Regular Cell Complexes in Total Positivity

Patricia Hersh
North Carolina State University

(Paper with same title, in: Inventiones Math., 197 (2014), 57-114.)

(See http://www4.ncsu.edu/~uplhersh for slides, including appendix with more details)
Topological Aspects of Total Positivity

- Lusztig initiated study of Totally nonnegative, real part of (matrix) Schubert varieties, flag varieties, ...
  (i.e. part with minors all nonnegative in spaces of matrices or of flags)
- Conjecturally/provably homeomorphic to closed balls (after deconing)
- Proving this:
  - puts restrictions on relations among (exponentiated) Chevalley generators,
  - reveals structure in canonical bases; a motivation for cluster algebras.
- **Main Result of Talk:** Proof of Fomin-Shapiro Conjecture via new tools exploiting interplay of combinatorial data & topological data.
Deducing Topological Structure from Combinatorics + Caching One Topology?

\[ F(K) = e_1 \]

e.g.

\[ K = \text{ball} \]

\[ K' = \text{IRP}^2 \]

\[ F(K') = \text{"closure poset" or "face poset"} \]

\[ (u \leq v \iff u \leq \overline{v}) \]

Notations: A CW complex: cells \( e_a \), characteristic maps \( f_a : B^{\dim(e_a)} \to \bigcup_{\beta \in \overline{e_a}} \beta \)

\( \uparrow \) attaching maps \( f_a |_{\beta B^{\dim(e_a)}} \)
Recall: a poset is graded if \( u \leq v \) in \( P \) implies minimal paths \( u \to v \) all same length.

\[ \text{e.g. } \begin{array}{c}
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast \\
\ast 
\end{array} \]

is not graded

- A graded poset is thin if each rank 2 interval has exactly 4 elements.

Recall: A CW complex is regular if the attaching map for each cell is a homeomorphism, i.e. cell closures are closed balls.

- \( K \) regular \( \Rightarrow K \simeq \Delta (F(K) - \sigma_0) = \text{sd}K \)
  (setting where combinatorial determines top.)
**Defn (Björner):** A finite, graded poset $P$ is **CW poset** if

- $P$ has unique min’l elt. $\hat{0}$
- $P$ has additional element(s)
- $x \neq \hat{0}$ \implies $\Delta(\hat{0}, x) \cong S^{\text{rank}(x)-2}$

**Thm (Björner):** $P$ is CW poset if and only if there exists a regular CW complex $K$ with $P = F(K)$.

A **Goal of Mine:** Use combinatorics of $F(K)$ + manageable topological info (radim. one cell incidences) to understand $K$. 
Some Examples of CW Posets

- Shellable & thin (Danaraj-Klee)
- Bruhat order (Björner & Wachs)
- Closure poset for double Bruhat decomp. of totally nonneg. part of flag variety (Williams)
- Closure poset of triangulation of double suspension of homology sphere with “big cell” glued in (due to work of J. Cannon & R. Edwards) (hence the focus of CW posets on intervals $(\delta, u)$)
The Bruhat order is a partial order on a Coxeter group $W$ with $u \leq v \iff$ there exists reduced expressions (i.e., products of minimal number of adjacent transpositions) $r(u)$ and $r(v)$ with $r(u)$ subexpression of $r(v)$.

E.g., $W = S_3$ with generators $s_1 = (1,2)$ and $s_2 = (2,3)$.

- Closure poset for Schubert cell decompositions of flag varieties $G/P$
- Reduced word $(i_1, \ldots, i_d)$ for $s_1 s_2 \cdots s_d$
**Question (Bernstein):** Find regular CW complexes naturally arising from rep'n theory which are homeomorphic to closed balls and have the (lower) Bruhat intervals as closure posets.

**Conjectural Solution (Fomin & Shapiro):** The Bruhat stratification of $\mathbf{Lk}(\text{id})$ in totally nonnegative, real part of unipotent radical in semisimple, simply connected algebraic group defined and split over $\mathbb{R}$.

**Thm (Fomin-Shapiro):** This has Bruhat order as closure poset. Has desired homological properties.
**Theorem (H.):** Fomin-Shapiro

Conjecture indeed holds.

**Special Case (Running Example for Talk):** Space of totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed, positive constant, stratified by which minors are positive and which are 0.

