Combinatorics Meets Topology: Möbius Functions, Euler Characteristic & Beyond

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Plan: discuss how counting problems can be solved using topology.
Counting by Inclusion-Exclusion

e.g. "counting" points in the \( \mathbb{R}^2 \) complement of \( \mathcal{M} \)
yields: counted \( 1-1-1-1+2 = 0 \) times

\[
\begin{align*}
\mathbb{R}^2 & - \left/ \ell_1 \right. - \left/ \ell_2 \right. - \left/ \ell_3 \right. + 2\rho
\end{align*}
\]

° Coefficients 1, -1, -1, 1, 2 in such inclusion-exclusion counting formula
given by "Möbius function" \( M \) (upcoming)
Now let's define a function $M$ to calculate these coefficients (to generalize this counting technique).

We need:

- $M(\mathbb{R}^2, \mathbb{R}^2) = 1 = \text{coef. of } \mathbb{R}^2$
- $M(\mathbb{R}^2, l_1) = -1 = \text{coef. of } l_1$
  
  (so $M(\mathbb{R}^2, \mathbb{R}^2) + M(\mathbb{R}^2, l_1) = 0$)
- $M(\mathbb{R}^2, l_2) = -1$
- $M(\mathbb{R}^2, l_3) = -1$
- $M(\mathbb{R}^2, p) = 2$

  (so $M(\mathbb{R}^2, \mathbb{R}^2) + \sum_{i=1}^{3} M(\mathbb{R}^2, l_i) + M(\mathbb{R}^2, p) = 0$)
**Second Example:**

Qn: How many students at Duke haven't studied any of the languages French, German or Spanish?

**Ans:** \(|D| - |F| - |G| - |Sp| + |F \cap G| + |F \cup Sp| + |G \cap Sp| - |F \cap G \cap Sp|\)

Size of #French #German #Spanish Duke students students students

Coefficients again calculated recursively.
Def'n: Möbius function $M_p(x, y)$ of a "partially ordered set" $P$ is defined recursively:

$$M_p(x, x) = 1$$
and

$$M_p(x, y) = -\sum_{x \not\equiv z \not\equiv y} M_p(x, z)$$

Poset $P\uparrow$

$p = l_1, l_2, l_3$

$\mathbb{R}^2$

Its Möbius fn $M_p(\mathbb{R}^2, -)$

coefficients in $\mathbb{R}^2 - l_1 - l_2 - l_3 + 2p$

(we say $u \preceq v \iff$ draw upward path $u$ to $v$ for $v$ subset of $u$)
Partially Ordered Sets
(Posets) More Generally

Unlike the integers where any \( u, v \) satisfy \( u \leq v \) or \( v \leq u \) (or both if \( u = v \)), some sets only allow comparison of some of the pairs of elements.

e.g. 1. Subsets of \( \mathbb{Z} \) with \( S \subseteq T \iff S \subseteq T \)

\( \iff \) if and only if

2. Positive integers \( \omega / d \leq n \)

\( \iff d \) is a divisor of \( n \) (e.g. \( 2 \leq 6 \) but \( 2 \not\mid 5 \))
**Poset of Subsets of \( 1, 2, \ldots, n^3 \)**

**its Möbius Function**

\[(1-1)^n = 0^n = 0 \text{ for } n \geq 1\]

\[(x+y)(x+y) \ldots (x+y) \text{ for } x=1 \neq y=-1\]

\[\sum (\binom{n}{k}) y^k x^{n-k} \text{ for } x=1 \neq y=-1\]

\[\sum (-1)^k \binom{n}{k}\]

**e.g.,** \(\binom{4}{2} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 0\)

**Proof by Induction yields:** \(M(\emptyset, n) = (1)^r k(n)\)
**Simplicial Complexes**

A *simplicial complex* (e.g. a triangulation) is made of vertices (called "0-simplices"), edges ("1-simplices"), solid triangles ("2-simplices"), solid tetrahedra ("3-simplices"), etc.
Realization: any simplicial complex with vertices $v_0, v_1, \ldots, v_n$ can be drawn in $\mathbb{R}^n$ by letting $v_0 = (0, 0, \ldots, 0)$ and for $1 \leq i \leq n$ letting $v_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ \(\uparrow\) adding edges, with spot \(\wedge\) etc. as needed.

