



# Diagonal Supersymmetry for Coinvariant Rings



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## Abstract

We show that bosonic-fermionic coinvariant rings have a natural  $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[\mathfrak{S}_n]$ -module structure. Their character series are a sum of super Schur functions  $s_\lambda(\mathbf{q}/\mathbf{u})$  times  $\mathfrak{S}_n$  Frobenius characters with universal coefficients, which do not depend on  $k, j$ . This proves the ‘‘Diagonal Supersymmetry’’ conjecture of Bergeron [2020].

## History

The classical coinvariant ring  $R_n^{(1,0)} = \mathbb{C}[x] / \langle \mathbb{C}[x]_{+}^{\mathfrak{S}_n} \rangle$  is the quotient of a polynomial ring in  $n$  variables  $x = \{x_1, \dots, x_n\}$  by  $\mathfrak{S}_n$ -invariant polynomials with no constant term. It has dimension  $n!$ , Hilbert series  $[n]_q!$ , and Frobenius series

$$\sum_{\lambda \vdash n} \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} s_\lambda.$$

Haiman [1994] introduced the diagonal coinvariant ring

$R_n^{(2,0)} = \mathbb{C}[x, y] / \langle \mathbb{C}[x, y]_{+}^{\mathfrak{S}_n} \rangle$ , which extends the classical coinvariant ring to two sets of  $n$  variables.  $\mathfrak{S}_n$  acts diagonally by permuting the indices of the variables. Haiman [2002] proved that  $R_n^{(2,0)}$  has dimension  $(n+1)^{n-1}$  and Frobenius series  $\nabla_{q,t}(e_n)$ .

## Bosonic-fermionic coinvariant rings

We study coinvariant rings with  $k$  sets of  $n$  commuting (bosonic) variables  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ , where  $\mathbf{x}^{(i)} = \{x_1^{(i)}, \dots, x_n^{(i)}\}$ , and  $j$  sets of  $n$  anticommuting (fermionic) variables  $\boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}$ , where  $\boldsymbol{\theta}^{(i)} = \{\theta_1^{(i)}, \dots, \theta_n^{(i)}\}$ . Denote this by

$$R_n^{(k,j)} = \frac{\mathbb{C}[x^{(1)}, \dots, x^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]}{\langle \mathbb{C}[x^{(1)}, \dots, x^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}]_{+}^{\mathfrak{S}_n} \rangle},$$

where  $(-)_+^{\mathfrak{S}_n}$  denotes those polynomials without constant term, which are invariant under the diagonal action of  $\mathfrak{S}_n$ .

The *multigraded Frobenius series* is defined by

$$\text{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\substack{r_1, \dots, r_k \\ s_1, \dots, s_j \geq 0}} F \text{char} \left( (R_n^{(k,j)})_{r_1, \dots, r_k, s_1, \dots, s_j} \right) q_1^{r_1} \cdots q_k^{r_k} u_1^{s_1} \cdots u_j^{s_j},$$

where  $\mathbf{q}$  denotes  $q_1, \dots, q_k$ , and  $\mathbf{u}$  denotes  $u_1, \dots, u_j$ .  $R_n^{(k,j)}$  is a  $\text{GL}_k \times \text{GL}_j \times \mathfrak{S}_n$ -module. Take  $\text{GL}_k$  characters,  $\text{GL}_j$  characters, and Frobenius characters to determine its multigraded Frobenius series.

**Theorem** (Bergeron [2020]). For partitions  $\lambda$  with  $\ell(\lambda) \leq n$ , and  $\mu \vdash n$ , there exist nonnegative integers  $c_{\lambda\nu\mu}$  such that for any  $(k, j)$ ,

$$\text{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\substack{\lambda, \nu \\ \ell(\lambda) \leq k, \ell(\nu) \leq j}} \sum_{\mu \vdash n} c_{\lambda\nu\mu} s_\lambda(\mathbf{q}) s_\nu(\mathbf{u}) s_\mu.$$

## Main results

Define  $P(k, j, n) = \{\lambda \mid \ell(\lambda) \leq n, \lambda_{k+1} \leq j\}$ .

**Theorem.** Fix  $n \geq 1$ . As  $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[\mathfrak{S}_n]$ -modules:

$$R_n^{(k,j)} \cong \bigoplus_{\lambda \in P(k,j,n)} \bigoplus_{\mu \vdash n} (U_{k|j}^\lambda \otimes N^\mu)^{\oplus c_{\lambda\mu}},$$

where  $U_{k|j}^\lambda$  are simple  $U(\mathfrak{gl}(k|j))$ -modules with character  $s_\lambda(\mathbf{q}/\mathbf{u})$  and  $N^\mu$  are simple  $\mathbb{C}[\mathfrak{S}_n]$ -modules, for some nonnegative integer coefficients  $c_{\lambda\mu}$ .

