Transversal numbers of polytopes, spheres, and pure simplicial complexes

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joint work with

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Hypergraphs and transversals

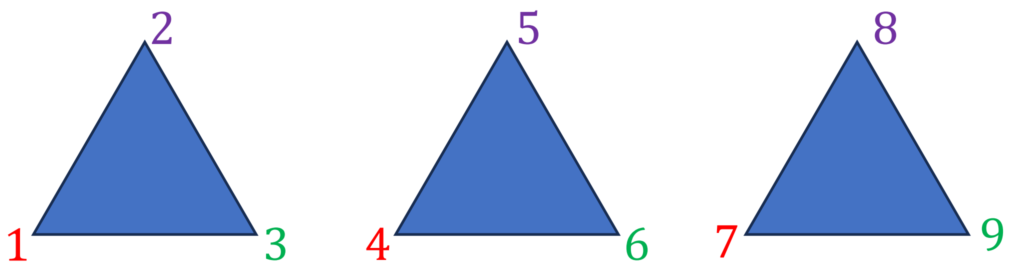
We start with some definitions

- A hypergraph H is a pair (V, E) where V is the vertex set and $E \subset 2^V$
- H is d-uniform if every element of E has size d

- A transversal of H is a subset of V that intersects every element of E
- The transversal number $T(H)$ of H is the minimum size of a transversal of H
- The transversal ratio $\tau(H)$ of H is $\frac{T(H)}{|V|}$

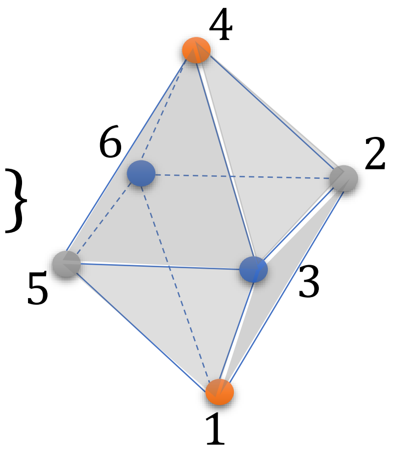
Examples

- $H = (V, E)$, where $V = [9], E = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}\}$



$T(H) = 3$

- $V = [6], E = \{123, 126, 156, 135, 234, 246, 456, 345\}$



$T(H) = 2$

$\boldsymbol{d}$-Uniform hypergraphs

$T(H)$ is related to other combinatorial invariants of H, e.g. (strong and weak) independence number, (strong and weak) chromatic number

Turán's problem (1961): Determine $f(n, m, d) \coloneqq \max T(H)$, where $H = (V, E)$ ranges over all d-uniform hypergraphs with $|V| = n$, $|E| = m$, and $d \geq 2$

A lot of work has been done when m is at most linear in n:

Alon, 1990: $T(H) \leq \frac{\ln d}{d}(n + m)$

Chvátal-McDiarmid, 1992: $T(H) \leq \ (n + \left\lfloor \frac{d}{2} \right\rfloor m)/\lfloor \frac{3d}{2} \rfloor$

We are interested in d-uniform hypergraphs with $m \gg n$ and in hypergraphs coming from topology/geometry

Pure simplicial complexes

- A simplicial complex Δ on a finite vertex set $V(\Delta) = [n]$ is a collection of subsets of $[n]$ that is closed under inclusion: $F \in \Delta, G \subset F \Rightarrow G \in \Delta$
- Elements F of Δ are called faces; $\dim F = |F| - 1$ (so a vertex has dimension 0); maximal under inclusion faces are facets
- $\dim \Delta = \max\{\dim F : F \in \Delta\}$
- Δ is pure if all facets of Δ have the same dimension as Δ

Note: $\{d$-uniform hypergraphs$\} \Leftrightarrow \{$pure $(d-1)$-dimensional complexes$\}$

We define $T(\Delta) = T(\text{hypergraph of facets of } \Delta)$, and similarly for $\tau(\Delta)$