\[\text{e.g.} \]

- We can also think of simplicial complexes abstractly, letting a face be a collection of the vertices in it, requiring for $S$ a face \(\uparrow\) $T \subseteq S$ then $T$ must also be a face.
Higher Dimensional Simplicial Complexes (\textit{\#} Cell Complexes)

4-simplex
(needs 4-dim'l space to fit in)

4-dim'l cube: (not a simplicial complex)

\[ (0,0,0,0) \]
\[ (0,0,1,0) \]
\[ (0,1,0,0) \]
\[ (1,0,0,0) \]
(Reduced) Euler Characteristic

- The reduced Euler characteristic of $K$, denoted $\tilde{\chi}(K) = -1 + \#\text{vertices} - \#\text{edges} + \#\text{triangles} = \ldots$

  e.g., $\tilde{\chi}(\bigtriangleup) = -1 + 4 - 6 + 4 = 1$
  $\tilde{\chi}(\text{tetrahedron}) = -1 + 5 - 9 + 6 = 1$

Adding faces without changing "Topology" won't change $\tilde{\chi}$!

\[ \tilde{\chi} = -1 + 3 - 3 + 1 = -1 + 4 - 5 + 2 = -1 + 5 - 8 + 4 \]
**Defn:** The order complex of a poset $P$ is the abstract simplicial complex, denoted $\Delta(P)$, whose $i$-dimensional faces are the $(i+1)$-"chains" $v_0 < \ldots < v_i$ in $P$

**e.g.**

$P = \{a_1, b_1, o\}$ \hspace{1cm} $\Delta(P) = \{a_2, b_2, a_0, b_0, o\}$

**Key Property** (due to Hall; popularized by Rota):

$M_P(x, y) = \chi(\Delta_P(x, y)) = -1 + \#\text{vertices} - \#\text{edges} + \#2\text{-faces} - \ldots$\n
$= -1 + \beta_0 - \beta_1 + \beta_2 - \ldots$

- $\beta_i = \#i\text{-dim}\text{'}\text{ hole boundaries}$
- $(u, v) = \sum_{z \in P | u < z < v} z$
- $\Delta_P(u, v) = \Delta(\exists z \in P | u < z < v)$
Techniques Yielding Möbius Functions (\& Poset Topology)

- (lexicographic) shellability
- EL-labelings (Anders Björner)
- CL-labelings (Anders Björner & Michelle Wachs)

\[ \Delta(P) \cong \]

(telling us \( \tilde{x} \) hence \( \mu \))

- Lexicographic discrete Morse functions (Babson-H.)
  (for other topoL. types)
Intersection Posets

E.g.,

$A = \mathbb{R}^2$

(l intersection poset)

$H = \mathbb{R}^3$
Intersection Poset $L_A$ for $A = \{x_i = x_j \mid 1 \leq i < j \leq n\}$ the "Partition Lattice"

\[ \hat{\theta} = 1234 \]

\[ \hat{\theta} = 1234 \]

\[ M_{\Pi_4}(\hat{\theta}, \hat{\theta}) = -6 \]
Some Applications of Möbius Functions & "Shellability"

1. Shellability of intersection posets of hyperplane arrangements due to shellability of "geometric lattices"

(Anders Björner) & "geometric semilattices"

(Michelle James Wachs Walker), yielding Möbius fns of "intersection posets" of hyperplane arrangements

\[ \\text{\textasciitilde useful e.g. for...} \]
2. Zaslavsky: region counting formulas for the complement of $IR$-hyperplane aren't $A$

$\# \text{regions} = \sum_{u \in L_A} |M(0,u)|$

$\# \text{bdd regions} = | \sum_{u \in L_A} M(0,u) |$

$M(1^{R^2}, 1^{R^2}) = 1$

$M(1^{R^2}, H_i) = -1$ for $i = 1, 2, 3$

$M(1^{R^2}, H_1 \cap H_2 \cap H_3) = 2$

$L_A = "\text{intersection poset}"$

E.g. $\# \text{regions} = 1 + 3 + 2$

$\# \text{bdd regions} = 1 - 3 + 2$
3. Björner-Lovász-Yao: lower bound via Möbius funs for deciding if there are \( k \) equal coordinates in \( \vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \) by pairwise coord. comparisons, i.e. deciding whether \( \vec{z} \) lies on "\( k \)-equal arr't" of subspaces \( x_1 = \ldots = x_k \).
• lower bd on \# leaves (and hence on \(\log_3(\text{depth})\)) was given in terms of betti \#’s (i.e., \# holes in each dimension) in topological space \(\mathbb{R}^n\) — k-equal subspace arrangement

\[\sim\]