Taking  $\mathfrak{gl}(k|j)$  and Frobenius characters yields a character series in terms of super Schur functions and Schur functions. A *super Schur function* is

$$s_\lambda(\mathbf{q}/\mathbf{u}) := \sum_{\nu \subseteq \lambda} s_\nu(\mathbf{q}) s_{\lambda/\nu}(\mathbf{u}).$$

A priori, the coefficients are functions of  $(k, j)$ . We show that they are independent of  $(k, j)$ , proving a conjecture of Bergeron [2020].

**Theorem** (Diagonal Supersymmetry). Fix  $n \geq 1$ . For partitions  $\lambda$  with  $\ell(\lambda) \leq n$ , and  $\mu \vdash n$ , there exist nonnegative integer coefficients  $c_{\lambda\mu}$  such that for any  $(k, j)$ ,

$$\text{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_\lambda(\mathbf{q}/\mathbf{u}) s_\mu.$$

**Proposition.** The Frobenius series of  $R_n^{(1,0)}$  and  $R_n^{(0,1)}$  each do not contain enough information to determine the other. No Frobenius series of  $R_n^{(2,0)}$ ,  $R_n^{(1,1)}$ , or  $R_n^{(0,2)}$  contains enough information to determine another.

## Proof sketch

Let  $V = \mathbb{C}^n$ . The  $s_\lambda(\mathbf{q}/\mathbf{u})$  are characters of the simple  $\mathfrak{gl}(k|j)$ -modules  $U_{k|j}^\lambda$  which appear in the Howe decomposition of

$$\text{Sym}(\mathbb{C}^{k|j} \otimes V) = \mathbb{C}[x^{(1)}, \dots, x^{(k)}, \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(j)}].$$

**Proposition** ( $(\mathfrak{gl}(k|j), \text{GL}(n))$ -Howe Duality).

$$\text{Sym}(\mathbb{C}^{k|j} \otimes V) \cong \bigoplus_{\lambda \in P(k,j,n)} U_{k|j}^\lambda \otimes U_n^\lambda,$$

where  $U_{k|j}^\lambda$  is a simple  $\mathfrak{gl}(k|j)$ -module with character  $s_\lambda(\mathbf{q}/\mathbf{u})$ , and  $U_n^\lambda$  is a simple  $\text{GL}(n)$ -module with character  $s_\lambda$ .

- Define actions of  $\mathfrak{gl}(k|j)$  and  $\mathfrak{S}_n$  on  $\mathbb{C}^{k|j} \otimes V$ , which extend to  $\text{Sym}(\mathbb{C}^{k|j} \otimes V)$ .
- The actions commute with each other and give  $\text{Sym}(\mathbb{C}^{k|j} \otimes V)$  a  $U(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[\mathfrak{S}_n]$ -module structure.
- Use Howe duality to get  $\mathfrak{gl}(k|j)$  irreps (not semisimple in general).
- Restrict  $\text{GL}(n)$ -modules to  $\mathfrak{S}_n$ -modules, with some multiplicities  $d_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .
- The quotient ring  $R_n^{(k,j)}$  has a quotient module structure, so it has the same decomposition, with coefficients  $0 \leq c_{\lambda\mu} \leq d_{\lambda\mu}$ .
- Take  $\mathfrak{gl}(k|j)$ -characters and Frobenius characters.
- Show that the coefficients  $c_{\lambda\mu}$  are stable as  $k$  and  $j$  vary.

## Applications

Results extend to finite groups  $G \subseteq \text{GL}(n)$ . For dihedral groups, we can say more.

**Theorem.** The multigraded character series for  $R_{\mathfrak{S}_2(n)}^{(k,j)}$  is given by

$$\begin{aligned} \text{char}(R_{\mathfrak{S}_2(n)}^{(k,j)}; \mathbf{q}; \mathbf{u}) &= \chi_1 + s_{(1,1)}(\mathbf{q}/\mathbf{u}) \chi_2 + s_{(n)}(\mathbf{q}/\mathbf{u}) \chi_2 \\ &+ \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (s_{(i)}(\mathbf{q}/\mathbf{u}) + s_{(n-i)}(\mathbf{q}/\mathbf{u})) \chi^i \\ &+ \begin{cases} s_{(\frac{n}{2})}(\mathbf{q}/\mathbf{u}) (\chi_3 + \chi_4) & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases} \end{aligned}$$

## Coefficients

It is hard to determine the coefficients  $c_{\lambda\mu}$ .

**Proposition.**

$$\langle \text{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}), s_{(n)} \rangle = 1.$$

$$\langle \text{Frob}(R_n^{(k,j)}; \mathbf{q}; \mathbf{u}), s_{(n-1,1)} \rangle = \sum_{i=1}^{n-1} s_{(i)}(\mathbf{q}/\mathbf{u}).$$

## References

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