From Poset Topology to Combinatorial Representation Theory

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Counting Topologically

e.g., "counting" points in the $\mathbb{R}^2$ complement of $\mathbb{R}^2$ yields:

$\mathbb{R}^2 - l_1 - l_2 - l_3 + 2\mathbb{R}$

* Coefficients 1, -1, -1, 2 in such inclusion-exclusion counting formula given by "Möbius function" $\mu$
Defn: Möbius function $M_p(x,y)$ of partially ordered set (poset) $P$ is defined recursively: $M_p(x,x) = 1$ and $M_p(x,y) = -\sum M_p(x,z)$ (so $\sum M_p(x,z) = 0$) (for $x \neq y$) $x \leq z < y$.

=poset P: $p = l_1, l_2, l_3$

- $l_1$
- $l_2$
- $l_3$
- $\mathbb{R}^2$

- 2
- 1
- 1
- 1
- -1
- -1
- -1

Coefficients in $\mathbb{R}^2 - l_1 - l_2 - l_3 + 2P$
Working over $\mathbb{F}_2$: $$\# \text{pts} = 2^2 - 8 - 8 - 8 + 2$$

$$\sum_{\mathbf{u} \in \Lambda} M(\hat{0}, \mathbf{u}) g^{\dim V - r(h(\mathbf{u}))}$$

characteristic poly. = of the arrangement

e.g. chromatic poly given by hyperplanes $x_i = x_j \leq 1 \forall \mathbb{R}^n$

for each edge $e_{ij}$ in graph $G$

Applications:

- Complexity theory (Björner - Lovász - Yao)
- Number theory (Church - Ellenberg - Farb
- etc.
**Defn:** The order complex (or nerve) of a poset $P$ is the abstract simplicial complex $\Delta(P)$ whose $i$-dim' faces are the $(i+1)$-chains $v_0<v_1<\ldots<v_i$ in $P$.

**Thm (Hall; Popularized by Rota):**

$M_p(u,v) = \bigcap \left\{ \Delta(v_i, v) \right\}$

subposet $\{ z \in P \mid u < z \leq v \}$
Some Techniques in Poset Topology

- Quillen fiber lemma
  Use $f: P \to Q$ to show $\Delta(P) \cong \Delta(Q)$

- (lexicographic) shellability
  (Björner & Wachs)
  $\Rightarrow \Delta(P) \cong \text{wedge of spheres}$

- Lexic. discrete Morse fun's
  (Babson - H. ~2001)
  - Betti # bds, etc. more generally
Theorem (Zaslavsky):

\[
\text{# regions} = \sum_{u \in \mathcal{L}_A} |M(0, u)| \\
\text{#bdd regions} = \left| \sum_{u \in \mathcal{L}_A} M(0, u) \right|
\]

\[A = H_1 \cap H_2 \cap H_3\]

\[M(\mathbb{R}^2, \mathbb{R}^2) = 1\]
\[M(\mathbb{R}^2, H_i) = -1 \text{ for } i = 1, 2, 3\]
\[M(\mathbb{R}^2, H_1 \cap H_2 \cap H_3) = 2\]

E.g. \text{# regions} = 1 + 3 + 2 \\
\text{#bdd regions} = 1 - 3 + 2 \\
\mathcal{L}_A = "intersection poset"
Goresky-MacPherson Formula
\[ \tilde{H}^i(M_A) \cong \bigoplus_{x \in B_{N}^{\geq 0}} \tilde{H}_{\text{codim}(x)-2-i}^{c}(\partial, x) \]

subspace aren't complement as groups intersection poset

**Pf:** Stratified Morse theory

**Thm (Björner):** Intersection posets of central hyperplane arrangements are “shellable”, giving formula for \( M \) in terms of “matroid theory” (a theory capturing commonalities of linear algebra/spanning trees)
M as Topological Shadow

- A graded poset with $\emptyset \neq \uparrow$ is Eulerian if $M(u,v) = (-1)^{\text{rk}(v)-\text{rk}(u)}$ for all $u \leq v$.