Subspaces like \(x_1 = x_2 = x_3\)
for \(k = 3\)

• Mark Goresky & Robert MacPherson showed how to compute these betti \#’s from poset order complexes

• Björner & Wachs found shellings for these poset order complexes, namely intersection posets for “k-equal arrangement”
Appendix: Some Additional Slides Giving Further Details & Touching Upon Some Further Topics...
Technique: Shellability

- Simplicial complex is pure of dim. \( d \) if all maximal faces ("facets") are \( d \)-dimensional
- Simplicial complex is shellable if there is total order \( F_1, F_2, \ldots, F_k \), a shelling, on facets satisfying conditions guaranteeing we can build up the complex by attaching facets in this order so each step either leaves topology (homology) unchanged or closes off a sphere \( S^i \), increasing \( \beta_i \) by 1.
**Technique 1**: **Lexicographic Shellability**

(Anders Björner & Michelle Wachs)

A poset $P$ is **EL-shellable** if it admits labeling $\lambda$ of its cover relations $x < y$ w/in integers (called an **EL-labeling**) s.t. $u < v$ implies:

1. there is unique saturated chain $u < u_1 < \ldots < u_k < v$ s.t.
   $\lambda(u, u_1) \leq \lambda(u_1, u_2) \leq \ldots \leq \lambda(u_k, v)$ and

2. $(\lambda(u, u_1), \lambda(u_1, u_2), \ldots, \lambda(u_k, v))$
   is lexicographically smaller than the label sequences on all other saturated chains from $u$ to $v$. 
Thm (Björner): EL-labeling $\Rightarrow$ Shelling

Idea: Lexicographic order on maximal chains (breaking ties arbitrarily) induces shelling order on corresponding facets of $\Delta(P)$.

"descents in labeling" $\leftrightarrow$ overlap of facets

"descending" $\leftrightarrow$ facets attaching along entire boundary $\leftrightarrow$ spheres

$M_P(u,v) = \pm \# \text{descending chains } u \to v$ (for $P$ graded)
Example: Intersection Posets of Hyperplane Arrangements

- Choose any total order $H_1, H_2, \ldots, H_k$ on hyperplanes (resp. "atoms")

- Label $u <_L v$ with $\min \{ i \mid H_i \neq u \text{ and } H_i \leq v \}$

e.g.

$A =$

$A_1 H_2 H_3$

$L_A =$

$H_1 \wedge H_2 \wedge H_3$

$1 2 3$

$1 \mathbb{R}^2$
Intersection Poset $L_A$ for $A = \{x_i \leq x_j \mid 1 \leq i < j \leq n\}$ the "Partition Lattice"

$\hat{\mathbf{e}} = 1234$

$\Pi_4 = 1234$

$M_{\Pi_4}(\hat{\mathbf{e}}, \hat{\mathbf{e}}) = -6$
Cell Complexes \& their Face Posets

e.g.

\[ K = \text{ball} \]

\[ K' = \mathbb{RP}^2 \]

\[ F(K) = e_1, e_2 \]

\[ F(K') \quad \text{“closure poset” or “face poset”} \]

\[ (u \preceq v \iff u \subseteq \overline{v}) \]

\[ \mathbb{RP}^2 \text{ has different 1st homology group than ball} \]

\text{A Goal of Mine: Use combinatorics of } F(K) + \text{limited topological info to understand } K \]
"Topological Proof" of Möbius Function for Poset of Subsets

\[ K \]

\[ F(K) \]

\[ \text{poset of subsets} \]

\[ -1 + \beta_0 - \beta_1 + \beta_2 - \ldots \]

with \( \beta_0 = 1 = \beta_1 \) (\( \forall \) for \( B_n \) then \( \beta_0 = \beta_{n-2} = 1 \))

\[ K \cong \sigma_\mathcal{L}(K) = \Delta(K) \]
Discrete Morse Theory
(due to Forman reformulated by Chari)

Given any regular CW complex $\Delta$, construct an acyclic matching a.k.a. Morse matching on its face poset, i.e.,

an edge orientation s.t. "up edges" give a matching and directed graph has no cycles.