**Concrete Realization:** products of certain elementary matrices, by results of Whitney & Lusztig.
The Totally Nonnegative Part of a Space of Matrices

- $x_i(t) = \exp(t e_i)$ (type A)
  - $\exp(t e_i)$ (general finite type)

- $f(i, \ldots, i_d) : \mathbb{R}^d_{\geq 0} \rightarrow M_{n \times n} \subseteq \mathbb{R}^{n^2}$

  $(t, \ldots, t_d) \rightarrow x_{i_1}(t_1) \ldots x_{i_d}(t_d)$

  e.g. $f(1,2,1)(t_1, t_2, t_3) = x_1(t_1) x_2(t_2) x_1(t_3)$
  
  $= \begin{pmatrix} 1 & t_1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

  
  $= \begin{pmatrix} 1 & t_1 + t_3 & t_1 t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$
"Picture" of Map $f_{(i_1, i_2, i_0)}$

$\mathbb{R}_{\geq 0}^3 \cap (\Sigma t_i = 1 \text{ hyperplane})$

$f_{(i_1, i_2, i_3)}(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \end{pmatrix} \begin{pmatrix} 1 & t_3 \\ 1 & 1 \end{pmatrix}$

$t_2 = 0$

$f_{(i_1, i_2, i_3)}(t_1, 0, t_3) = \begin{pmatrix} 1 & t_1 \\ 1 & t_3 \end{pmatrix} = \chi_i(t_1 + t_3)

Non-injectivity: results from "modified nil-moves" $\chi_i(u) \chi_i(v) \Rightarrow \chi_i(u + v)$ directly.

& after "long braid moves" in Hecke algebra.
1st Ingredient to Fomin–Shapiro Conj:

O-Hecke Algebra Captures which Simplex Faces have Same Image under $f_{(i_{1}, \cdots, i_{d})}$

(1) $\chi_{i_{1}}(t_{1})\chi_{i_{2}}(t_{2}) = \chi_{i_{1}}(t_{1} + t_{2})$

"nil-move"

\[ \chi_{i_{1}}^{2} = \chi_{i_{1}} \] (O-Hecke alg. reln, up to sign)

(2) $\chi_{i_{1}}(t_{1})\chi_{i_{2}}(t_{2})\chi_{i_{3}}(t_{3}) = \chi_{i_{1}}(t_{1})\left(\frac{t_{2}t_{3}}{t_{1}t_{3}}\right)\chi_{i_{2}}(t_{1} + t_{3})\chi_{i_{3}}(t_{1}t_{3})$

(type A) assuming $t_{1} + t_{3} > 0$

$\chi_{i_{1}}\chi_{i_{2}}, \chi_{i_{2}} = \chi_{i_{1}}\chi_{i_{2}} \chi_{i_{1}}$

(similar relation holds outside type A)

"long braid move" with enrichment from parameters

Fibers as Curves:

(1): \[ \begin{array}{c}
\text{triangle} \\
\end{array} \]

(2): then (1): \[ \begin{array}{c}
\text{curved triangle} \\
\end{array} \]
Indexing Faces of Preimage by Words in O-Hecke Algebra

Key Observation About $f_{(i_1,...,i_d)}$:
\[
\text{im}(F_i) = \text{im}(F_j) \iff x(F_i) = x(F_j)
\]
equal as O-Hecke algebra elements

**Thm (Lusztig):** If $(i_1,...,i_d)$ is reduced, then $f_{(i_1,...,i_d)}$ is homeomorphism on $\mathbb{R}_d^+$

**Upshot:** $f_{(i_1,...,i_d)}$ restricts to homeomorphism on each face given by reduced subword.
Faces indexed by non-reduced subwords (or some reduced ones) are redundant.
Properties of Change-of-Coordinates Map Given by Braid Moves

e.g. \((t_1, t_2, t_3) \mapsto \left(\frac{t_2 t_3}{t_1 + t_3}, t_i + t_3, \frac{t_i t_2}{t_1 + t_3}\right)\)
in type A

- Tropicalizes to change-of-basis map for Lusztig's canonical bases:
  \((a, b, c) \mapsto (b + c - \min(a, c), \min(a, c), a + b - \min(a, c))\)

- A motivation for development of cluster algebras (and mutation)

Exercise: check this is an involution.
Proof Strategy (for FS-Conjecture & for images of "nice" maps from polytopes)

Set-up: Continuous, surjective fn $f: P \rightarrow Y$
from convex polytope $P$ (eg. $\Delta_1$) s.t. $f$ maps $\text{int}(P)$ homeomorphically to $\text{int}(Y)$.