- A graded poset $P$ is a CW poset if
  1. $\emptyset \in P$
  2. $P$ has at least one other element
  3. $\Delta(\emptyset, u) \cong S^{\text{rk}(u)-2}$ for $u \neq \emptyset$

\[ \begin{array}{c}
\text{e.g.} \\
\begin{array}{c}
\bigcirc = \begin{array}{c}
\square \neq \big| \\
\text{homeomorphic}
\end{array}
\end{array}
\end{array} \]
Thm (Björner): \( P \) is CW poset \( \Rightarrow \) there exists "regular" CW complex \( K \) with \( P \) as poset of closure relations, which implies \( \Delta(P) = \text{sd}(K) \cong K \).

Cor: CW Poset \( \Rightarrow \) Eulerian

a "regular CW complex" due to \( K \)
For methods to study homeom. type of $K$ via poset topology of $F(K)+$ topol. data about codim. one incidences, see:


- Gives representation theoretic/ geometric explanation for strong Bruhat interval being a CW poset.
Weak Bruhat Order: Another Partial Order on Permutations

$u \prec v$ iff $u$ obtained from $v$ by adjacent transposition $s_i = (i, i+1)$ sorting pair of letters in positions $i$ and $i+1$
**General Defn:** (left) weak order on Coxeter group $W$ is partial order with $u \prec v \iff v = s_i u$ for $u, v \in W$ s.t. $\text{length}(v) > \text{length}(u)$ for $\text{length}(u) := \min \{ r \mid u = s_{i_1} \ldots s_{i_r} \}$

**e.g.** $W = S_n$ with relations:

\[ s_i^2 = e, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \quad (\text{for } |j-i| > 1) \]

"braid rel's"
(Left) Weak Bruhat Order for $S_3$

$321 = s_1 s_2 s_1 = s_2 s_1 s_2$

$231 = s_2 s_1$

$123 = e$

$s_1$

$s_2$

$s_1 s_2 = 312$

$S_2 = 132$

(Also is Cayley graph)

"saturated chains" from $e$ to $w$  $\leftrightarrow$  "reduced expressions" for $w$

(paths upward)  $\leftrightarrow$  (minimal length)

(from $e$ to $w$)  $\leftrightarrow$  (factorizations)

**Connectedness under Braid Moves**

**Thm:** (see e.g. Björner-Brenti book) Let $(W,S)$ be Coxeter system; let $w \in W$. Then every two reduced expressions for $w$ are connected via braid moves.

c.g. $s_2s_3s_2s_1 \rightarrow s_3s_2s_3s_1 \rightarrow s_3s_2s_1s_3$

$w = s_2s_3s_2s_1 = s_3s_2s_3s_1 = s_3s_2s_1s_3$

**Left weak order:**

```
  s_2/\ s_3
  \    /
  s_3  s_2
  \   /
  s_2  s_3
  \ /   /
  s_1  s_3
```

The diagram illustrates the weak order relations among the generators $s_1, s_2, s_3$.
Thm (Edelman & Björner): Weak Bruhat order has $\Delta(u, v) = \text{ball or sphere}$, hence $M(u, v) = 0, \pm 1$ for all $u \leq v$.

Idea: Use "Quillen fiber lemma" (a.k.a. Quillen Theorem A)
A topological-combinatorial tool:

**Quillen Fiber Lemma:** Given a poset map \( f: P \rightarrow Q \) s.t. \( g \in Q \Rightarrow \Delta (\{ p \in P \mid f(p) \leq g \}) \) is contractible, then \( \Delta (P) \simeq \Delta (Q) \).

**Remark:** Used extensively in finite group theory to characterize groups via subgp lattice \( \triangleright \) in combinatorics.
Crystal Graphs

**Purpose:** Study rep'n theory of Kac-Moody algebras (e.g. affine Lie algebras) & their characters via combinatorics of "crystal graphs"

