}_{53142}$$

Monomial Positivity

Say $w \in S_n$ is **increasing after** k if $w(k{+}1) < \cdots < w(n)$ and **reversed up to** k if $w(i) = n + 1 - i$ for $i \leq k$.

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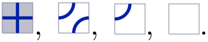
Question: How do we understand the monomials of $\mathfrak{S}_w$?

Pipe Dreams

A **pipe dream** is a diagram comprised of the tiles



filling the $n \times n$ grid so that wires enter from the left, exit on the top and no pair of wires crosses more than once. To obtain the **permutation** of a pipe dream, label the wires by row and read their order at the top.

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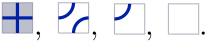

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Theorem (Billey–Jockusch–Stanley '93/Bergeron–Billey '94)

$$\mathfrak{S}_w = \sum_{P \in \mathcal{PD}(w)} \prod_i x_i^{\#\boxplus' s \text{ in row i}}$$

The easiest proof is via cotransition with induction on the number of reversed rows.

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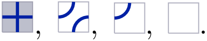

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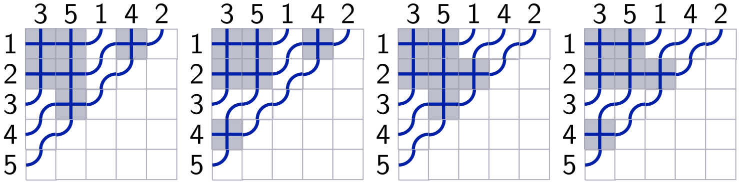
Example: The pipe dreams realizing the permutation $w = 35142$ are:



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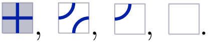

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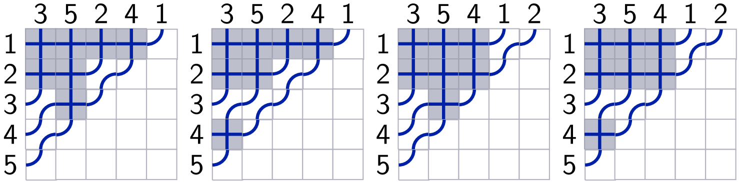
Example: The pipe dreams realizing summands in $x_1 \cdot \mathfrak{S}_w$ are:



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Affine spaces have trivial cohomology, but the torus T of invertible diagonal matrices acts on M_n.

Theorem (Fulton '92)

The equivariant cohomology class $[MX_w]_T = \mathfrak{S}_w$.

Defining ideals

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The **Schubert ideal** I_w is the ideal corresponding to MX_w. Fulton defines it via certain minors in the generic matrix $\mathcal{Z}_n = (z_{ij})_{i,j=1}^n$.

Example

For $w = 35142$, the matrix of rank bounds (ignoring trivial ones) is

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & 1 & 2 & 2 \\ 1 & \mathbf{1} & 2 & 3 \end{bmatrix}$$

Bold entries correspond to **essential** rank conditions. Then I_w is generated by 1×1–minors in $\mathcal{Z}_{[2],[2]}$ and 2×2–minors in $\mathcal{Z}_{[2],[4]}$ and $\mathcal{Z}_{[4],[2]}$:

$$I_w = (z_{11}, z_{12}, z_{21}, z_{22}, z_{31}z_{42} - z_{32}z_{41}, z_{13}z_{24} - z_{14}z_{23}).$$

Term Orders and Initial Ideals

A **term order** is a total order of monomials that respects degree and has the property that $m_1 < m_2$ implies $m_1 \cdot m_3 < m_2 \cdot m_3$. The **initial term** of a polynomial f with respect to $<$ is the $<$–maximal monomial $\mathrm{in}_<(f)$.

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A **Gröbner basis** for I with respect to $<$ is a generating set $\{f_1, \ldots, f_k\}$ for I so that $\{\mathrm{in}_<(f_1), \ldots, \mathrm{in}_<(f_k)\}$ generates $\mathrm{in}_<(I)$.

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Example: Let $<$ choose antidiagonals of minors. Then

$$\mathrm{in}_<(I_{35142}) = (z_{11}, z_{12}, z_{21}, z_{22}, z_{41}z_{32}, z_{14}z_{23})$$

so the Fulton generators are a Gröbner basis.

Knutson–Miller '05

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Take-away: Knutson and Miller give a geometric explanation for the pipe dream formula!