(A matching is a collection of graph edges s.t. no vertex is in more than one edge)
**Theorem (Forman):** $\Delta \sim \Delta^M$ a CW complex comprised of the unmatched cells, called critical cells.

**e.g.** $\bullet \sim \bigcirc$

- same topological structure
- (same homology groups + more!)

**Idea:** Find pairs of faces where one can be "pulled across" other eliminating both without changing topology, via moves called "collapses".

$v_2 \circ \circ v_3 \rightarrow \bigcirc \rightarrow v_2 = v_1 \circ = v_3$
First Examples

1. Boolean algebra of subsets of $\Sigma^{1,2,\ldots,n}_3$, face poset of simplex, matching $\Sigma^3 \times \Sigma^3$ with $SU(3) \times SU(3)$ AS

Diagram:

- Base pt
- Critical O-cell
- Matching edge in "reduced homology" version of discrete Morse theory
2. Any union of acyclic matchings on \( F(\Delta_2 \setminus \Delta_1), F(\Delta_3 \setminus \Delta_2), \ldots \), for \( \Delta_1 \leq \Delta_2 \leq \ldots \leq \Delta_k = \Delta \) a filtration of subcomplexes is an acyclic matching for \( \Delta \)

c.g. \( \overline{F_1} \leq \overline{F_1} \cup \overline{F_2} \leq \overline{F_1} \cup \overline{F_2} \cup \overline{F_3} \)

3. Shelling \( \Rightarrow \) Discrete Morse fn

whose critical cells are the maximal faces attaching along their entire boundary
Explanation for $\Delta \simeq \Delta^m$: Matching edges specify (internal) elementary collapses preserving homotopy type

Some Consequences of $\Delta \simeq \Delta^m$:
1. If $F(\Delta)$ has complete acyclic matching (w/ $\emptyset \in F(\Delta)$) then $\Delta$ is collapsible.

Recall: Some contractible complexes are not collapsible. e.g. dunce cap

\[\text{Diagram:}\]

[Diagram notation]
2. \( \tilde{\chi}(\Delta) = \tilde{\chi}(\Delta^m) \)

\[= -1 + \#0\text{-cells} - \#1\text{-cells} + \#2\text{-cells} \cdots \]

\[= -1 + \beta_0 - \beta_1 + \beta_2 \cdots \]

**For Posets:** \( M_p(x,y) = \tilde{\chi}(\Delta(x,y)) = \tilde{\chi}(\Delta^m(x,y)) \)

3. **Morse Inequalities:**

1. \( \beta_i(\Delta) \leq \tilde{m}_i(\Delta) = \# \text{i-dim critical cells} \)

2. \( \sum_{i \leq j} \beta_i(\Delta) \leq \sum_{i \leq j} \tilde{m}_i(\Delta) \)

   (for each \( j \leq \dim(\Delta) \))

**Rk:** "Greedy" matchings tend to satisfy acyclicity requirement.
Question (H.): Is there a good way to "complete the square":

- lexicographic
- shelling

\[ \Rightarrow \ ?? \]

\[ \downarrow \]

- shelling

\[ \Rightarrow \] discrete Morse function

To understand posets that fail to be shellable (e.g. not wedge of spheres)?

**Proposed Answer (Eric Babson & P.H.):**

"lexicographic discrete Morse fn's"
Research on "f-vectors"

If \( f_i(\Delta) = \# \text{i-dimensional faces in } \Delta \)

then which vectors arise as

\((f_0(\Delta), f_1(\Delta), f_2(\Delta), \ldots, f_{\dim(\Delta)}(\Delta))\)

for some \( \Delta \)?

e.g. \( f_0(\Delta) = 4 \Rightarrow f_1(\Delta) \leq \binom{4}{2} = 6 \)

• "Kruskal-Katona Thm" for simplicial complexes
• Richard Stanley used "commutative algebra" for spheres
• Isabella Novik for "homology spheres"
Topological "Pathologies"

eg. Alexander horned ball: