From Poset Topology to Combinatorial Representation Theory

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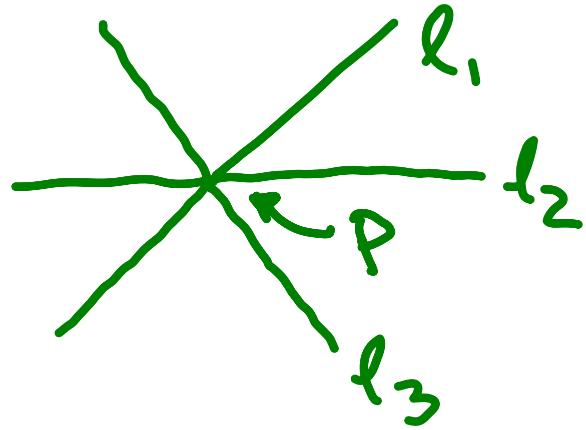
Counting Topologically

e.g. "counting" points in the $\mathbb{R}^2$ complement of $\rightsquigarrow$ yields:



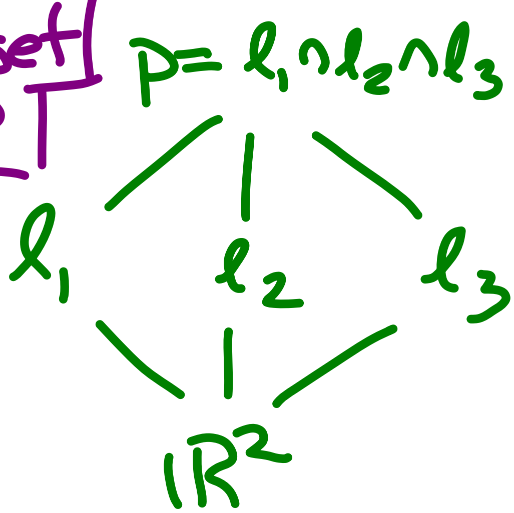
$$\mathbb{R}^2 - \ell_1 - \ell_2 - \ell_3 + 2p$$

- Coefficients $1, -1, -1, -1, 2$ in such inclusion-exclusion counting formula given by "Möbius function" μ

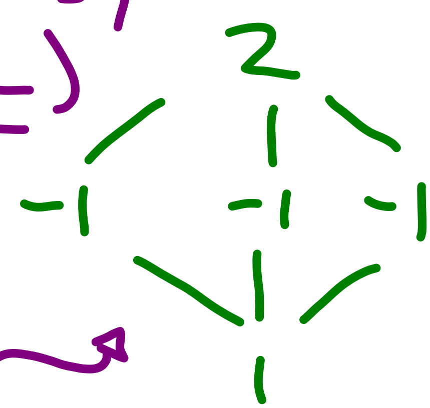
Defn: Möbius function $\mu_P(x,y)$ of partially ordered set (poset) P is defined recursively: $\mu_P(x,x)=1$

and $\mu_P(x,y) = -\sum_{x \le z < y} \mu_P(x,z)$ (for $x \neq y$) (so $\sum_{x \le z \le y} \mu_P(x,z) = 0$)

$\boxed{\text{Poset } P}$



$\boxed{\text{Its Möbius fn } \mu_P(\mathbb{R}^2, -)}$



coefficients in $\mathbb{R}^2 - \ell_1 - \ell_2 - \ell_3 + 2P$

Working over $\mathbb{F}_q$: $\#\text{pts} = q^2 - q - q - q + 2$

$$\sum_{u \in L_A} \mu(\hat{0}, u)\, q^{\dim V - \operatorname{rk}(u)} =: \begin{array}{l}\text{characteristic poly.} \\ \text{of the arrangement}\end{array}$$

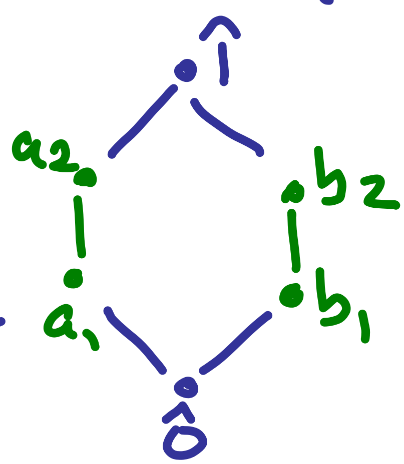
e.g. chromatic poly given by hyperplanes $x_i = x_j \subseteq \mathbb{R}^n$ for each edge e_{ij} in graph G

Applications:

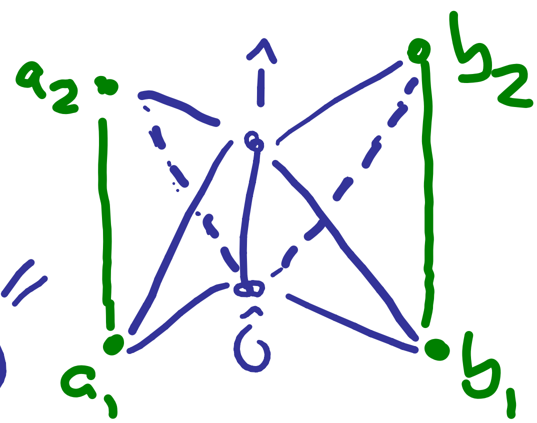
- Complexity theory (Björner–Lovász–Yao)
- Number theory (Church–Ellenberg–Farb)
- etc.

Defn: The order complex (or nerve) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dim'l faces are the $(i+1)$-chains $v_0 < v_1 < \dots < v_i$ in P.

e.g. $P =$



$\Delta(P) =$



Thm (Hall; Popularized by Rota):

$$\mu_P(u,v) = \tilde{\chi}(\Delta(\underbrace{u,v}_{\substack{\parallel \\ \text{subposet } \{z \in P \,|\, u<z<v\}}}))$$

Some Techniques in Poset Topology

- Quillen fiber lemma

  Use $f: P \to Q$ to show $\Delta(P) \simeq \Delta(Q)$

- (Lexicographic) shellability

  (Björner & Wachs)

  $\Rightarrow \Delta(P) \simeq$ wedge of spheres

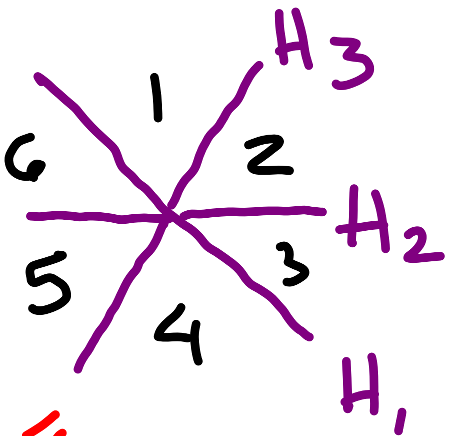
- Lexic. discrete Morse fn's

  (Babson-H. ~2001)

