Posets Arising as 1-Skeleta of Simple Polytopes, the Nonrevisiting Path Conjecture & Poset Topology

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Linear Programming

- Given a matrix $A$ & vectors $\vec{b}, \vec{c}$ seek $\max \{ \vec{c} \cdot \vec{x} | A \vec{x} \leq \vec{b} \}$

- $\{ \vec{x} | A \vec{x} \leq \vec{b} \}$ is polytope $P$ if set is bounded
Solving Linear Programs via Simplex Method

- Define $G(P, \vec{c})$ := directed graph on 1-skeleton of $P$ with $x_1 \rightarrow x_2 \iff \vec{c} \cdot \vec{x}_1 < \vec{c} \cdot \vec{x}_2$

- $\max \{ \vec{c} \cdot \vec{x} | A\vec{x} \leq \vec{b} \}$ = sink of $G(P, \vec{c})$

Simplex Method: walk from some vertex $v \in G(P, \vec{c})$ following arrows $v \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow s$ to sink $s$

- $\min \{ \vec{c} \cdot \vec{x} | A\vec{x} = \vec{b} \}$ = source of $G(P, \vec{c})$
  (also may walk to source)
An Example: Traveling Salesman Problem

Polytope Vertices:

\((1,0,1,1,0,1), (1,1,0,0,1,1), e_{12}, e_{14}, e_{23}, e_{34}, (0,1,1,1,1,0)\)

Cost Vector:

\(\hat{c} = (2.5, 7, 1.1, 3.4, 8, 6)\)
**Pivot Rule**: method to choose which out arrow $\rightarrow v \Rightarrow s$ to follow from $v$ towards sinks $s$.

**Key Questions**:

1. What is typical complexity of simplex method (path length)?
2. What is worst case? (i.e., diameter of $G(\mathcal{P}, \mathcal{E})$)
Quick Background on Polytopes

• A **polytope** in $\mathbb{R}^d$ is convex hull of finite # vertices, or equivalently a bounded set that is an intersection of half spaces.

• A polytope is **simple** if for each vertex $v$ and each collection $e_1, e_2, \ldots, e_r$ of edges emanating out from $v$ there is an $r$-dim'l face containing all these edges.

  e.g. $v \xrightleftharpoons{e_1} e_2 \leftrightarrow e_3$ but not $v \xrightleftharpoons{e_1} e_2 \leftrightarrow e_3$
**Hirsch Conjecture:** For d-dim'l polytopes with n facets (max'l faces), diameter of 1-skeleton graph, denoted $\Delta(d,n)$, satisfies $\Delta(d,n) \leq n-d$.

**Francisco Santos:** After many decades eluding many people, he constructed counterexamples ("spindles"): polytopes s.t. there exist vertices $v,w$ s.t. each facet includes $v$ or $w$. 
Nonrevisiting path conjecture:

For each \( u,v \) in polytope \( P \), there is path \( u \to v \) not revisiting any facet is has left.

Non-Revis. Path Conj \( \Rightarrow \) Hirsch Conj,

- nonrevisiting path leaves a facet at each step and still belongs to all facets at its conclusion.

Strong Monotone Path Conjecture:

existence of directed path of length \( \leq n-d \) from any vertex to vertex \( v \) maximizing \( z \cdot v \) with cost increasing each step.
Our Plan

Impose further conditions on $P$ and $\bar{c}$ that will imply a corollary of the following which we hope might also hold:

For each $u, v \in P$, each directed path from $u$ to $v$ never revisits any facet it has left.

This property would make all pivot rules efficient for $P$ and $\bar{c}$. 
**New Def'n:** \( G(P, \leq) \) has the Hasse diagram property if it is a Hasse diagram of finite poset, i.e. \( v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_r \) for \( r \geq 3 \) directed path precludes \( v_1 \rightarrow v_r \)

\[ \begin{array}{c}
  v_1 \\
  \uparrow \\
  v_2 \\
  \uparrow \\
  v_3 \\
  \vdots \\
  \uparrow \\
  v_r \\
\end{array} \]

**Note:** precludes d-simplices as faces for \( d \geq 2 \)
Important Non-Examples: "Klee-Minty Cubes"

e.g. $n=3$

$\vec{c} = (0,0,1)$

- path visits all vertices!

Figure modelled after one in Gärtner-Henk-Ziegler paper

$n$-dimensional Klee-Minty cube

$$\Delta = \{ (x_1, x_2, x_3) \mid 0 \leq x_i \leq 1/3 \text{ for } i \geq 1, x_1 < x_2 < 1-3x_3, x_1 < x_2 < 1-3x_3 \}$$
Note: Klee-Minty cubes violate Hasse diagram property in a way that seems to be at the heart of what leads to existence of "long" directed path (visiting all $2^n$ vertices) in it.

Our hope: Hasse diagram property precludes such issues.
Lemma: Given $F \subseteq G$ with $\dim(G) = \dim(F) + 1$ in simple polytope $P$ with generic $\vec{c}$ s.t. $G(P, \vec{c})$ is a Hasse diagram, then there does not exist $v, w \in F$ with directed path $P_F$ from $v$ to $w$ in $F$, outward oriented edge $v$ to $G \setminus F$ and inward oriented edge $G \setminus F$ to $w$. 
**Corollary:** Monotonicity of out-degrees & partial outward directions.

3 violate Hasse diagram (at switch from out to in)

**Corollary:** For each face $F \leq P$ with $\partial E \in F$ or $\hat{1} \in F$, directed paths cannot revisit $F$ after departing from it.
Recall: A poset $L$ is a lattice if for each $u, v \in L$ there exists unique least upper bound ("join") for $u$ and $v$, denoted $u \vee v$, and unique greatest lower bound for $u$ and $v$ ("meet"), $u \wedge v$.

Note: For $P$ simple, $G(P, \mathcal{E})$ Hasse diagram, an upper bound for $u, v$ both covering $x$ is sinks of unique 2-face containing $x, u, v$. 
"Pseudo-joins" in a Polytope

Let $P$ be a simple polytope with generic cost vector $\vec{c}$ such that $G(P, \vec{c})$ is Hasse diagram of poset $L$ with $x_1, x_2, \ldots, x_r \in L$ s.t. there exists $u \in L$ with $u \prec x_i$ for $i = 1, 2, \ldots, r$. Define pseudo-join of $x_1, x_2, \ldots, x_r$ as sink of unique r-face of $P$ containing $x_1, x_2, \ldots, x_r$. 
Lemma: For \( P \) a simple polytope and \( \vec{c} \) a generic cost vector s.t. \( G(P,\vec{c}) \) is Hasse diagram of poset \( L \), let \( S, T \subseteq \{a_1, \ldots, a_n\} \) be distinct sets of atoms. Then \( \text{pseudojoin}(S) \neq \text{pseudojoin}(T) \).

For \( L \) a lattice, this also holds for atoms in each interval \( [u, v] \subseteq L \).
Idea:

1. Reduce to $S \in T$ with $|T| = |S| + 1$

   - $S_1 \not\subset S_2 \not\subset S_3$ with $\mathsf{psj}(S_1) = \mathsf{psj}(S_3) 
   \Rightarrow \mathsf{psj}(S_2) = \mathsf{psj}(S_3)$

   - $S_1 \not\subset S_2 \not\subset S_2 \not\subset S_1$, then use $S_1 \cap S_2 \not\subset S_1$ with $\mathsf{psj}(S_1 \cap S_2) = \mathsf{psj}(S_1)$

2. Use codim one nonrevisiting lemma

3. For $[u, v]$, use that $v$ is an upper bound for the atoms of $[u, v]$ w.l.o.g. in $[u, v]$
**Note:** Since pseudo-join of $x_1, \ldots, x_r$ is an upper bound, there exists directed path from $x_i \vdash_v u \vdash_v x_r$ to pseudo-join$(x_1, \ldots, x_r)$.

**Thm:** Let $P$ be a simple polytope and $\vec{c}$ be generic cost vector with $G(P, \vec{c})$ Hasse diagram of finite lattice. Then 
$pseudo-join(x_1, x_2, \ldots, x_r) = x_i \vdash_v u \vdash_v x_r$

**Pf:** induction on $r$ with $r=2$ base case especially tricky part.
Idea for $r=2$ case:

- $y^r = \text{point of reentry}$
- $2$-face $F$
- $y_1 = y_i = \text{departure point}$

* $y^r \in F \Rightarrow \text{psj}(x, y^r) \in F$

strict inequality $\Rightarrow \exists$ smaller $k-i=x=y_k \in F$

by induction on length longest path to $\uparrow$
Idea for Inductive Step:
Induct on |S| with |S| = 2
base case as just discussed.

\[ T \subseteq S \]
\[ J(T) \leq J(S) \]
\[ \Rightarrow \]
\[ \forall \theta T \]
\[ \psi_y(S, \exists a_r \exists b_r) = J(S, \exists a_r \exists b_r) \]
Defn: The order complex (or nerve) of a poset $P$ is the abstract simplicial complex $\Delta(P)$ whose $i$-dim'l faces are the $(i+1)$-chains $v_0 < v_1 < \ldots < v_i$ in $P$.

\[ P = \begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} \quad \Delta(P) = \begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} \]

Thm (Hall; Popularized by Rota):
\[ M_p(u,v) = \vDash (\Delta(u,v)) \]
subposet $\{z \in P | u < z < v\}$
A topological-combinatorial tool:

**Quillen Fiber Lemma:** Given a poset map \( f : P \to Q \) s.t. \( g \in Q \Rightarrow \Delta(\{ p \in P | f(p) \leq g \}) \) is contractible, then \( \Delta(P) \simeq \Delta(Q) \).

**Remark:** Used extensively in finite group theory to characterize groups via subgp lattice \& in combinatorics.
**Theorem:** Let $P$ be a simple polytope with generic cost vector $\tilde{c}$ such that $G(P,\tilde{c})$ is the Hasse diagram of a finite lattice $L$. Then each open interval $(y,u) = \{ z \in L | u < z < v \}$ has order complex homotopy equivalent to a ball or a sphere.

**Applications:**

- permutahedra $\rightarrow$ weak order
- associahedra $\rightarrow$ Tamari lattice
- generalized associahedra $\rightarrow$ Cambrian lattices
Permutahedron is Weak Order

- cost vector $\bar{c}$ any strictly ascending vector such as $\bar{c}=(1,2,3,4)$.

- Homotopy type 1st due to Edelman (type A) & Björner
**Associahedron as Tamari Lattice**

- Use Loday's realization
- Poset of binary trees with cover relations: $\vee < \wedge$
  
  $((a,b),c) < (a,(b,c))$

- Homotopy type is $\mathbb{Z}$ due to Björner & Welsh via nonpure lexicographic shellability
Generalized Associahedra as Cambrian Lattices

- related to cluster algebras
- homotopy type 1st due to Nathan Reading
Some Remarks

1. Any zonotope $P$ with generic cost vector $z$ yields $G(P,z)$ with non-revisiting property, hence Hasse diagram property.

2. Given shelling on simplicial polytope $X$, this induces "facial order" on vertices of $X^*$. For $G(X^*)$ Hasse diagram of lattice, pseudo-joins equal joins, and distinct $\Delta(u,v) = \text{ball or sphere}$. 
Some Further Questions

Qn 1: Does P simple + G(P, π) Hasse diagram of lattice ⇒ no directed path can revisit face it has departed? (If not, variations?)

Qn 2: Variations on these hypotheses? Non-simple polytopes?

Qn 3: Structure of posets of joins/pseudo-joins for non-simple polytopes?

Qn 4: Other interesting examples?