Crystal Graphs Arising as Posets

- poset $\leftrightarrow$ basis vectors for the elts $\leftrightarrow$ various weight spaces (guaranteed to exist by properties (of Kashiwara's "crystal basis")
- cover reln's $\leftrightarrow$ crystal (lowering) fns
$(\text{Type A})$ Crystals of Highest Weight Reps and their Kashiwara Lowering Operators

e.g. $\lambda = \square$

Integer partition $(2, 1)$

$[2, 2]_3 \xrightarrow{f_2} [2, 3]_3 \xrightarrow{f_1}$

$[1, 3]_3 \xleftarrow{f_2}$

weight $(1, 0, 2)$

$x_1 x_3$

changes weight by $(0, -1, 1)$

weight $(1, 1, 1)$

$x_1 x_2 x_3$

$[1, 2]_2 \xrightarrow{f_2}$

weight $(1, 2, 0)$

$x_1 x_2$

changes weight by $(1, 1, 0)$

weight $(2, 1, 0)$

$x_1 x_2 x_3$

weight $\rightarrow$

$(2, 0, 1)$

$x_1^2 x_3$
Type A crystal for highest weight rep in of shape $\lambda$

1. $\hat{\Delta} = 111_{-1}$ of shape $\lambda$
   $\begin{array}{c}
   22_{-2} \\
   33_{-1}
   \end{array}$ "highest weight vector"

2. $u \rightarrow v$ has $v$ obtained from $u$ by incrementing to $i+1$ rightmost $i$ not in "parenthesization pair" with an $i+1$
e.g. $11111444 \rightarrow 11111444$

$f_3 \quad 2233 \quad 444$

$344 \quad \square$ $f_3 \quad 2233 \quad 444$

$34433444 \rightarrow \square 44433444$

**Parenthesization Pairs**: Read leftmost column bottom to top, then subsequent columns likewise, ignoring all but i’s & i+1’s. Pair up consecutive i+1 then i, deleting both. Repeat pairing/deleting until no more such pairs.
"character" of crystal
= x_1^3 x_2 + x_1^2 x_2^2 + ... = weight(x_1) + weight(x_2) + ...

character of rep 'n
= ...
Stambridge: "g-crystals"
(Crystals of highest weight reps in simply laced case)

\[ f_i(x) f_j(x) \Rightarrow f_j f_i(x) \]

b) likewise for \( e_i, e_j \)
"raising operators":
\[ f_i(x) = y \]
\[ f_i \dagger \dagger e_i \]
\[ x = \hat{e}_i(y) \]

c) reln's depend on location
Two Types of Motivation

1. Given quantized enveloping algebra \( U = U^+ \otimes_{\mathcal{O}(U)} U^0 \otimes_{\mathcal{O}(U)} U^+ \), the canonical basis (or crystal basis) \( \mathcal{B} \) has the remarkable property that each highest weight module \( V_\lambda \) has a basis \( \{ v_{\lambda, b} \mid v_{\lambda, b} \neq 0 \} \), i.e. the elements of crystal poset.

2. Technique to prove "Schur positivity": crystal character is a Schur function
Transferring structure from weak order to crystals via poset map called "key"

- Related to "Key polynomials" of Lascoux & Schützenberger

- Schubert polynomials (describing cohomology classes in Schubert calc.) are positive sums of Key polynomials
Right key "r" of a "KM"-crystal

\[ r(\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 \end{pmatrix}) = s_3 s_2 s_1 s_2 \]

\[ r(\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}) = s_3 s_1 s_2 \\ s_1 s_2 s_1 \]

\[ r(\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}) = s_1 s_2 \]

\[ r(\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}) = s_2 s_1 \]

\[ r(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}) = s_1 \]

\[ r(\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}) = s_2 \]

**r:** crystal poset \[ \Rightarrow \text{weak Bruhat order} \]

\[ u \preceq v \Rightarrow r(u) \leq r(v) \]

**key:** \[ r(\emptyset) = e \]
New Algorithm to Calculate Right Key of a KM-Crystal

(1) $\text{key}(\hat{o}) = e$

(2) if $\hat{o} \xrightarrow{\varepsilon} a$, then $\text{key}(a) = s_i$
   (i.e. $\hat{o} \prec a$)

(3) if $v$ covers 2 or more elements then $\text{key}(v) = \bigvee_{u \in \{u' \xrightarrow{\varepsilon} v\}} \text{key}(u)$
   (for join taken in weak order)

(4) if $u \xrightarrow{\varepsilon} v$ and $v$ does not cover any other elements, then:
   (a) $\text{key}(v) = \text{key}(u)$ if $\exists u' \xrightarrow{\varepsilon} u$
   (b) $\text{key}(v) = s_i \cdot \text{key}(u)$ otherwise
Thm (H.-Lenart): For \( \hat{\omega} = \text{highest weight vector in symmet. KM-crystal} \), \( \Delta(\hat{\omega}, u) \neq \text{ball unless } u = \min (k^{-1}(\omega_0 W_f)) \) where \( \Delta(\hat{\omega}, u) \neq S^{|\hat{\omega}| - 2} \).