  - Betti # bds, etc. more generally

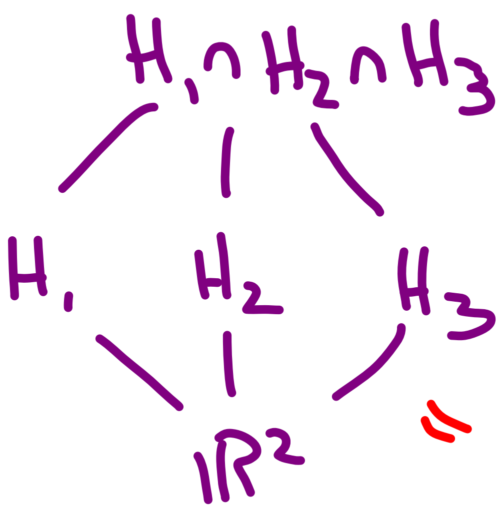
Theorem (Zaslavsky):

$A =$



$$\#\text{regions} = \sum_{u \in L_A} |\mu(\vec{0}, u)|$$

$$\#\text{bdd regions} = \left| \sum_{u \in L_A} \mu(\vec{0}, u) \right|$$



$= L_A =$ "intersection poset"

e.g. $\#\text{regions} = 1+3+2$

$\#\text{bdd regions} = 1-3+2$

$\mu(\mathbb{R}^2, \mathbb{R}^2) = 1$

$\mu(\mathbb{R}^2, H_i) = -1$ for $i = 1,2,3$

$\mu(\mathbb{R}^2, H_1 \cap H_2 \cap H_3) = 2$

Goresky-MacPherson formula

$$\tilde{H}^i(M_A) \cong \bigoplus_{x \in L_A^{>0}} \tilde{H}_{\operatorname{codim}(x)-2-i}(\hat{0}, x)$$

subspace arr't complement

as groups

intersection poset

Pf: Stratified Morse theory

Thm (Björner): Intersection posets of central hyperplane arrangements are "shellable", giving formula for M in terms of "matroid theory" (a theory capturing commonalities of linear algebra/spanning trees)

μ as Topological Shadow

- A graded poset with $\hat{0} \neq \hat{1}$ is **Eulerian** if $\mu(u,v) = (-1)^{rk(v)-rk(u)}$ for all $u \leq v$.

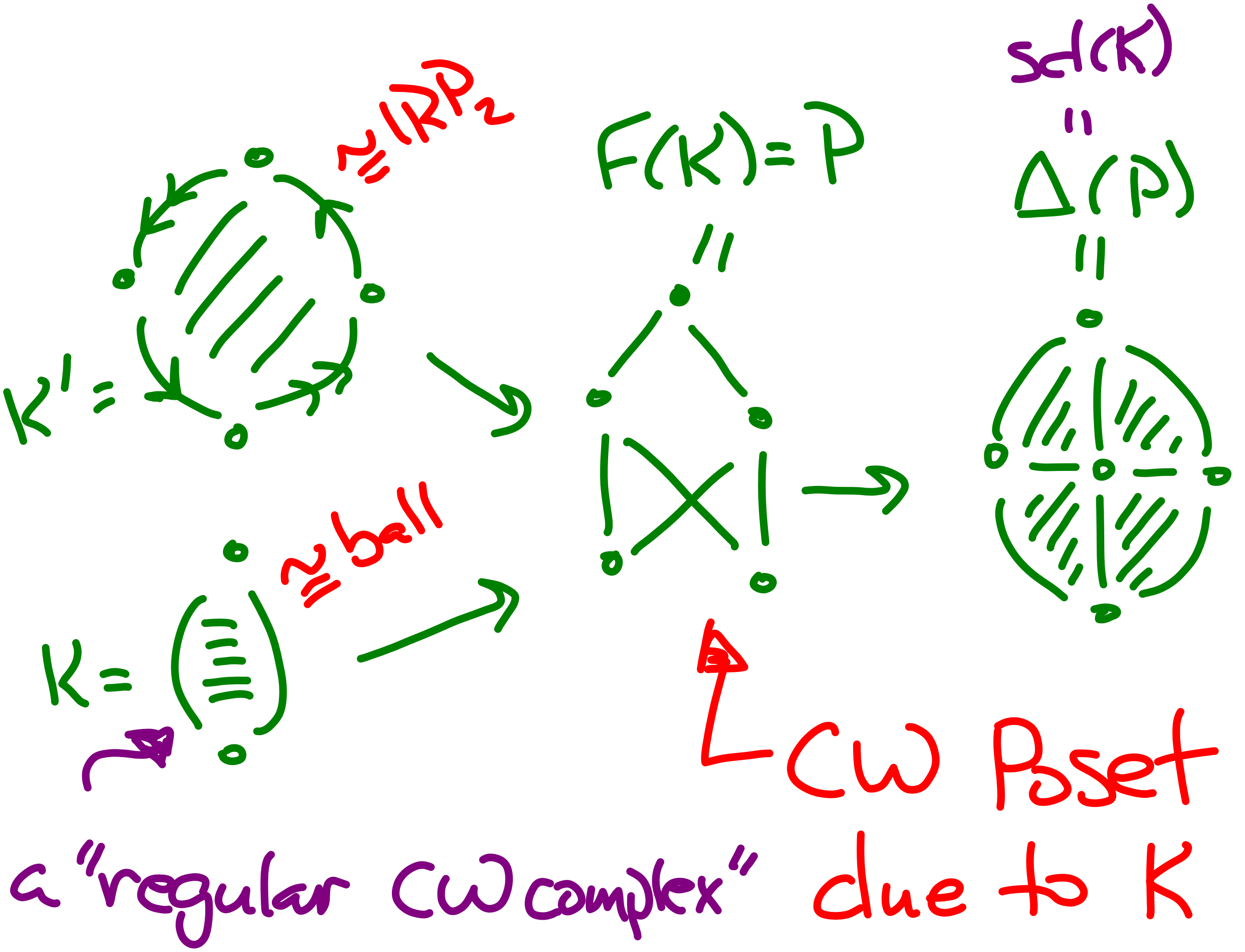
- A graded poset P is a **CW poset** if
  (1) $\hat{0} \in P$
  (2) P has at least one other element
  (3) $\Delta(\hat{0},u) \cong S^{rk(u)-2}$ for $u \neq \hat{0}$

$\uparrow$ homeomorphic

e.g.

Thm (Björner): P is CW poset $\iff$ there exists "regular" CW complex K with P as poset of closure rel'ns, which implies $\Delta(P) = sd(K) \cong K$.

