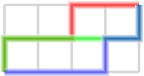
Cyclic partial orders, Parke-Taylor polytopes, and the magic number conjecture for the amplituhedron

Lauren K. Williams, Harvard

noncrossing lattice paths



plane partition

3	3	2	2
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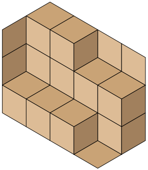
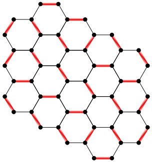
rhombic tiling



perfect matching



Based on: arXiv:2404.03026,

joint with Matteo Parisi, Melissa Sherman-Bennett, and Ran Tessler

Outline

- Partial cyclic orders and bicolored subdivisions
- Applications to Parke-Taylor identities and Parke-Taylor polytopes
- What is the amplituhedron?
- Magic number conjecture for the amplituhedron
- Proof of Magic number conjecture when $m = 2$

Partial and total cyclic orders

A *(partial) cyclic order* on a finite set X is a ternary relation $C \subset X^3$ such that for all $a, b, c, d \in X$:

$$(a,b,c) \in C \implies (c,a,b) \in C \qquad \textit{cyclicity}$$
$$(a,b,c) \in C \implies (c,b,a) \notin C \qquad \textit{asymmetry}$$
$$(a,b,c) \in C \text{ and } (a,c,d) \in C \implies (a,b,d) \in C \qquad \textit{transitivity}$$

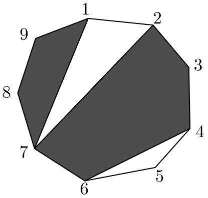
Ex: The triples $\{(2,5,7),(5,7,6),(1,8,7),(8,7,2)\}$ determine a partial cyclic order on [8].

A cyclic order C is *total* if for all $a, b, c \in X$, either $(a,b,c) \in C$ or $(a,c,b) \in C$.

Informally, a total cyclic order C on $[n]$ is a way of placing $1, \ldots, n$ on a circle, just as a total order is a way of placing $1, \ldots, n$ on a line.

Bicolored subdivisions and cyclic orders

- A *bicolored subdivision* τ of an *n*-gon is a subdivision of the polygon into smaller polygons (black or white) in which every edge connects two vertices of the *n*-gon.



- The *area* of a bicolored subdivision τ is the number of black triangles in any refinement of τ to a triangulation. Here: area$(\tau) = 5$.
- We can read off a cyclic order C_τ from τ, by reading vertices of white (respectively, black) polygons clockwise (resp counterclockwise).
- The C_τ from our example requires that $(1, 2, 7)$, $(4, 5, 6)$, $(1, 9, 8, 7)$, and $(2, 7, 6, 4, 3)$ are circularly ordered.
- A *circular extension* of C_τ is a total circular order compatible with C_τ. E.g. one circular extension of our example is: $(1\,9\,8\,2\,7\,6\,4\,5\,3)$.

The Grassmannian and Plücker coordinates

The **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix C.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(C)$ is the minor of the $k \times k$ submatrix of C in column set I.

Grassmannian identities from bicolored subdivisions

- Given a permutation $w = w_1 \dots w_n$, define the *Parke-Taylor function*

$$\mathrm{PT}(w) := \frac{1}{P_{w_1 w_2} P_{w_2 w_3} \dots P_{w_n w_1}},$$

where the P_{ij} are Plücker coordinates on the Grassmannian $\mathrm{Gr}_{2,n}^{\circ}$.

We get the following identity.

Theorem (Parisi–ShermanBennett–Tessler–W)

Let τ be a bicolored subdivision, and let C_τ be the cyclic partial order. Then

$$\sum_{w \in \mathrm{Ext}(C_\tau)} \mathrm{PT}(w) = (-1)^k \, \mathrm{PT}(\mathbf{I}_n),$$

where $k = \mathrm{area}(\tau)$, $\mathbf{I}_n$ is the identity permutation,
and the sum is over all circular extensions (w) of C_τ.

Grassmannian identities from bicolored subdivisions

The *Parke-Taylor function* is $\mathrm{PT}(w_1 \dots w_n) := \frac{1}{P_{w_1 w_2} P_{w_2 w_3} \dots P_{w_n w_1}}$.

Theorem (P–SB–T–W)

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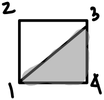


Example:

The circular extensions of C_τ are $(1243), (1423)$,

so Thm says $\frac{1}{P_{12}P_{24}P_{43}P_{31}} + \frac{1}{P_{14}P_{42}P_{23}P_{31}} = (-1)\frac{1}{P_{12}P_{23}P_{34}P_{41}}$.

(Rk: 3-term Plücker relation)

Parke-Taylor identities from bicolored subdivisions

Theorem (P–SB–T–W)

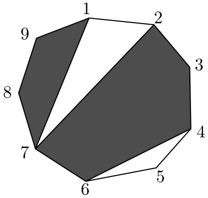
Let τ be a bicolored subdivision, and let C_τ be the cyclic partial order. Then

$$\sum_{w \in \mathrm{Ext}(C_\tau)} \mathrm{PT}(w) = (-1)^k \, \mathrm{PT}(\mathbf{I}_n),$$

where $k = \mathrm{area}(\tau)$, and the sum is over all circular extensions (w) of C_τ.

- PT functions related to: cohomology of $\mathcal{M}_{0,n}$ and *scattering eqns* (Cachazo-He-Yuan); Lie polynomials (Frost-Mason); non-planar plabic graphs (Arkani-Hamed-Bourjaily-Cachazo-Postnikov-Trnka).
- Thm above implies the $U(1)$ *decoupling identities* and *shuffle identities* for Parke-Taylor functions.
- There are some analogous results for linear extensions of posets due to Curtis Greene, in connection to the Murnaghan-Nakayama formula (rep theory of S_n).

Bicolored subdivisions and Parke-Taylor polytopes



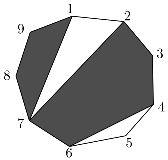
- We can associate a *Parke-Taylor polytope* $\Gamma_\tau \subset \mathbb{R}^{n-1}$ to each bicolored subdivision on $[n]$: for any *compatible arc* $i \to j$ with $i < j$,

$$\text{area}(i \to j) \leq x_i + x_{i+1} + \cdots + x_{j-1} \leq \text{area}(i \to j) + 1.$$

- A *compatible arc* is an edge of a polygon or lies entirely inside a black or white polygon.
- $\text{area}(i \to j)$ is the "black area" to the left of the arc.
- Above, $3 \to 7$ is a compatible arc. Gives inequality:

$$2 \leq x_3 + x_4 + x_5 + x_6 \leq 3.$$

Decompositions of Parke-Taylor polytopes



We've seen how each bicolored subdivision τ gives rise to:
a partial cyclic order C_τ and a Parke-Taylor polytope Γ_τ.

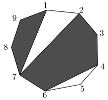
Theorem (Parisi–Sherman-Bennett–Tessler–W.)

Let τ be a bicolored subdivision. Then the Parke-Taylor polytope Γ_τ has a triangulation

$$\Gamma_\tau = \bigcup \Delta_{(w)}$$

into unit simplices $\Delta_{(w)}$, where w ranges over all circular extensions of the partial cyclic order C_τ. In particular, the normalized volume of Γ_τ equals the number of circular extensions of C_τ.

Decompositions of Parke-Taylor polytopes



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Let τ be a bicolored subdivision. Then the Parke-Taylor polytope Γ_τ has a triangulation

$$\Gamma_\tau = \bigcup \Delta_{(w)}$$

into unit simplices $\Delta_{(w)}$, where w ranges over circular extensions of C_τ.

- Reminiscent of Stanley's result that the volume of the *order polytope* of a poset P equals the number of linear extensions of P.
- Related work of Ayyer–Josuat-Verges–Ramassamy, and D'Leon–Hanusa–Morales–Yip.
- Followup work of Yuhan Jiang and Bullock-Jiang on h^*-vectors of positroid and alcoved polytopes.

What is the amplituhedron?

Recall: the **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix C.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(C)$ is the minor of the $k \times k$ submatrix of C in column set I.

The *matroid* associated to $C \in Gr_{k,n}$ is $\mathcal{M}(C) := \{I \in \binom{[n]}{k} \mid p_I(C) \neq 0.\}$

Gelfand-Goresky-MacPherson-Serganova '87 introduced the *matroid stratification* of $Gr_{k,n}$.

Given $\mathcal{M} \subset \binom{[n]}{k}$, let $S_{\mathcal{M}} = \{C \in Gr_{k,n} \mid p_I(C) \neq 0 \text{ iff } I \in \mathcal{M}\}$.

$$\text{Matroid stratification: } Gr_{k,n} = \sqcup_{\mathcal{M}} S_{\mathcal{M}}.$$

However, the topology of matroid strata is terrible –
Mnev's *universality theorem* (1987).

What is the positive Grassmannian?

Background: 1994 Lusztig total positivity for G/P, 1997 Rietsch, 2006 Postnikov preprint on *totally non-negative* (TNN) or "positive" Grassmannian.

Let $Gr_{k,n}^{\geq 0}$ be subset of $Gr_{k,n}(\mathbb{R})$ where Plucker coords $p_I \geq 0$ for all I.

Inspired by matroid stratification, one can partition $Gr_{k,n}^{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}} := \{C \in Gr_{k,n}^{\geq 0} \mid p_I(C) > 0 \text{ iff } I \in \mathcal{M}\}$.

In contrast to terrible topology of matroid strata ...

(Postnikov, see also Rietsch) If $S_{\mathcal{M}}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. So we have *positroid cell decomposition*

$$Gr_{k,n}^{\geq 0} = \sqcup S_{\mathcal{M}}.$$

There's a classification of the (nonempty) cells.

What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$, Arkani-Hamed–Trnka (2013).

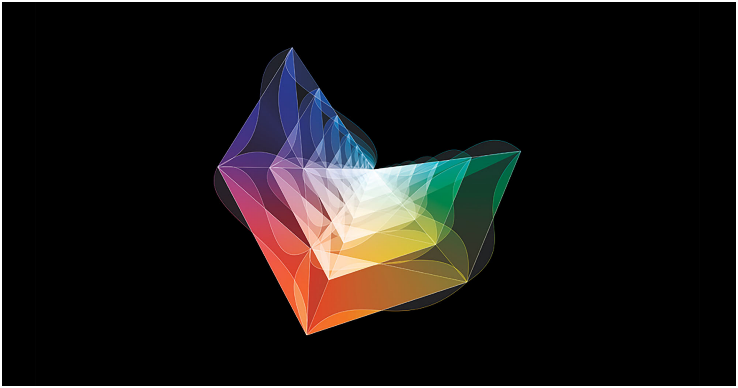
Fix n, k, m with $k + m \leq n$.
Let $Z \in \text{Mat}_{n,k+m}^{>0}$ be an $n \times (k+m)$ matrix with max'l minors positive.
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to span(CZ).
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Motivation for the amplituhedron ($\mathcal{N} = 4$ SYM):

- the recurrence of Britto–Cachazo–Feng–Witten (2005) expresses scattering amplitudes as sums of rat'l functions of momenta. Indiv terms have "spurious poles" – singularities not present in amplitude.
- Hodges (2009) observed that in some cases, the amplitude is the volume of a polytope, with spurious poles arising from internal boundaries of a triangulation of the polytope. Asked if in general each amplitude is the volume of some geometric object.
- AH–T found the amplituhedron as the answer to this question; BCFW recurrence is interpreted as "triangulation" of $\mathcal{A}_{n,k,4}(Z)$.

What is the amplituhedron?

- A "jewel at the heart of quantum physics" – Wired Magazine.

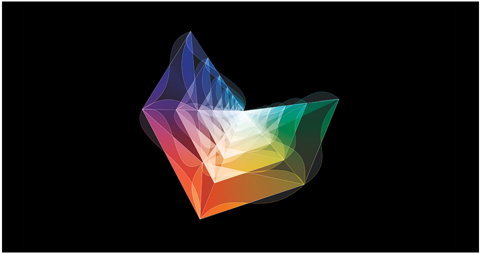


What is the amplituhedron?

- #10 among the 100 top stories of 2013, Discover Magazine.

What is the amplituhedron?

"One of the 25 best inventions of the year 2013," Time Magazine.[1]



"The new method represents probabilities as pyramid-like structures, then combines the pyramids into one elegant gemstone-like structure called an amplituhedron,. . . "

[1] Other best inventions included: the nest protect smoke alarm, a new atomic clock, the driverless (toy) car, and the cronut.

What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$

Fix n, k, m with $k + m \leq n$, let $Z \in \text{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- If $m = n - k$, $\mathcal{A}_{n,k,m}(Z) = Gr_{k,n}^{\geq 0}$.
- If $k = 1$ and $m = 2$, $\mathcal{A}_{n,k,m} \subset Gr_{1,3}$ is equivalent to an n-gon in $\mathbb{RP}^2$:
- For $k = 1$ and general m, n, get cyclic polytope in $\mathbb{RP}^m$.
- For $m = 1$ and general k, n, get bounded complex of cyclic hyperplane arrangement in $\mathbb{R}^k$ (Karp-W.)