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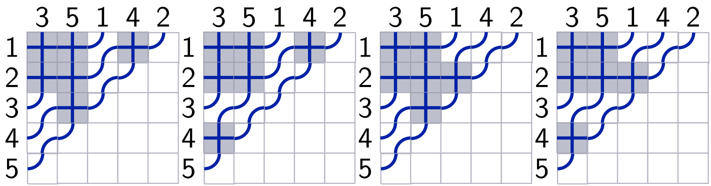
These correspond to the pipe dreams of w:



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Key Lemma: For w reversed up to $a-1$ but not a, let $b = w(a)$. Then

$$J_w + (z_{ab}) = \bigcap_{u \in \Phi(w)} J_u.$$

This precisely mirrors the cotransition proof of pipe dreams.

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$$\begin{aligned} \mathrm{in}_<(I_w) + (z_{ab}) &\subseteq \mathrm{in}_<(I_w + (z_{ab})) \subseteq \mathrm{in}_<(\cap_{u \in \Phi(w)} I_u) \\ &\subseteq \cap_{u \in \Phi(w)} \mathrm{in}_<(I_u) = \cap_{u \in \Phi(w)} J_u = J_w + (z_{ab}). \end{aligned}$$

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There is an inductive proof of Knutson–Miller via cotransition from Marberg–Pawlowski '22. Let J_w be the (for now) conjectural initial ideal.

Key Lemma: For w reversed up to $a-1$ but not a, let $b = w(a)$. Then

$$J_w + (z_{ab}) = \bigcap_{u \in \Phi(w)} J_u.$$

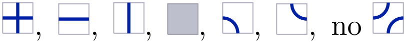
This precisely mirrors the cotransition proof of pipe dreams. Then

$$\begin{aligned} \mathrm{in}_<(I_w) + (z_{ab}) &\subseteq \mathrm{in}_<(I_w + (z_{ab})) \subseteq \mathrm{in}_<(\cap_{u \in \Phi(w)} I_u) \\ &\subseteq \cap_{u \in \Phi(w)} \mathrm{in}_<(I_u) = \cap_{u \in \Phi(w)} J_u = J_w + (z_{ab}). \end{aligned}$$

Since these have the same equivariant class, containment must be equality. 'Divide out' by (z_{ab}) to finish.

Bumpless Pipe Dreams

A **bumpless pipe dream** is a diagram comprised of the tiles



filling the $n \times n$ grid so that wires enter from the right, exit on top and no pair of wires crosses more than once. To obtain the **permutation** of a bumpless pipe dream, label the wires by row *from the bottom* and read their order at the top.

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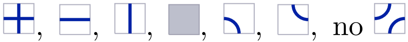

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Theorem (Lam–Lee–Shimozono '16/Lascoux '03)

$$\mathfrak{S}_w = \sum_{P \in \mathcal{BPD}(w)} \prod_i x_i^{\#\,\blacksquare' s \text{ in row } n+1-i}$$

This is most easily proved using the transition equations.

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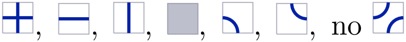

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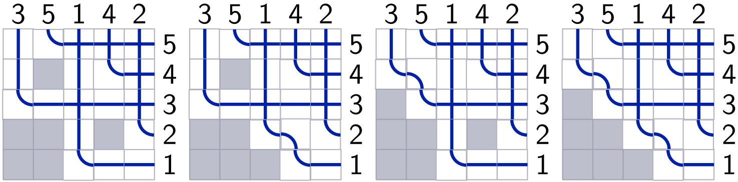
Example: The bumpless pipe dreams for $w = 35142$ are



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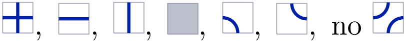

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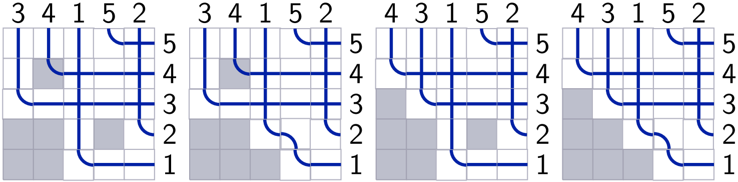
Example: Transition for w gives $v = 35124$ and $u = 35214$:



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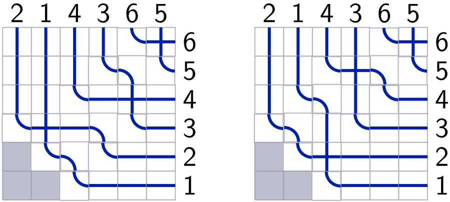

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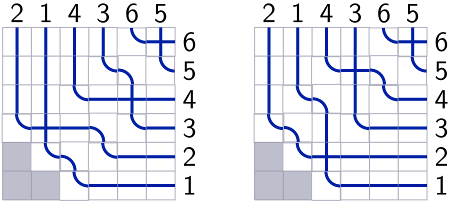

This means to understand the bumpless pipe dream formula geometrically, we must interpret multiplicities.