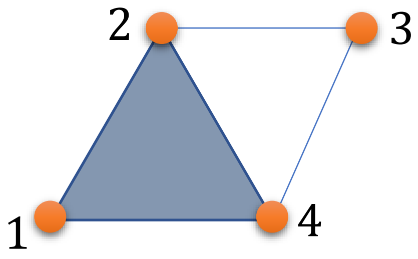
Geometric realizations

Simplicial complex Δ $\rightarrow$ Topological space $||\Delta||$ = geometric realization of Δ

$V = \{1,2,\ldots,n\} \rightarrow e_1, e_2, \ldots, e_n \in \mathbb{R}^n$,

$F = \{i_1,\ldots,i_k\} \in \Delta \rightarrow T_F \coloneqq \mathrm{conv}\{\mathrm{e}_{i_1},\ldots,e_{i_k}\}$

$||\Delta|| \coloneqq \cup_{F\in\Delta} T_F$ (T_F is a geometric simplex)

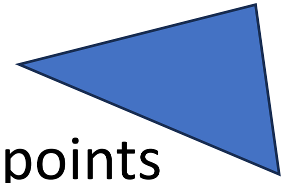


Δ is called a simplicial sphere if $||\Delta||$ is homeomorphic to a sphere

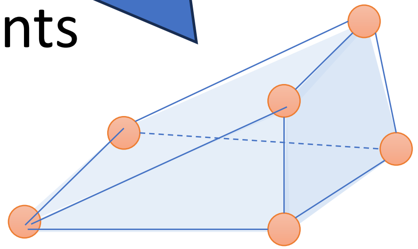
Simplicial spheres are pure simplicial complexes

Polytopes

A polytope P is the convex hull of finitely many points in $\mathbb{R}^d$.



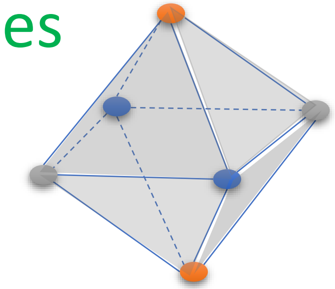
Example: a simplex is the convex hull of affinely independent points



The dimension of a polytope P is the dimension of its affine hull

A (proper) face of P is the intersection of P with a supporting hyperplane

A face F of P is itself a polytope. Any polytope P has finitely many faces
A $(d-1)$-face of a d-polytope is called a facet; 0-faces are vertices



A polytope P is simplicial if all proper faces of P are simplices

Simplicial polytopes vs simplicial spheres

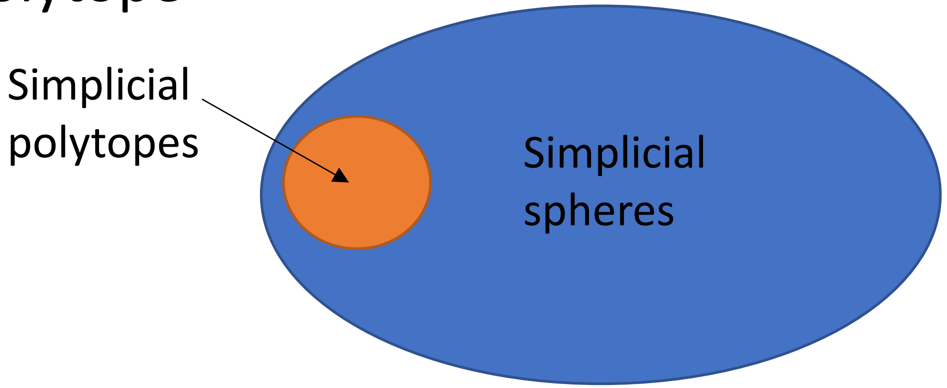
Definition Let P be a *simplicial* polytope. The *boundary complex* of P is

$$\partial P := \{\text{vertex sets of proper faces of } P\}$$

- The boundary complex of any simplicial d-polytope is a simplicial $(d-1)$-sphere
- Every simplicial 2-sphere is the boundary of a 3-polytope