Cor: CW Poset $\Rightarrow$ Eulerian



For methods to study homeom. type of K via poset topology of $F(K)$ + topol. data about codim. one incidences, see:

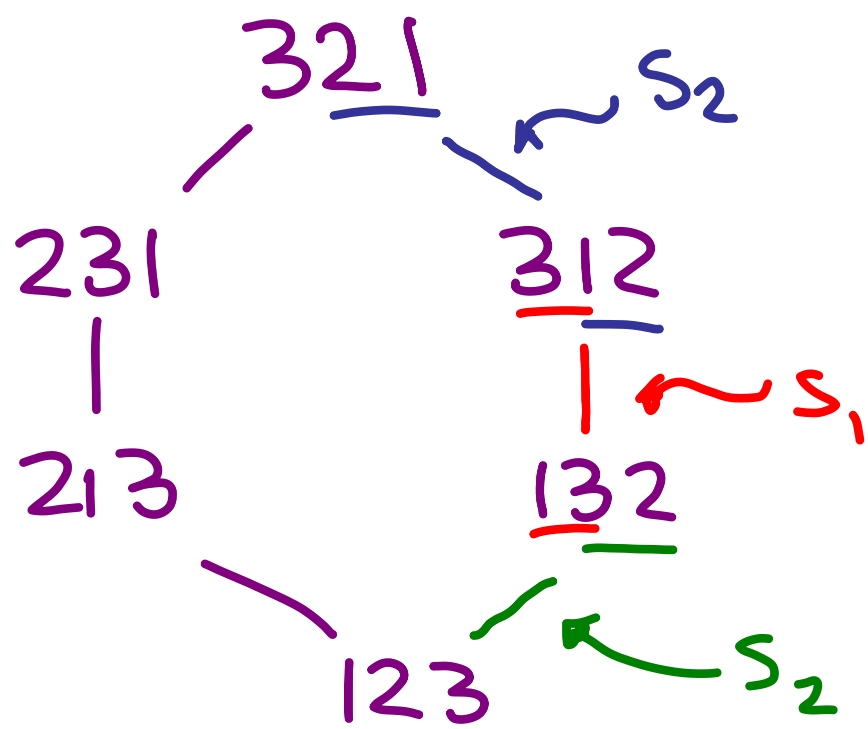
P. H., "Regular CW complexes in total positivity", Invent. Math., 197 (2014), no. 1, 57-114.

- Gives representation theoretic/geometric explanation for strong Bruhat order being a CW poset.

Weak Bruhat Order: Another Partial Order on Permutations

$u \lessdot v$ iff u obtained from v by adjacent transposition $s_i = (i, i+1)$ sorting pair of letters in positions i & $i+1$

e.g.



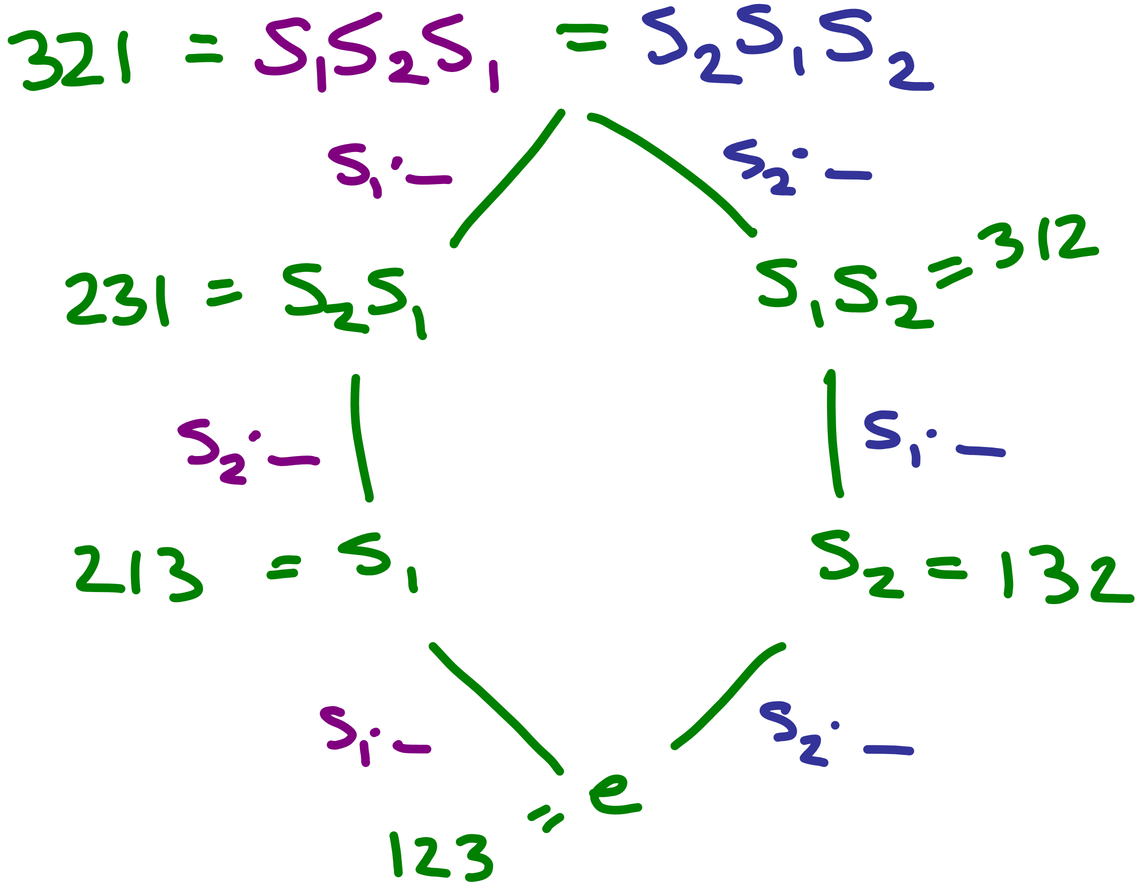
General Def'n: (left) weak order on Coxeter group W is partial order with $u \lessdot v \iff v = s_i u$ for $u, v \in W$ s.t. length$(v) >$ length(u) for length$(u) := \min\{r \mid u = s_{i_1} \cdots s_{i_r}\}$

e.g. $W = S_n$ with relations:

$$s_i^2 = e \;\;\&\;\; \underbrace{s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \;\;\&\;\; s_i s_j = s_j s_i}_{\text{"braid rel'ns"}}$$

(for $|j-i| > 1$)

(Left) Weak Bruhat Order for S_3



(also is Cayley graph)

"saturated chains" from e to ω $\longleftrightarrow$ "reduced expressions" for ω

(paths upward from e to ω)

(minimal length factorizations for ω)

Connectedness under Braid Moves

Thm (see e.g. Björner-Brenti book): Let (W,S) be Coxeter system & let $w \in W$. Then every two reduced expressions for w are connected via braid moves.