Klein–Weigandt

Let $<_{\mathrm{lex}}$ be lexicographic with $z_{nn} > \cdots > z_{n1} > \cdots > z_{1n} > \cdots > z_{11}$.
For a bumpless pipe dream P, let $I_P = (z_{ij} : P(i,j) = \square)$.

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Theorem (Klein–Weigandt)

For every permutation w:

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The equivariant class is the sum of minimal primes with multiplicity, so this proves the bumpless pipe dream formula geometrically!

Proof Idea

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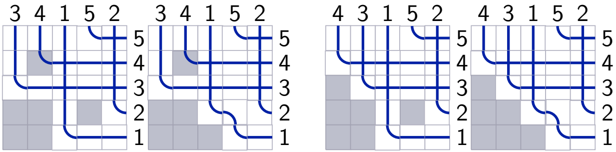

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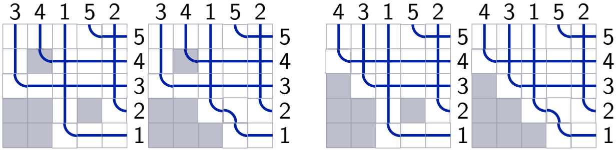

Ideal-level picture: Isolate bpds based on whether they have variable z_{42}:

$$v: \ \mathrm{in}_<(I_{35124}) + (z_{42}) = I_{P_1} \cap I_{P_2}, \quad u: \ \mathrm{in}_<(I_{35214}) = I_{P_3} \cap I_{P_4}$$

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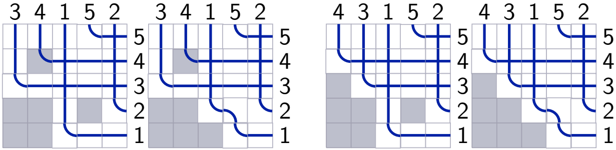

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by induction. Then show $\mathrm{in}_<(I_{35142}) = (\mathrm{in}_<(I_{35124}) + (z_{42})) \cap \mathrm{in}_<(I_{35214})$. This is called geometric vertex decomposition / linkage. Multiplicities appear when different paths converge on the same weights, but each step is multiplicity free.

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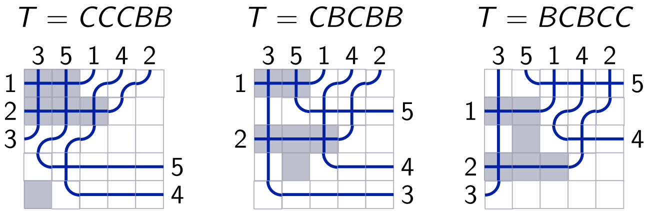

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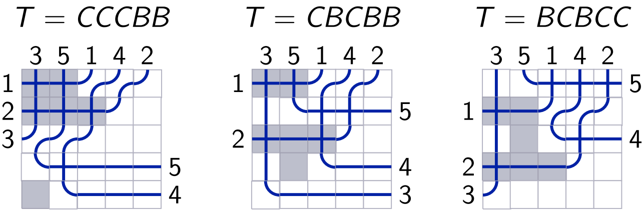

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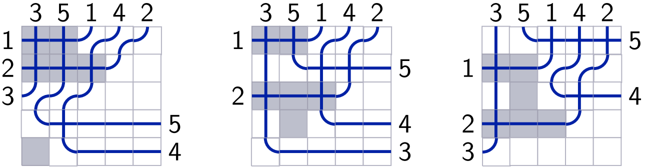
$T = CCCBB$ $\quad$ $T = CBCBB$ $\quad$ $T = BCBCC$



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Theorem (Knutson–Udell '26+)

For $w \in S_n$ and $T \in \{C, B\}^n$, $\mathfrak{S}_w = \sum_{P \in \mathcal{HPD}(w)} \mathrm{wt}(P)$.

Hybrid Term Orders

For an ordered list $x_1, \ldots, x_n$ of variables and $S \subseteq [n]$, a **successive term order** tests variables in order, lexing if $i \in S$ and revlexing otherwise.

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Note: For $T = C \ldots C$, $<_T$ is an antidiagonal term order. For $T = B \ldots B$, $<_T$ is the Klein–Weigandt term order.

H–Udell–Yu '26+

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- bijectively at the level of HPDs;
- algebraically at the level of ideals.

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- For what w and T is $\mathrm{in}_{<_T}(I_w)$ prime? Can we find Gröbner bases?

Thank You!