However

Theorem (Goodman-Pollack, 1986; Kalai, 1988; Pfeifle-Ziegler, 2004) For $d \geq 4$, most of simplicial $(d-1)$-spheres are not realizable as the boundary complex of a simplicial d-polytope



Main questions for this talk

What can we say about the transversal numbers and transversal ratios of

simplicial polytopes $\subset$ simplicial spheres $\subset$ pure complexes

Define:

$$\tau_d^P(\mathrm{n}) = \max\{\tau(\Delta) : \Delta = \partial P,\, P \text{ is a simplicial } d\text{-polytope with } n \text{ vertices}\}$$

$$\tau_d^S(\mathrm{n}) = \max\{\tau(\Delta) : \Delta \text{ is a simplicial } (d-1)\text{-sphere with } n \text{ vertices}\}$$

$$\tau_d^P = \limsup_{n\to\infty} \tau_d^P(\mathrm{n}), \text{ and } \tau_d^S = \limsup_{n\to\infty} \tau_d^S(\mathrm{n})$$

How do τ_d^P and τ_d^S behave? [This problem was raised by Alon, Kalai, Matoušek, Meshulam, and also by Briggs, Dobbins, Lee]

Small dimensions

- $d = 2$: $T(n\text{-gon}) = \lceil \frac{n}{2} \rceil$, so $\tau_2^P = \tau_2^S = \frac{1}{2}$
- $d = 3$: The graph of a 3-polytope is planar, hence 4-colorable. The union of any two color-sets is then a transversal. Thus $\tau_3^P = \tau_3^S \leq \frac{1}{2}$

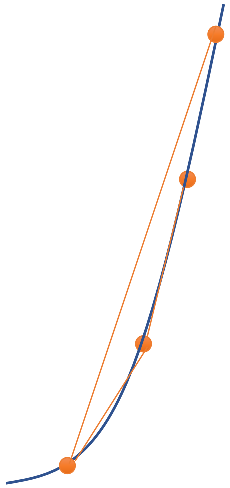
Theorem (Briggs-Dobbins-Lee): $\tau_3^P = \tau_3^S = \frac{1}{2}$

Proof: construct simplicial 3-polytopes with all four color-sets of the same size (one such example is the regular icosahedron)

Cyclic polytopes

Moment curve: $M = M_d : \mathbb{R} \to \mathbb{R}^d$
$t \mapsto (\mathrm{t}, \mathrm{t}^2, \mathrm{t}^3, \dots, t^d)$

Let $n > d$. The cyclic polytope, $C(d, n)$, is defined as
$\mathrm{conv}(M(1), M(2), \dots, M(n))$



Theorem (Gale evenness condition)

- $C_d(n)$ is a d-dimensional simplicial polytope on n vertices
- A d-subset F of $[n]$ forms a facet of $\partial C(d, n)$ if and only if any two elements of $[n] \setminus F$ are separated by an even number of elements from F
- In particular, $\partial C_d(n)$ is $\lfloor \frac{d}{2} \rfloor$-neighborly: every set of $\leq \lfloor \frac{d}{2} \rfloor$ vertices forms a face

Transversal numbers of cyclic polytopes

The Gale evenness condition implies that **most** of facets of $C(2k,n)$ are of the form $\{i_1, i_1+1, i_2, i_2+1, \dots, i_k, i_k+1\} \subset [n]$

While **all** facets of $C(2k+1,n)$ are of the form

$$\{1, i_1, i_1+1, \dots, i_k, i_k+1\} \subset [n] \text{ or } \{i_1, i_1+1, \dots, i_k, i_k+1, n\} \subset [n]$$

Theorem (Briggs-Dobbins-Lee, 2023)

- $T\big(C(2k,n)\big) = \left\lceil \frac{n-2k}{2} \right\rceil + 1$ but $T\big(C(2k+1,n)\big) = 2$. In particular, $\tau_{2k}^{P} \geq 1/2$ for all $k \geq 2$
- There exists a non-polytopal 3-sphere Δ with 21 vertices and $T(\Delta) = 11$. In fact, $\tau_4^S \geq 11/21$

Siblings of cyclic polytopes

Recall: most of facets of $C(2k,n)$ are of the form

$$\{i_1, i_1+1, i_2, i_2+1, \dots, i_k, i_k+1\} \subset [n]$$

In fact, the subcomplex of $\partial C(2k,n)$ generated by these facets is a simplicial $(2k-1)$-**ball**, $B(2k,n)$, and the boundary of this ball is $\partial C(2k-1,n)$

New definition: Let $k \geq 2$. Let $\Gamma(2k+1,n)$ and $\Gamma(2k+2,n)$, be complexes whose facets are all sets of the form $\{i_1, i_1+1, i_2, i_2+1, \dots, i_k, i_k+1, i_k+2\} \subset [n]$, and $\{i_1, i_1+1, i_2, i_2+1, \dots, i_k, i_k+1, i_k+2, i_k+3\} \subset [n]$, respectively

Theorem (N-Zheng, 2024+)

$\Gamma(2k+1,n)$ and $\Gamma(2k+2,n)$ are simplicial **balls**; their boundary complexes are **polytopal** spheres

Transversal numbers of odd-dimensional polytopes

We denote the corresponding polytopes by $D(2k, n)$ and $D(2k+1, n)$ and call them “siblings of cyclic polytopes”

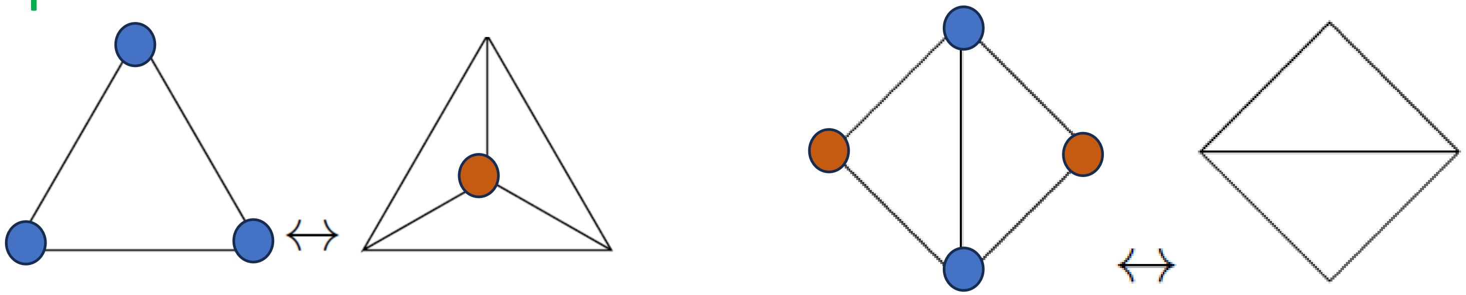
Theorem (N-Zheng, 2024+):

- $D(2k, n)$ and $D(2k+1, n)$ are k-neighborly polytopes of dimension $2k$ and $2k+1$, respectively
- $T\big(D(2k, n)\big) = \frac{n}{2} - O(1)$ and $T\big(D(2k+1, n)\big) = \frac{2n}{5} - O(1)$

In particular, $\tau^P_{2k+1} \geq 2/5$ for all $k \geq 2$

Bistellar flips

- A bistellar flip on a pure simplicial complex Δ of dimension $d-1$ replaces an induced subcomplex of Δ of the form $\bar{A} * \partial\bar{B}$ with $\partial\bar{A} * \bar{B}$. Here $|A| + |B| = d+1$, and $\bar{A}$ and $\bar{B}$ are simplices on A and B.
- Example: $d = 3$



Theorem (Pachner, 1991)

Every two PL $(d-1)$-spheres can be connected by a sequence of flips

Transversal numbers of small-dimensional spheres

So far, we saw that for $k \geq 2$,

$\tau^S_{2k} \geq \tau^P_{2k} \geq 1/2$ and $\tau^S_{2k+1} \geq \tau^P_{2k+1} \geq 2/5$, while $\tau^S_4 \geq 11/21$

By applying bistellar flips to cyclic polytopes and their siblings, we obtain

Theorem (N-Zheng, 2024+) $\tau^S_4 \geq 5/8, \tau^S_5 \geq 1/2, \tau^S_6 \geq 6/11$

Proof idea for $d = 4$: apply flips to $\partial C(4,8m)$ in a way that (1) many of the sets $\{i, i+1, j, j+1\}$ remain facets, and (2) all sets of the form $\{1,3,5,7\} + 8i$, $\{2,4,6,8\} + 8i$ $(0 \leq i \leq m-1)$ become new facets

Back to pure complexes

- Recall that Turán's problem asks to investigate transversal numbers of pure complexes of dimension $d-1$ with n vertices and m facets
- Also, observe that $C(d,n)$ has $\binom{n-\lfloor\frac{d+1}{2}\rfloor}{n-d}+\binom{n-\lfloor\frac{d+2}{2}\rfloor}{n-d}\approx n^{\lfloor\frac{d}{2}\rfloor}$ facets

That's why we are interested in the case of $m \gg n$, and, in particular, $m \approx n^{d/2}$

Proposition (N-Zheng)

Let Δ be a pure complex of dimension $d-1$ with n vertices and m facets. Then for n sufficiently large, $T(\Delta) \leq n+1-\frac{1}{e}nm^{-1/d}$

Corollary: If Δ is a $(2k-1)$-dimensional Eulerian complex with n vertices, then $T(\Delta) \leq n - c_k\sqrt{n}$, where c_k is a constant independent of n

Transversal numbers of pure complexes

Notation: $f(n) = \Theta(g(n))$ means that there exist constants $C_1, C_2 > 0$ s.t. $C_1 g(n) \le f(n) \le C_2 g(n)$ for all $n \gg 0$

Theorem (N-Zheng)

Let $d \ge 2$. For all $n \gg 0$, there exists a pure simplicial complex $\Delta(d,n)$ of dimension $d-1$ with n vertices and $\Theta(n^{(d+1)/2})$ facets whose transversal number is $n - \Theta(\sqrt{n})$. In particular, $\lim_{n\to\infty} \tau\big(\Delta(d,n)\big) = 1$

(There are similar bounds for complexes with $\Theta(n^{d/2})$ facets)

Proof: For $d = 2k$, let $\Delta(d,n)$ be the complex whose facets are

$$\{\{i_1, i_1+\ell, i_2, i_2+\ell, \dots, i_k, i_k+\ell\} \subset [n] : 1 \le \ell \le \sqrt{n}\}$$

Summary and open problems

As we saw, for $k \geq 2$, $\tau_{2k}^{S} \geq \tau_{2k}^{P} \geq 1/2$ and $\tau_{2k+1}^{S} \geq \tau_{2k+1}^{P} \geq 2/5$

In addition, $\tau_4^S \geq 5/8,\ \tau_5^S \geq 1/2,\ \tau_6^S \geq 6/11$

The biggest open problem is that we have **no** non-trivial **upper** bounds on τ_d^P and τ_d^S !

On the other hand, there exist pure complexes with parameters $(d, n, n^{\frac{d}{2}})$ and τ approaching 1

Open problems

1. Is $\tau_d^P = \tau_d^S$ for all $d \geq 4$?
2. For a fixed $d \geq 4$, are τ_d^P and τ_d^S bounded away from 1?
3. Is $\lim_{d\to\infty} \tau_d^P = \lim_{d\to\infty} \tau_d^S = 1$?
4. Are the sequences $\{\tau_{2k}^P\}, \{\tau_{2k+1}^P\}, \{\tau_{2k}^S\}, \{\tau_{2k+1}^S\}$ weakly increasing?
5. What can we say about transversal ratios of special classes of spheres such as, for instance, flag spheres?

There are many more remaining mysteries, but let me stop here

THANK YOU!