We'd like to "triangulate" or "tile" the amplituhedron

Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ a continuous surjective map onto km-dim'l amplituhedron $\mathcal{A}_{n,k,m}(Z)$.

A *tiling* of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{\overline{\tilde{Z}(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of km-dimensional cells, such that:

- $\tilde{Z}$ is injective on each S_π for $\pi \in \mathcal{C}$ ($\overline{\tilde{Z}(S_\pi)}$ a *tile*)
- their union equals $\mathcal{A}_{n,k,m}(Z)$
- their interiors are pairwise disjoint

We will work with all-Z tilings, coming from collections of cells that give tilings for all Z.

Motivation:
the "volume" of the amplituhedron computes scattering amplitudes;
AH-T conjectured that certain "BCFW cells" give a tiling of $\mathcal{A}_{n,k,4}(Z)$;
(proved for the "standard" BCFW tiling by EvenZohar–Lakrec–Tessler and generalized to all BCFW tilings by EZ–L–P–SB–T–W.)

Tilings of the amplituhedron

Tilings have been studied in special cases. Their cardinalities are interesting!

special case	cardinality of tiling of $\mathcal{A}_{n,k,m}$	explanation
$m = 0$ or $k = 0$	1	$\mathcal{A}$ is a point
$k + m = n$	1	$\mathcal{A} \cong \mathrm{Gr}_{k,n}^{\geq 0}$
$m = 1$	$\binom{n-1}{k}$	Karp-W.
$m = 2$	$\binom{n-2}{k}$	AH-T-T, Bao-He, P-SB-W
$m = 4$	$\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$	AH-T, EZ–L–T, EZ–L–P–SB–T–W
$k = 1$, m even	$\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$	$\mathcal{A} \cong$ cyclic polytope $C(n,m)$

Tilings of the amplituhedron

Observation (Karp-Zhang-W, 2017)

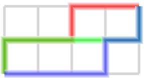
$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

All known tilings of $\mathcal{A}_{n,k,m}$ for even m have cardinality $M(k, n-k-m, \frac{m}{2})$. Call this prediction the *Magic Number Conjecture*.

Remark: Consistent with results for $m=2, m=4, k=1$. Symmetries!

The number $M(a,b,c)$ counts: (In figure, $a,b,c = 2,4,3$.)

noncrossing lattice paths



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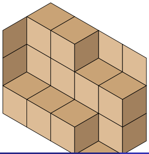

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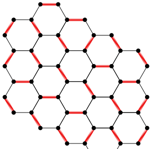


The magic number theorem for the $m = 2$ amplituhedron

Magic Number Theorem (P–SB–T–W)

All tilings of ampl. $\mathcal{A}_{n,k,2}(Z)$ have size $M(k, n-k-2, 1) = \binom{n-2}{k}$.

$k = 1$: Thm says that all triangulations of an n-gon have size $n-2$.

Ideas of the proof:

- The tiles Z_τ of $\mathcal{A}_{n,k,2}(Z)$ are in bijection with *bicolored subdivisions* τ of an n-gon with area k (P–SB–W).
- Just as each Parke-Taylor polytope has a decomposition into w-simplices where w ranges over circular extensions of C_τ, each tile has a decomposition into "w-chambers" where w ranges over circular extensions of C_τ.
- Use above decompositions to define a weight function on $\mathcal{A}_{n,k,2}(Z)$ and each tile, and show that this function is the same for ALL tiles.
- Therefore each tiling of $\mathcal{A}_{n,k,2}(Z)$ has the same size.

In more detail ...

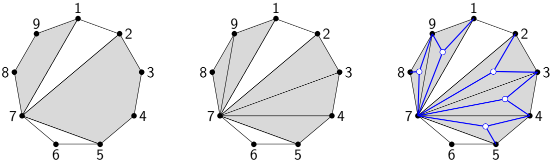
Tiles of the amplituhedron

Recall: $\overline{\tilde{Z}(S_\pi)}$ is a *tile* for $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ if $\tilde{Z}$ is injective on km-dim'l cell S_π. Lukowski–Parisi–Spradlin–Volovich conjectured:

Theorem (Parisi–Sherman-Bennett–W)

The tiles for $\mathcal{A}_{n,k,2}(Z) \leftrightarrow$ bicolored subdivisions of an n-gon with area k. To construct the cell S_π:

- Choose triangulation of black polygons into k black triangles.
- Put white vertex in every black triangle, connected to three vertices.
- Elements of S_π are the $k \times n$ *Kasteleyn matrices* with rows/columns indexed by the white and black vertices.



1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	*	*	*
*	0	0	0	0	0	*	0	*
0	*	*	0	0	0	*	0	0
0	0	*	*	0	0	*	0	0
0	0	0	*	*	0	*	0	0

Chambers of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$

Let $Z \in \mathrm{Mat}_{n,k+2}^{>0}$. Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+2}$ sending $C \mapsto CZ$.
Recall $\mathcal{A}_{n,k,2}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+2}$.

- Let $Z_1, \dots, Z_n$ be rows of Z. Let $Y \in Gr_{k,k+2}$ (viewed as matrix).
- Given $I = \{i_1 < i_2\} \subset [n]$, define the *twistor coordinate*

$$\langle YZ_I \rangle = \langle YZ_{i_1} Z_{i_2} \rangle := \det \begin{bmatrix} - & Y & - \\ - & Z_{i_1} & - \\ - & Z_{i_2} & - \end{bmatrix}$$

- Inspired by matroid stratification, we define the *amplituhedron sign stratification* – decompose $\mathcal{A}_{n,k,2}(Z)$ into pieces based on the signs of twistor coordinates. (Parisi–Sherman-Bennett–W.; Karp-W.)
- Call the top-dimensional pieces *chambers*.
- Thm: (P-SB-W) The nonempty chambers of $\mathcal{A}_{n,k,2}$ are naturally indexed by circular permutations on $[n]$ with k cyclic descents; call them *w-chambers* $\Delta_{(w)}^{Z}$.

The Magic Number Theorem for $\mathcal{A}_{n,k,2}(Z)$

- We define the *weight* $\Omega(\Delta^Z_{(w)})$ of any w-chamber to be $\Omega(\Delta^Z_{(w)}) := \mathrm{PT}(w)$.
- Given any region R which is a union of w-chambers, we define its *weight* as

$$\Omega(R) := \sum \Omega(\Delta^Z_{(w)}) = \sum \mathrm{PT}((w)),$$

where the sum is over all w-chambers $\Delta^Z_{(w)} \subset R$.

- Then for any tile Z_τ of $\mathcal{A}_{n,k,2}(Z)$,

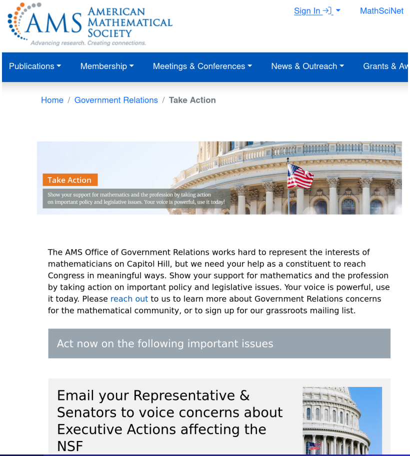
$$\Omega(Z_\tau) = \sum_{w \in \mathrm{Ext}(C_\tau)} \mathrm{PT}(w) = (-1)^k\,\mathrm{PT}(\mathbf{I}_n).$$

The point is: Ω is constant on tiles of $\mathcal{A}_{n,k,2}(Z)$.

- It is known that there is a tiling of $\mathcal{A}_{n,k,2}(Z)$ consisting of $\binom{n-2}{k}$ tiles, so $\Omega(\mathcal{A}_{n,k,2}(Z)) = (-1)^k \binom{n-2}{k}\,\mathrm{PT}(\mathbf{I}_n)$.
- It follows that all tilings have cardinality $\binom{n-2}{k}$.

An aside about the National Science Foundation

- The NSF is facing a potential budget cut of 66%.
- This has already had a major impact on REU's, graduate fellowships, postdoctoral fellowships, conferences, etc.
- Please call your senators and representatives!



Thank you!

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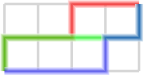

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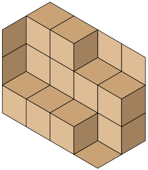

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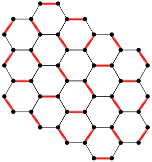

- The magic number conjecture for the $m = 2$ amplituhedron and Parke-Taylor identities `arXiv:2404.03026`, joint with Matteo Parisi, Melissa Sherman-Bennett, and Ran Tessler.

- "The $m = 2$ amplituhedron and the hypersimplex: signs, clusters, triangulations, Eulerian numbers, Communications of the AMS, 2023, joint with Matteo Parisi and Melissa Sherman-Bennett.