Proof: Use Quillen fibre lemma with

\[ f: \text{crystal} \rightarrow \text{Boolean algebra} \]

\[ \text{poset of subsets of } \{\xi_1, \ldots, n\} \]

\[ \chi \mapsto \max \{ \xi \leq \omega_0 (\{5\}) \leq \text{weak key}(x) \} \]

Corollary: \( M(\hat{\omega}, u) = 0, \pm 1 \)

highest weight vector
Thm (H.-Lenart): Given any lower interval \((\hat{0}, u)\) in a \(y\)-crystal, then set of saturated chains from \(\hat{0}\) to \(u\) is connected by "Stembridge moves", namely moves of the form: and

Note: Likewise in doubly-laced case via "Stemberg moves".
M_p(u,v) = 2 \quad u = 234

\text{not connected by "Stembridge moves"}
Arbitrarily High Rank
Disconnected Open Intervals

\[ V = \begin{array}{c|c|c|c|c|c|c|c} \hline 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ \hline 3 & 4 & 5 & 6 & \cdots & n+1 & n+1 \\ \hline \end{array} \]

\[ U = \begin{array}{c|c|c|c|c|c|c|c} \hline 1 & 2 & \cdots & n-3 & n-2(n-1) \\ \hline 3 & 4 & 5 & \cdots & n+1 & n+1 \\ \hline \end{array} \]

Label sequences: 1, 2, 2, 3, 3, 4, 4, ..., n-1, n-1, n
\& n, n, n, n, 2, 2, 1 in distinct components
Thm (H.-Lenart) There exist $u < v$ s.t. $M(u, v) = 2^j$ for all positive integers $j$.

Thm (H.-Lenart): $M(u, v) \neq 0, \pm 1$ in $\gamma$-crystal $\implies$ relation within $[u, v]$ not implied by Stembridge local relations.

Appendix: a few slides with extra details...
A crystal $B$ of type $\Phi$ is a nonempty set $B$ with raising $\xi$, lowering operators $e_i, f_i$ such that $\xi_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{\xi - \infty\}$

$\text{wt} : B \rightarrow \Lambda = \text{weight lattice of type } \Phi$

s.t.

(CA1) $x, y \in B$, then $e_i(x) = y \Leftrightarrow x = f_i(y)$

both implying $\text{wt}(y) = \text{wt}(x) + \alpha_i$

(CA2) $\varphi_i(x) = \xi_i(x) - 1$

$\xi_i(x) = \varphi_i(x) + 1$

$\langle \text{wt}(x), \alpha_i \rangle$
Non-Lattice Example:

\[ \frac{1123}{344} = (uuv)_1, \quad \frac{1223}{334} = (uuv)_2 \]
Examples with \( M(u,v) = 2^j \)

\( j=1: \quad u = \frac{1112}{234} \quad v = \frac{1123}{344} \)

\( j=2: \quad u^{(2)} = \begin{bmatrix} 1111 \\ 2222 \\ 6667 \\ 789 \end{bmatrix} \quad v^{(2)} = \begin{bmatrix} 1111 \\ 2222 \\ 6678 \\ 899 \end{bmatrix} \)

\[ u_+ = u+5 = \begin{bmatrix} 6667 \\ 789 \end{bmatrix} \quad v_+ = v+5 = \begin{bmatrix} 6678 \\ 899 \end{bmatrix} \]

\[ [u^{(2)}, v^{(2)}] \cong [u, v] \times [u, v] \]

So \( M(u^{(2)}, v^{(2)}) = 2^2 \)

\[ u^{(3)} = \begin{bmatrix} u_+ \\ u_{++} \end{bmatrix} \quad v^{(3)} = \begin{bmatrix} v_+ \\ v_{++} \end{bmatrix} \]

\[ [u^{(k)}, v^{(k)}] \cong [u, v] \times \ldots \times [u, v] \quad M = 2^k \]

\( k\text{-fold} \)