Step 1: Perform "collapses" on $2P$, each preserving regularity and homeomorphism type - via continuous, surjective collapsing functions $P \rightarrow P$ yielding $P/\sim$ with fewer cells s.t. $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$

Step 2: Prove $f: P/\sim \rightarrow Y$ is homeomorphism by new regularity criterion
Collapsing cell $\sigma$ onto cell $p \in \partial \sigma$ within $\partial \Sigma$

**Thm**: (M. Brown; Connell): Any topological manifold with boundary $\partial M$ has a collar (i.e. a nbhd homeomorphic to $\partial M \times [0,1]$).

**Fact**: Our collapses will preserve this (hence existence of collar) for: $\partial \Sigma \setminus \sigma = \emptyset$

Plan: Collapse $\sigma$ onto $p \in \partial \sigma$, stretching collar for $\partial \Sigma \setminus \sigma$ to cover $\partial \sigma \cdot p$.
2nd Ingredient: New Regularity Criterion

Preparatory Lemma (H.): Let \( K \) be a finite CW complex w/ characteristic maps \( \{ f_\alpha \} \). Suppose:

1. \( \forall \alpha, f_\alpha(2B^{\text{dim } \alpha}) \) is a union of open cells (surjectivity)

Non-Example:

2. \( \forall f_\alpha \), the preimages of the open cells of codim. one in \( \overline{f}_\alpha \) are dense in \( 2(B^{\text{dim } \alpha}) \)

Non-Example:

Then \( F(K) \) is graded by cell dimension.

Insightful feedback: Next theorem “spreads around” injectivity requirement.
Thm (H.) Let $K$ be finite CW complex w.r.t. characteristic maps $\exists f_\alpha \exists$. Then $K$ is regular w.r.t. $\exists f_\alpha \exists$ $\iff$

1. $K$ meets requirements of prop 1 for $F(K)$ to be graded by cell $\dim$.  
2. $F(K)$ is thin and each open interval $(u,v)$ for $\dim(v) - \dim(u) \geq 2$ is connected (as graph)

(Combinatorial condition)

Non-Example

$\Delta = \{p_1, \ldots, p_n\}$

$\Delta - \{p_i\}$

$\Delta - \{p_i, p_j\}$

$\Delta - \{p_i, p_j, p_k\}$

$(p, \tau) = \ldots$
(3) For each $\alpha$, the restriction of $f_\alpha$ to preimages of codim. one cells in $\bar{e}_\alpha$ is injective. (topological condition)

**Non-Example:**

(4) $\forall e_c \subseteq \bar{e}_\alpha$, $f_\alpha$ factors as continuous inclusion $i: B^{\dim \sigma} \to B^{\dim \alpha}$ followed by $f_\alpha$.

**Non-Example:**

**Notably Absent:** Injectivity requirement for $\{f_\alpha\}$ beyond codim. one.

**Proof:** Induction on difference in dim.
**3rd Ingredient:** Injectivity of Attaching Maps in Codimension One via Coxeter group exchange axiom

\[ S_1S_2S_1S_3 \]

\[ S_1S_2S_1S_3, \quad S_1S_2S_3, \quad -S_2S_1S_3, \quad S_1S_3S_3 \]

\[ 3214 \neq 2341 \neq 3142 \]

reduced subexpressions of reduced expression obtained by deleting one letter give distinct Coxeter group elements.

**In contrast:** fails in higher codimension.

\[ S_1S_2S_1S_3 \]

\[ S_1... \quad --S_1-- \]
4th Ingredient: (Mainly Combinatorial) Requirements Enabling Collapses Across Curves

There is a series of earlier face collapses

\[ \Delta \xrightarrow{\text{as } g_i \text{ is surjection onto } \Delta} \]

\[ K_0 \xrightarrow{g_1} K_1 \xrightarrow{g_2} K_2 \rightarrow \cdots \rightarrow K_i \]

(\(\Delta/\mathcal{R}_i\), (new cell structure))

polytope!

with closed cell of \(K_i\) covered by images of parallel line segments in \(K_0\) with family \(G_i\) of "parallel-like" curves satisfying:

- Distinct endpoints condition (DE):

  ![Distinct endpoints condition](image)

- Distinct initial points condition (DIP):

  ![Distinct initial points condition](image)

- "Least upper bound condition" (LU\(\bar{B}\))...
LUB: Condition to ensure Regularity is Preserved
(suggested by David Speyer)

If $A \neq B$ are 1Ded via face collapse of $F$, then all least upper bounds for $A \neq B$ just prior to collapse of $F$ must also be collapsed in this step.

E.g. Want to prevent:

![Diagram](image)

Note: conditions on which cells 1Ded yet; checkable with combinatorics of reduced/non-reduced words in $O$-Hecke algebra.
Collapsing "non-reduced" Face Across Curves

\[ f(1, 2, 3, 1, 2) \] face with \( t_3 = 0 \)

\[ x_1 x_2 - x_1 \quad (t_5 = 0) \] face

\[ -x_2 - x_1 x_2 \quad (t_1 = 0) \] face

- Collapse across curves
  - Identifying \( t_1 t_2 t_4 \) \& \( t_2 t_4 t_5 \) terms

\[ (t_1, t_2, 0, t_4, t_5) \xrightarrow{\text{only for } t_1 + t_4 > 0} x_1(t_1)x_2(t_2)x_1(t_4)x_2(t_5) \]

\[ x_2(t_1')x_1(t_2')x_2(t_4')x_2(t_5) \]

\[ x_2(t_1')x_1(t_2') x_2(t_4' + t_5) \]

\[ \text{for } t_1' = \frac{t_2 t_4}{t_1 + t_4}, \quad t_2' = t_1 + t_4, \quad t_4' = \frac{t_1 t_2}{t_1 + t_4} \]

Curves within fibers of trivial:

\[ t_1' = k, \quad t_2' = k_2 \quad t_4' + t_5 = k_3 \]
5th Ingredient: Deletion Pairs: How to Transfer Coxeter Group Properties to $Q$-Hecke Algebra

In a non-reduced expression $S_{i_1} \ldots S_{i_d}$, let $\{S_{i_r}, S_{i_t}\}$ be a deletion pair if $S_{i_r} \cdot S_{i_t}$ and $S_{i_r} \cdot S_{i_t}$ are reduced expressions while $S_{i_r} \cdot S_{i_t}$ is nonreduced.

**Key Coxeter Group Property:** Any two reduced expressions for same Weyl connected by series of braid moves ensures nonreduced expressions admit modified n!-moves.

**E.g.**

\[ x_1 x_2 x_1 \quad \Rightarrow \quad x_2 x_3 x_2 \]
\[ \underline{x_2 x_1} \quad \underline{x_2 x_3 x_2} \]
\[ x_2 x_1 \quad x_2 x_3 x_2 \]
\[ \underline{x_2 x_1} \quad \underline{x_2 x_3 x_2} \]
6th Ingredient: Cell Collapsing
Order (*Embedding) Enabling Proof
by Induction on Word Length

\[ (i_1, i_2, i_3, \ldots, i_p, i_{p+1}, \ldots, i_d) \]
\[ \text{im}(f(i_{r-1} - i_{r-2})) \]
all possible collapses done
not in any collapses yet

"The Fine Print":

**Collapsing Order:** greedily choose:
1. leftmost deletion pair, then
2. minimize \(t-r\), then
3. maximize cell dimension.

Repeat until all "non-reduced" cells collapsed.
Long Braid Move as Change of Coord’s Homeomorphism on Closed Cell

E.g., \( t_2 = 0 \) \( \Rightarrow \) \( \Delta \frac{1}{n} \) collapse

\( t_2 = 0 \) \( \Rightarrow t_i + t_j = 0 \) \( \Rightarrow \text{ch} \rightarrow \text{ch}(t_2 = 0) \)

Idea: Subwords of \( (i,j,...) \) and \( (j,i,...) \) do not admit any long braid moves. Thus:

\( \Delta \frac{m(i,j)}{s:j} \) \( \Rightarrow \) \( \frac{f(i,j,...)}{\text{homeom.}} \)

\( \frac{f_{(j,i,...)}^{-1}}{s:i} \) \( \Rightarrow \) \( \frac{f(i,j,...)}{\text{homeom.}} \)

\( \Delta \frac{m(i,j)}{s:j} \) \( \Rightarrow \) \( \frac{f(i,j,...)}{\text{homeom.}} \)
Summary: 1. Eliminate all cells indexed by non-reduced subwords of \((i_1, ..., i_d)\) via explicit collapses.

2. Thereby also identify pairs of cells indexed by subwords that are reduced words for some \(u \in \mathbb{W}\).

3. Justify collapses preserve homeomorphism type, regularity via combinatorics of reduced words.

4. Deduce from regularity criterion that resulting quotient space is homeomorphic to \(\text{im}(f_{i_1, ..., i_d})\) once all non-reduced cells eliminated.

Thus, Fomin-Shapiro Conjecture holds.
"Flow" on a Fiber From Collapsing Process to Base Point of Fiber

**Example:**

\[ x_1 \rightarrow x_1 \rightarrow x_2 \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_3 \]

\[ x_1(t_1) \rightarrow x_1(t_3) \rightarrow x_2(t_4) \rightarrow x_1(t_5) \rightarrow x_2(t_7) \rightarrow x_3(t_8) \rightarrow x_3(t_{10}) \]

\[ x_1(t_1 + t_3) \rightarrow x_3(t_5 + t_{10}) \]

\[ x_2(t'_1) \rightarrow x_1(t'_2) \rightarrow x_2(t'_3 + t_7) \]
A Follow-up Project:
(with Jim Davis & Ezra Miller)

Conjecture (Davis - H. - Miller): $f_{(i_1, \ldots, i_d)}'(p)$
for each $p \in Y^o$ is a regular CW complex
homeomorphic to a ball with closure
poset dual to face poset for interior of
"subword complex" $\Delta((i_1, \ldots, i_d), \omega)$.

Remark: Subword complexes previously
arose as Stanley-Reisner complexes of
initial ideals of coordinate rings
associated to matrix Schubert varieties.
A Poset Map (on Face Posets) induced by $f_{c;12...id}$ (implicit Deth'of Subword Complexes)

Boolean Algebra $B_n$ Bruhat Order

- Apply braid moves $\overset{i}{x_i} \rightarrow x_i$ to get reduced expression; replace $x_i$'s by $s_i$'s
- Fibers $f_2^{-1}(u) = \exists x \in B_n | \text{length}(x) = u_3$ are dual to face posets of subword complexes (fibers as in Quillen's Lemma A)
Davis-H-Miller Conjecture implies
Fomin-Shapiro Conjecture
(with Jim Davis & Ezra Miller)

Combining Top'l Results: Let \( g : B \rightarrow Z \) be a continuous surjection from ball \( B \) to Hausdorff space \( Z \) whose restriction to \( \text{int} (B) \) is an embedding. Suppose also:

1. \( g(\partial B) \subseteq \partial B = S^n \)
2. \( g(\partial B) \cap g(\text{int}(B)) = \emptyset \)
3. \( g^{-1}(p) \) is contractible \( \forall p \in g(\partial B) \).

Then \( Z \cong B \).

(Based on Kirby-Siebenmann + local contractibility of \( \text{Homeo}(S^n, S^n) \).)
Existing Proof of Fomin-Shapiro viewed from this Perspective

* factors $f_{(i_1, \ldots, i_d)}$ as composition of simple collapsing maps $g_i$
  (where requisite paths of homeomorphisms are easy to construct explicitly).

* regularity criterion shows
  $$
  f_{(i_1, \ldots, i_d)}(\Delta^n) \Rightarrow f_{(i_1, \ldots, i_d)}(\Delta^n)
  $$
  is indeed a homeomorphism.
Further Questions:

1. Analogous story for totally nonnegative part of: Grassmannian? loop group? flag variety?
   (partial results of Postnikov, Rietsch, Williams, Speyer, Marsh, ...)


Thank you!
Connection to Schubert Varieties & Bruhat Decompositions

- $Y^\circ = \text{image of } f_{(\iota, \iota, \iota)} : \mathbb{R}^d_{>0} \rightarrow M_{n \times n}$

- $Y_\omega = \overline{V}_\omega = \text{image from } \mathbb{R}^d_{>0}$
  
  = totally nonnegative part of $\overline{B^{-\omega} B^{-\omega}} \cap \text{ (unipotent)}$

- $Y_{\omega_0} = \text{totally nonnegative part of space of upper triangular matrices } \omega \text{ with } 1\text{'s}$
  
  (old result of Whitney-type A) on diagonal
A Motivation: Understanding Relations Among (Exponentiated) Chevalley Generators

Lie algebra

$\exp(t e_i) = \begin{pmatrix} 1 & t e_i & \frac{t^2 e_i^2}{2} & \frac{t^3 e_i^3}{6} & \cdots \end{pmatrix}$

We Prove: Only the “obvious” relations occur
Collapsing a Cell $\sigma$ onto a Cell $\bar{\rho}$

Act as ID outside $\triangle$, so also need ID at $\partial(\triangle)$.

Collar to be stretched across $\sigma$ (stretched to cover $s_1 \cup s_2$).

- Map segments $s_2$ in $\sigma$ onto endpoint in $\bar{\rho}$, stretch extension $s_1 \subseteq \text{collar}$ to cover $s_1 \cup s_2$, act as ID on $\bar{\rho} \times [0,1] \subseteq \text{collar}$.

- For $c \in \partial\sigma$, collapsing map on $c \times [0,1]$ will stretch $s_1$ to cover $s_1 \cup s_2$ and shorten $s_2 \cup s_3$ to cover $s_3$, as depicted next.
"Close-up" of bottom part of collapsing map

Key Observations:

(1) This type of collapse makes sense more generally, relying on existence of continuous fn $ln: \overline{\sigma} \to \mathbb{R}$ sending point to "length" of segment in $\overline{\sigma}$ containing it.

(2) These collapses are explicitly approximable by homeomorphisms:

Approximate segment to be stretched by
Verifying DE (Distinct Endpoints Condition) with Combinatorics (Gives Flavor of Many Lemmas)

Suppose collapse of $F$ uses curves starting in $G_1$ and ending in $G_2$

\[ \cdots X_{t_r} \cdots X_{t_k} \cdots X_{t_3} \cdots \]

$G_1 \leadsto G' \leadsto G_2$

If $G_1$ were already identified earlier with $G_2$ then there exists $G'$ with earlier steps identifying $G_1$ with $G'$ and $G'$ with $G_2$. But the former would have also identified $G_1 \cup \exists x_{s_3} F = F$ with the cell $G' \cup \exists x_{s_3} F = F'$ which was already collapsed in step identifying $G'$ with $G_2 \Rightarrow \ldots$
Ingredients in Relationship Between Fibers & Image of "Nice" Map:

- **CE-approx. theorem**: $g: \mathcal{E} \rightarrow \mathcal{E}$ as above may be approximated by homeomorphisms
  - Armentrout: $\dim 3$
  - Quinn: $\dim 4$
  - Kirby-Siebenmann: $\dim \geq 5$ (more generally)

- **Local Contractibility of Homeos $(S^m, S^m)$**: two homeomorphisms "close enough" to each other may be connected by path of homeomorphisms

**Idea**: $B \cong \text{metric ball} = \overline{S^3} \cup (0, 1] \times \partial B$.

Use path of homeomorphisms converging to $g|_{\partial B}$ to construct $f: B \rightarrow B$ with $f^*(p) = g^*(p)$ $\forall p \in B$ and $f|_{\partial B} = g|_{\partial B}$, so $g(B) = B/\sim = f(13) \cong B$. 
Checking Sphericity for $f_{(i_{1}, \ldots, i_{d})}(\partial \Delta^{d})$

1. Stratification has Bruhat intervals as closure posets, thus CW posets.
2. Induction on dimension $\Rightarrow$ cell closures in $f_{(i_{1}, \ldots, i_{d})}(\partial \mathcal{B})$ are balls, so $f_{(i_{1}, \ldots, i_{d})}(\partial \mathcal{B})$ is regular CW complex "K".
3. Hence, $K \cong \Delta(F(K) \setminus \partial) \cong$ sphere.
Subword Complexes (introduced by Knutson & Miller)

\( Q := \text{(not necessarily reduced) expression} \)

\( w := \text{Coxeter group element} \)

Facets of \( \Delta(Q, w) \) are the subwords of \( Q \) whose complements are reduced words for \( w \).

\( \Delta(Q, w) = \begin{cases} 
-1 \quad 12 \\
-2 \\
1 \\
\end{cases} \)

\( Q = (1, 2, 1, 2) \quad w = s_1 s_2 \)

Thm (Knutson - Miller): \( \Delta(Q, w) \) is vertex decomposable (hence shellable) ball or sphere.

More Generally? "Fibers" of Parametrization Maps for Nonneg Flag variety, loop groups, etc.?
Homotopy Type of Bruhat Intervals: New Proof by Quillen Fibre Lemma

Thus (Armstrong- H.): The poset map $f_{(i_1...i_d)}$ yields short proof of: $rk_v-rku-2$

$\Delta_{Bruihut}(u,v) \cong S$ for all $u \leq v$

Idea: fibers $f^1_2(u) = \{x \in B_n | f(x) \geq u^2\}$ are dual to face posets of subword complexes - proven to be balls by Allen Knutson & Ezra Miller.

Subword complexes previously arose as:

Stanley-Reisner complex for Gröbner degeneration of matrix Schubert variety ideal (Knutson and Miller)