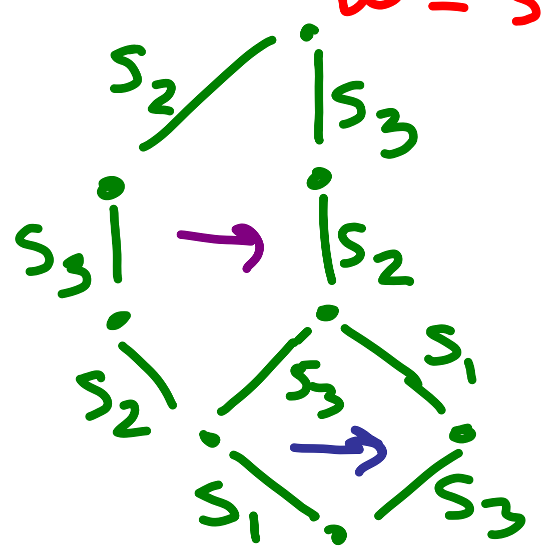
e.g. $\underbrace{s_2 s_3 s_2} s_1 \to s_3 s_2 s_3 s_1$

$\to s_3 s_2 s_1 s_3$

$w = s_2 s_3 s_2 s_1 = s_3 s_2 s_3 s_1$

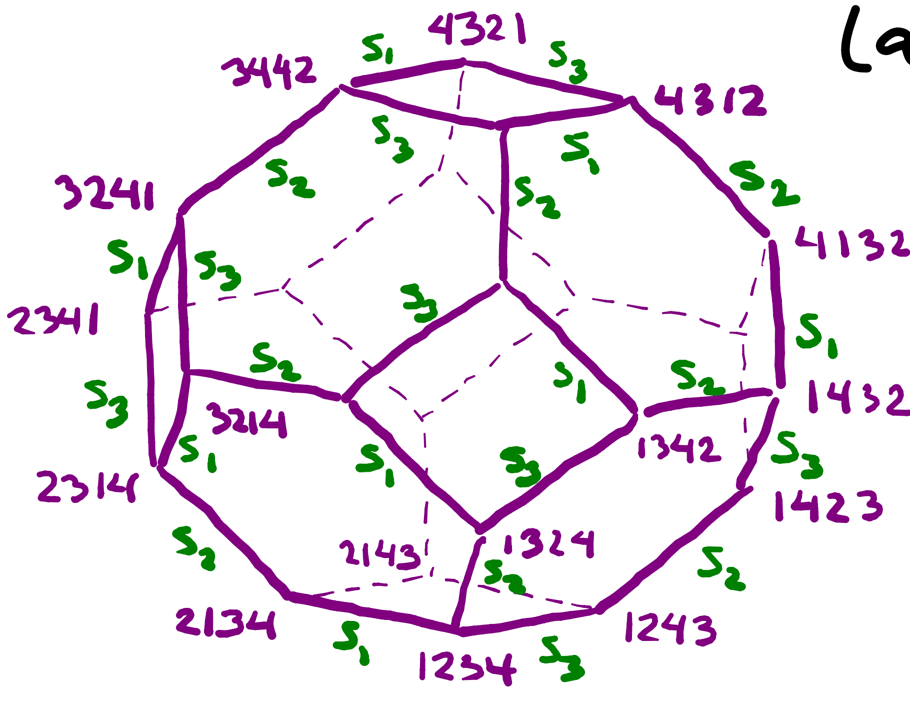
$= s_3 s_2 s_1 s_3$

left weak order:



Thm (Edelman & Björner): Weak Bruhat order has $\Delta(u,v) \simeq$ ball or sphere, hence $\mu(u,v) = 0, \pm 1$ for all $u \leq v$.

Idea: Use "Quillen fiber lemma" (a.k.a. Quillen Theorem A)



A topological-combinatorial tool:

Quillen Fiber Lemma: Given a poset map $f: P \to Q$ s.t. $q \in Q \Rightarrow \Delta(\{p \in P \mid f(p) \leq q\})$ is contractible, then $\Delta(P) \simeq \Delta(Q)$.

Remark: Used extensively in finite group theory to characterize groups via subgp lattice & in combinatorics.

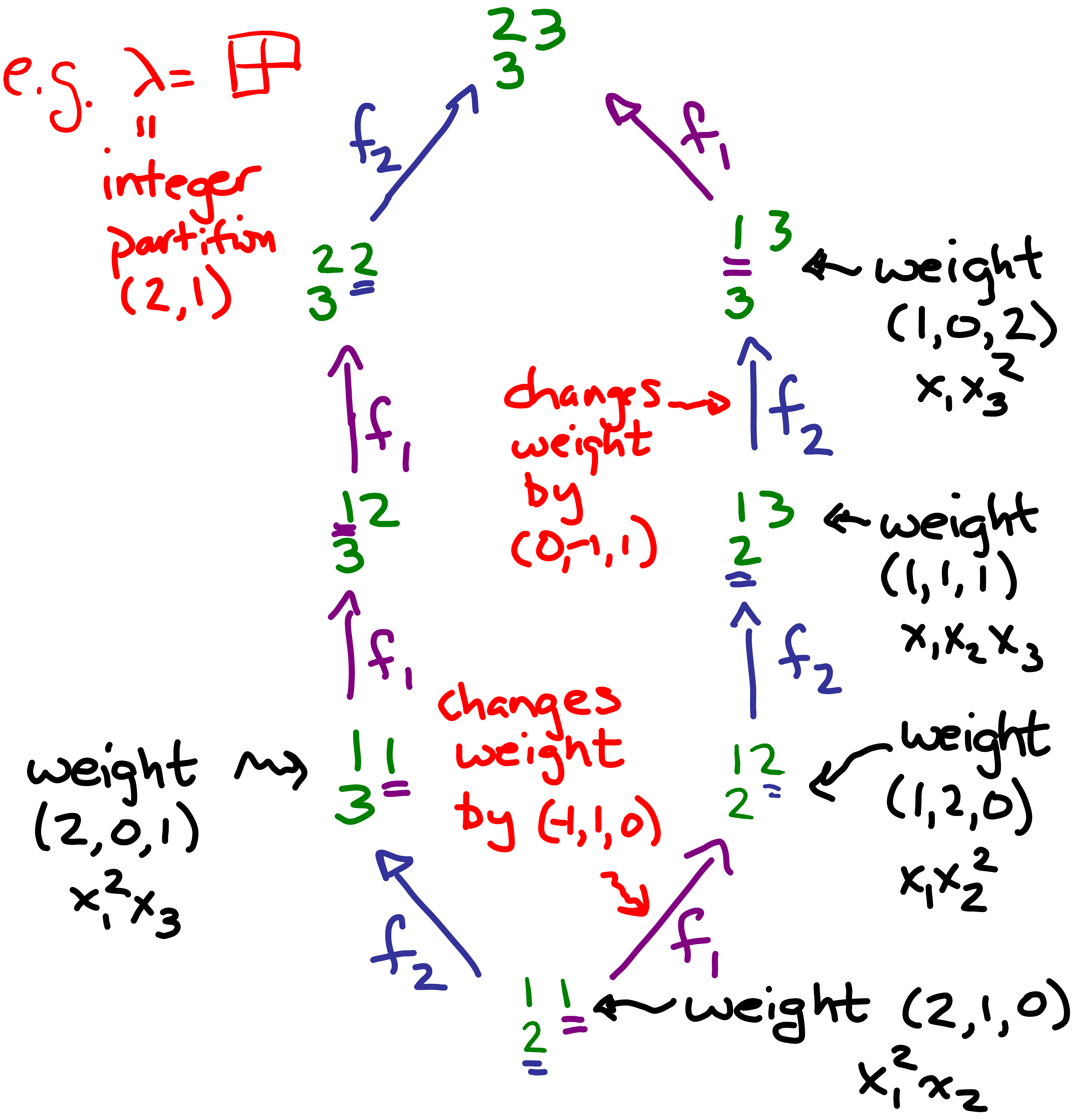
Crystal Graphs

Purpose: Study rep'n theory of Kac-Moody algebras (e.g. affine Lie algebras) & their characters via combinatorics of "crystal graphs"

Crystal Graphs Arising as Posets

- poset elts $\leftrightsquigarrow$ basis vectors for the various weight spaces
  (guaranteed to exist by properties of Kashiwara's "crystal basis")
- cover rel'ns $\leftrightsquigarrow$ crystal (lowering) f_i operators

(Type A) Crystals of Highest Weight Rep'ns & their Kashiwara Lowering Operators



Type A crystal for highest weight rep'n of shape λ

1. $\hat{0} = \begin{array}{l} 111\ldots1 \\ 22\ldots2 \\ 33\ldots \\ \vdots \end{array}$ of shape λ

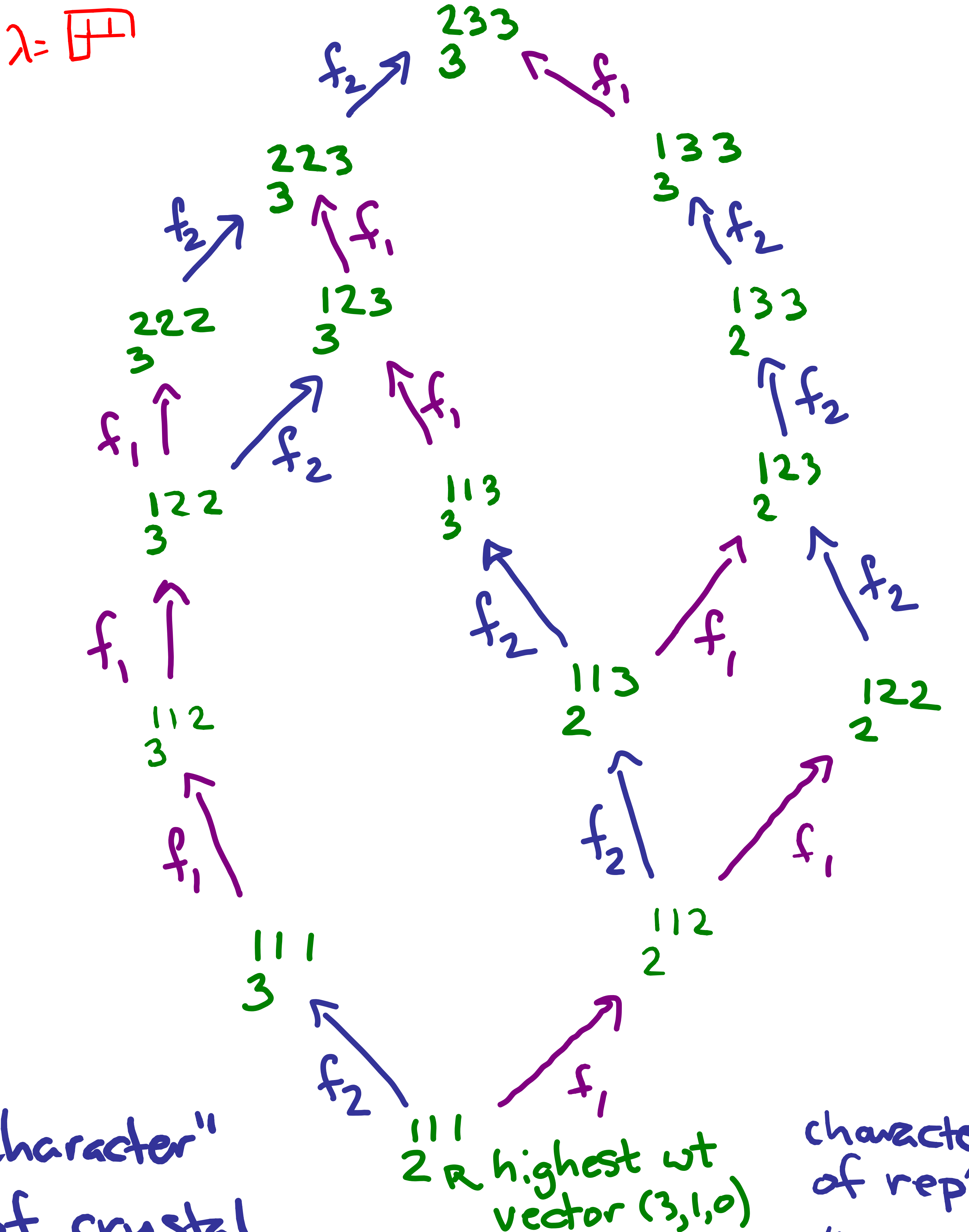
   "highest weight vector"

2. $u \xrightarrow[i]{} v$ has v obtained from u by incrementing to $i+1$ rightmost i not in "parenthesization pair" with an $i+1$

e.g. $\begin{array}{l} 11111444 \\ 2233 \\ \boxed{3}44 \end{array} \overset{f_3}{\rightsquigarrow} \begin{array}{l} 11111444 \\ 2233 \\ \boxed{4}44 \end{array}$

$\boxed{3}4433444 \rightsquigarrow \boxed{4}4433444$

<u>Parenthesization Pairs</u>: Read leftmost column bottom to top, then subsequent columns likewise, ignoring all but i's & $i+1$'s. Pair up consecutive $i+1$ then i, deleting both. Repeat pairing/deleting until no more such pairs.



"character" of crystal

character of rep'n

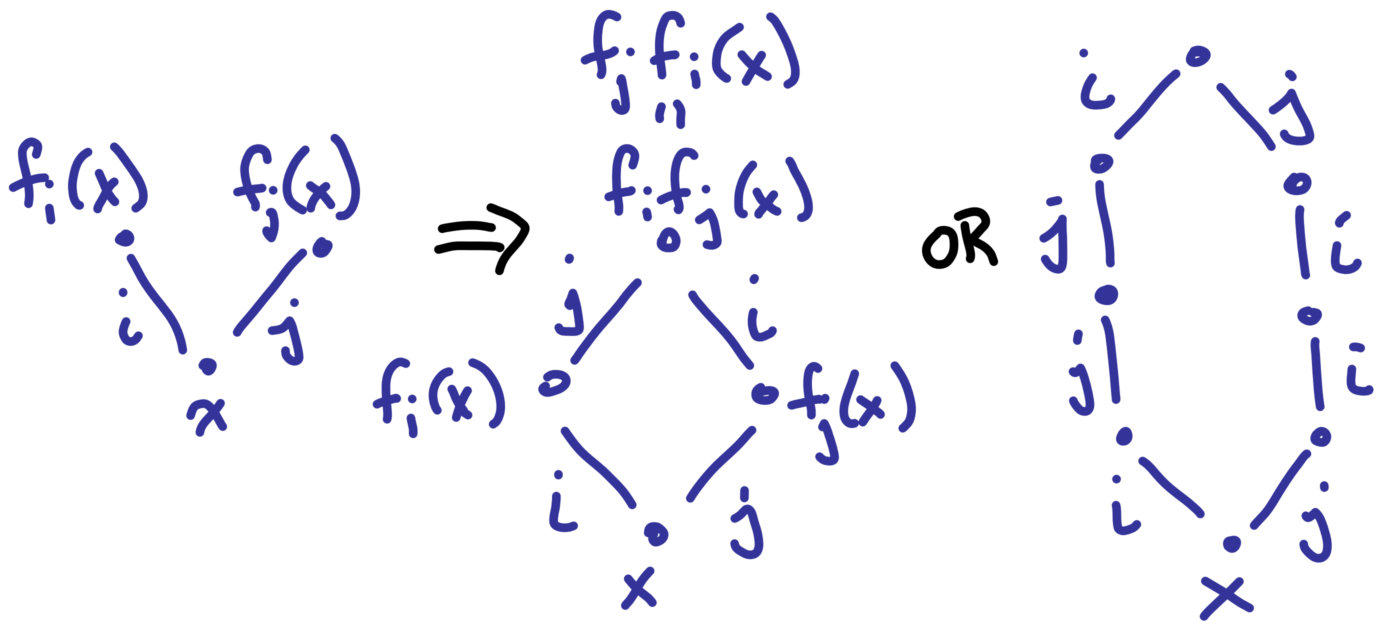
$\|$

$$= x_1^3 x_2 + x_1^2 x_2^2 + \dots = \mathrm{weight}\begin{pmatrix} 1\,1\,1 \\ 2 \end{pmatrix} + \mathrm{weight}\begin{pmatrix} 1\,1\,2 \\ 2 \end{pmatrix} + \dots$$

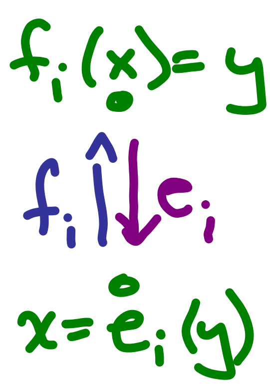
Stembridge: "$\mathfrak{g}$-crystals"

(Crystals of highest weight rep'ns in simply laced case)

a)



b) likewise for e_i, e_j "raising operators":



c) reln's depend on location

Two Types of Motivation

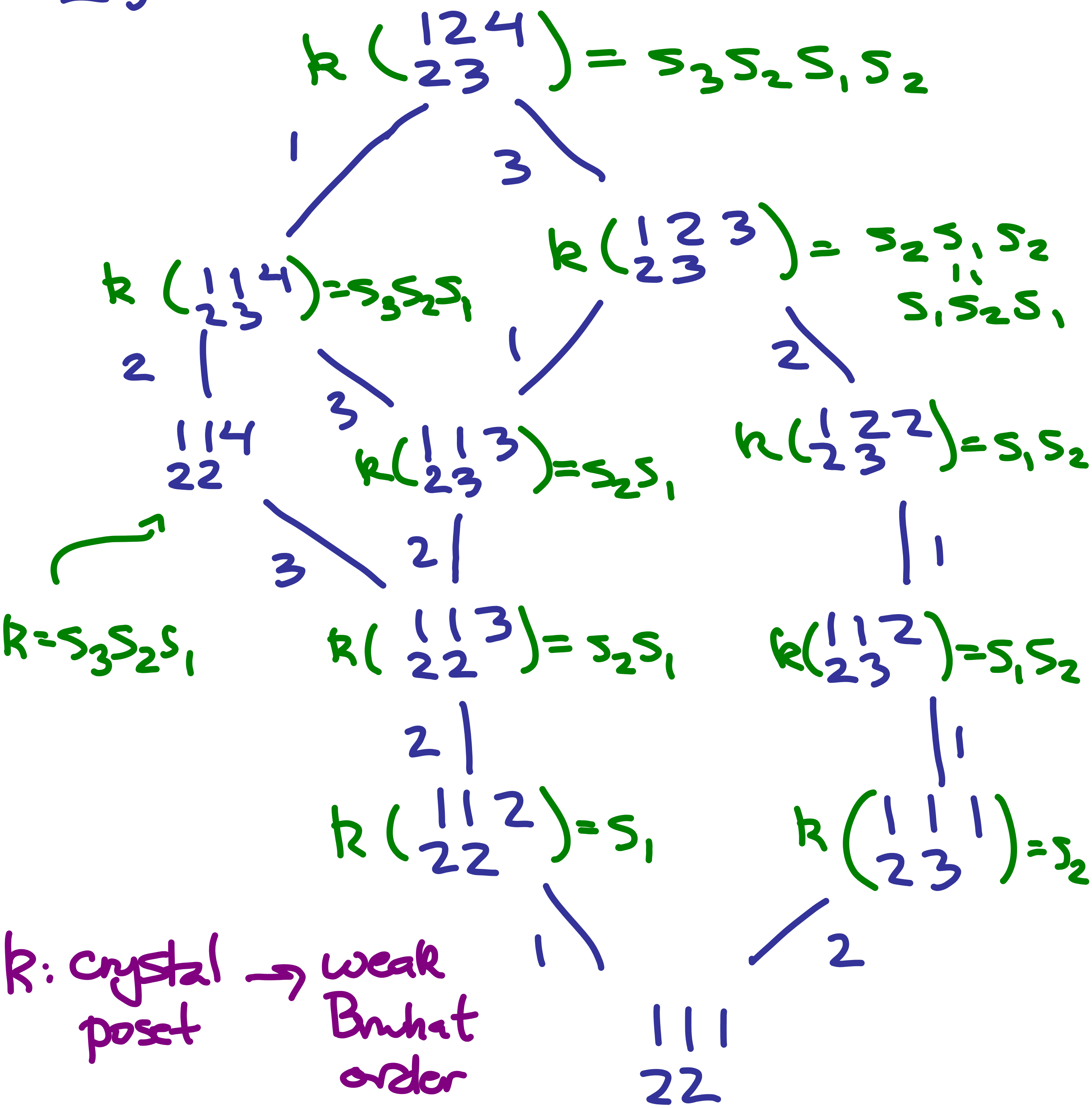
1. Given quantized enveloping algebra $U = U^- \otimes_{\mathbb{Q}(v)} U^0 \otimes_{\mathbb{Q}(v)} U^+$, the canonical basis (or crystal basis) B has the remarkable property that each highest weight module V_λ has a basis $\{v_\lambda b \mid v_\lambda b \neq 0\}$, i.e. the elements of crystal poset.

2. Technique to prove "Schur positivity": crystal character is a Schur function

Transferring Structure from Weak Order to Crystals via Poset Map Called "Key"

- related to "key polynomials" of Lascoux & Schützenberger
- Schubert polynomials (describing cohomology classes in Schubert calc.) are positive sums of key polynomials

Right key "k" of a "KM"-crystal



k: crystal poset $\rightarrow$ weak Bruhat order

$u\leq v \Rightarrow k(u)\leq k(v)$

key $k(\hat{0})=e$

New Algorithm to Calculate Right Key of a KM-Crystal

(1) $\mathrm{key}(\hat{0}) = e$

(2) if $\hat{0} \xrightarrow[i]{} a$, then $\mathrm{key}(a) = s_i$

(i.e. $\hat{0} \lessdot a$)

(3) if v covers 2 or more elements

then $\mathrm{key}(v) = \bigvee_{\{u \mid u \to v\}} \mathrm{key}(u)$

(for join taken in weak order)

(4) if $u \xrightarrow[i]{} v$ and v does not cover any other elements, then:

(a) $\mathrm{key}(v) = \mathrm{key}(u)$ if $\exists\, u' \xrightarrow[i]{} u$

(b) $\mathrm{key}(v) = s_i \cdot \mathrm{key}(u)$ otherwise

Thm (H.-Lenart): For $\hat{0}$ = highest weight vector in symmet. KM-crystal, $\Delta(\hat{0},u) \simeq$ ball unless $u = \min(R^{-1}(\omega_0(W_J)))$ where $\Delta(\hat{0},u) \simeq S^{|J|-2}$.

Proof: Use Quillen fibre lemma with

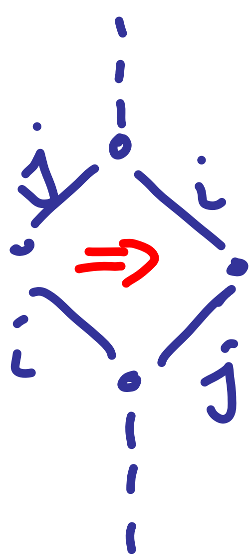
$$f : \text{crystal} \longrightarrow \text{Boolean algebra} \quad \left(\begin{array}{l}\text{poset of subsets}\\ \text{of } \{1,\ldots,n\}\end{array}\right)$$

$$x \longmapsto \max\{S \mid \omega_0(J_S) \leq_{weak} \text{key}(x)\}$$

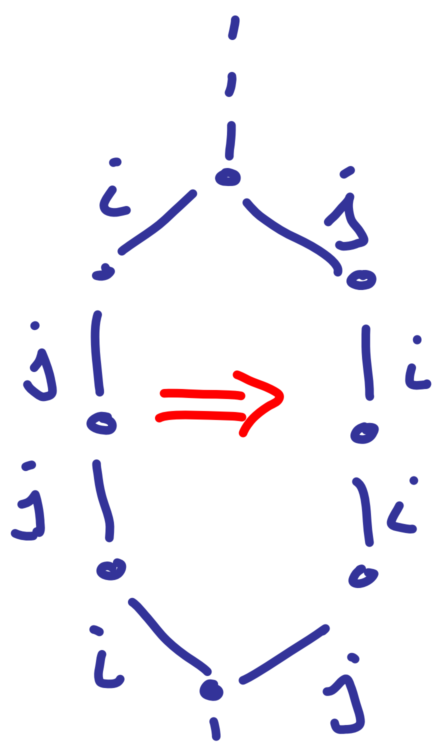
Corollary: $\mu(\hat{0},u) = 0, \pm 1$

highest weight vector

Thm (H.-Lenart): Given any lower interval $(\hat{0}, u)$ in a $\mathfrak{g}$-crystal, then set of saturated chains from $\hat{0}$ to u is connected by "Stembridge moves", namely moves of the form:



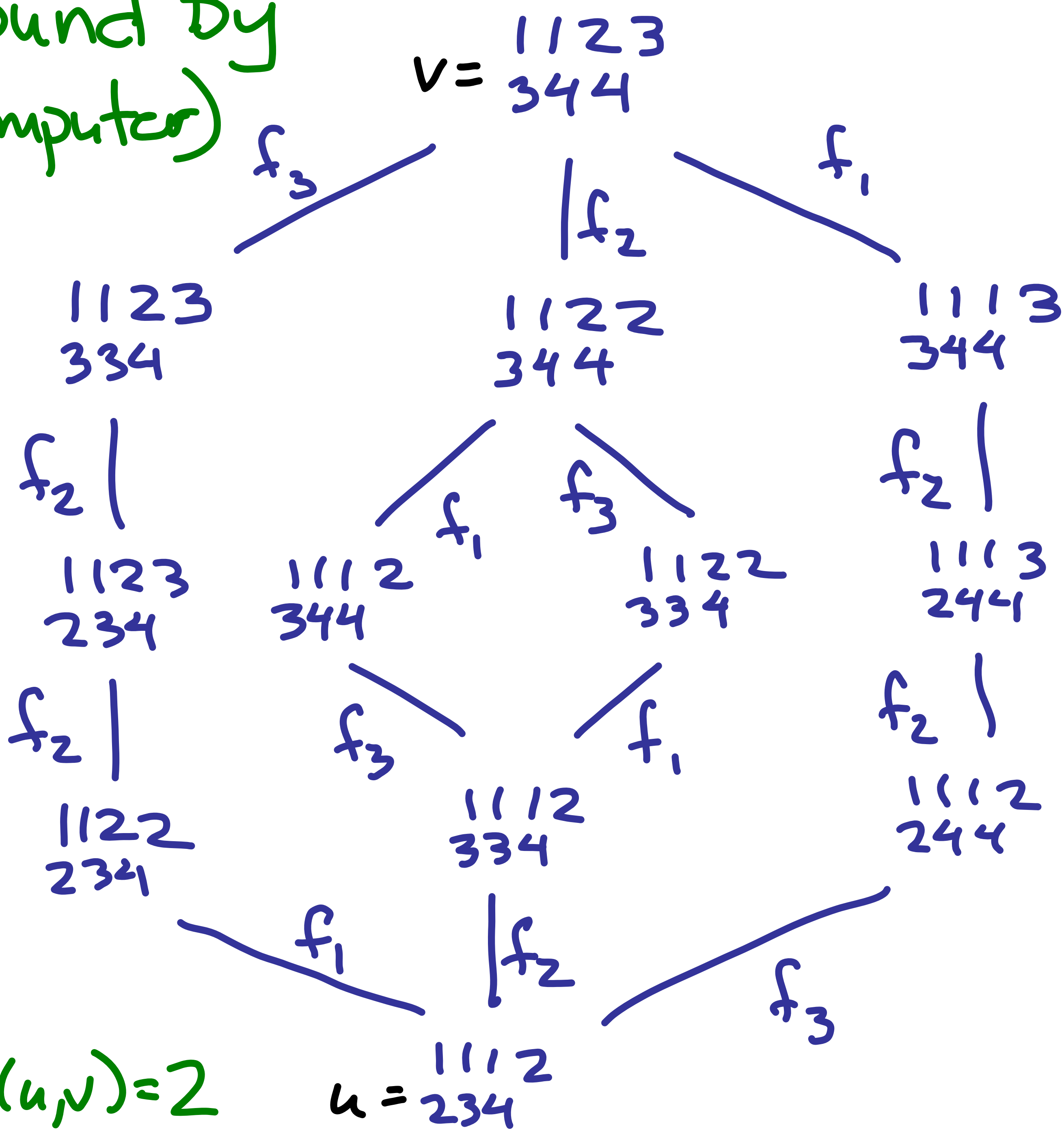
and



Note: Likewise in doubly-laced case via "Sternberg moves".

H.-Lenart: New Reln's

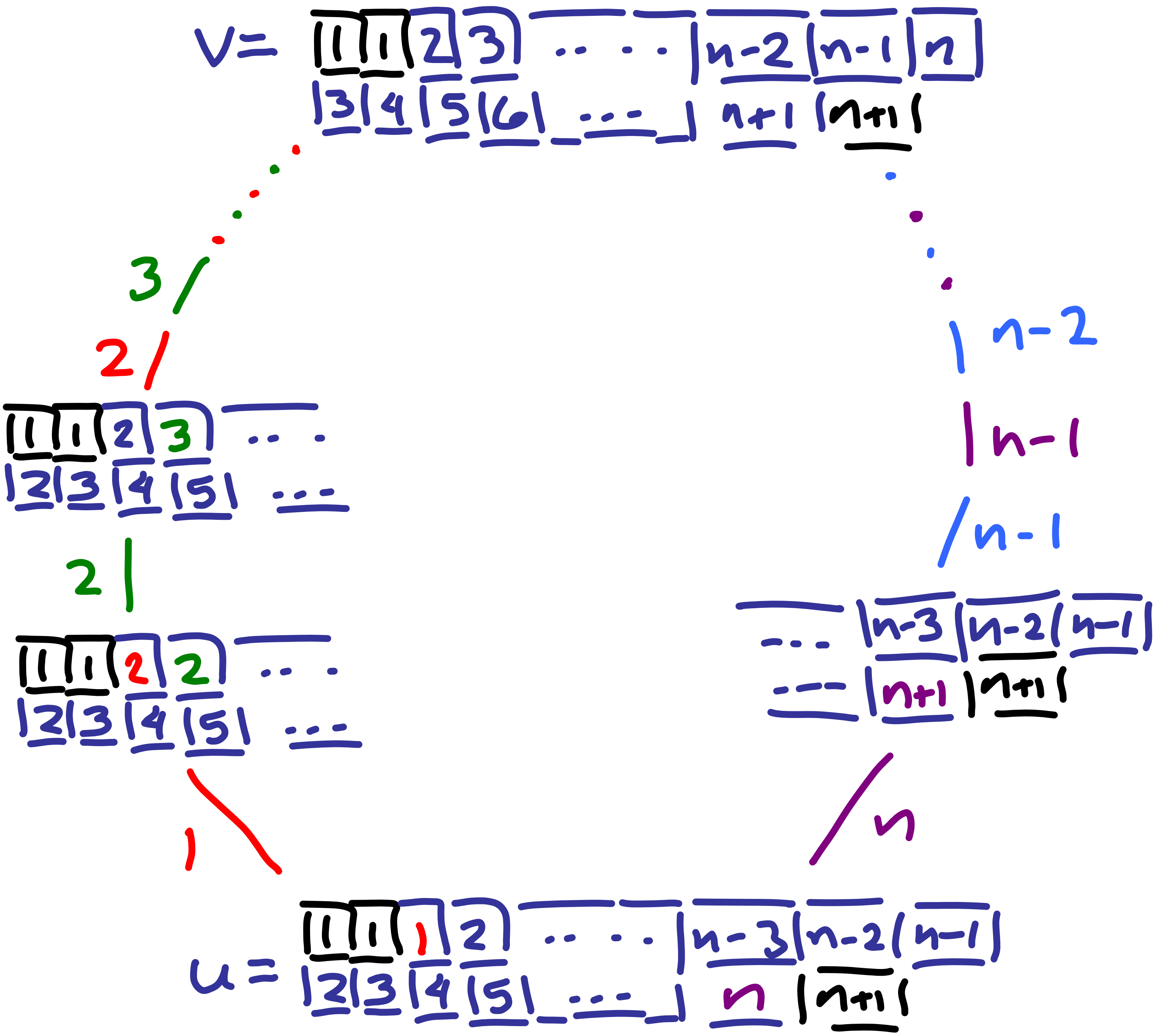
(found by computer)



$\mu_P(u,v) = 2$

& not connected by "Stembridge moves"

Arbitrarily High Rank Disconnected Open Intervals



label sequences: $1, 2, 2, 3, 3, 4, 4, \ldots, n-1, n-1, n$

& $n, n-1, n-1, \ldots, 2, 2, 1$ in distinct components

Thm (H.-Lenart) There exist $u<v$ s.t. $\mu(u,v)=2^j$ for all positive integers j.

Thm (H.-Lenart): $\mu(u,v)\neq 0,\pm 1$ in q-crystal $\Rightarrow$ relation within $[u,v]$ not implied by Stembridge local relations.

Proof: Theory of SB-labelings of H-Meszàròs.

Appendix: a few slides with extra details...

Crystals

A crystal B of type Φ is a nonempty set B with raising & lowering operators e_i, f_i & fns

$$\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$$

$$wt : B \to \Lambda = \text{weight lattice of type } \Phi$$

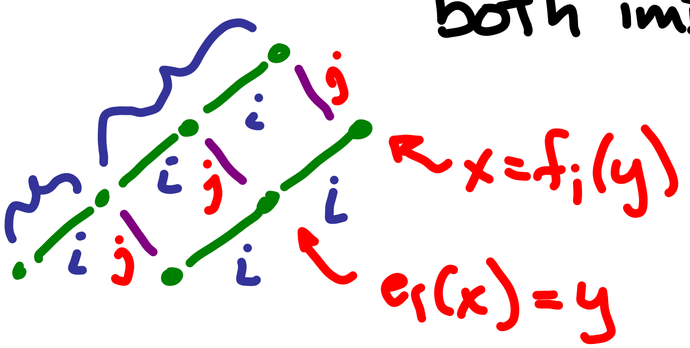
s.t.

(A1) $x, y \in B$, then $e_i(x) = y \iff x = f_i(y)$

both implying $wt(y) = wt(x) + \alpha_i$

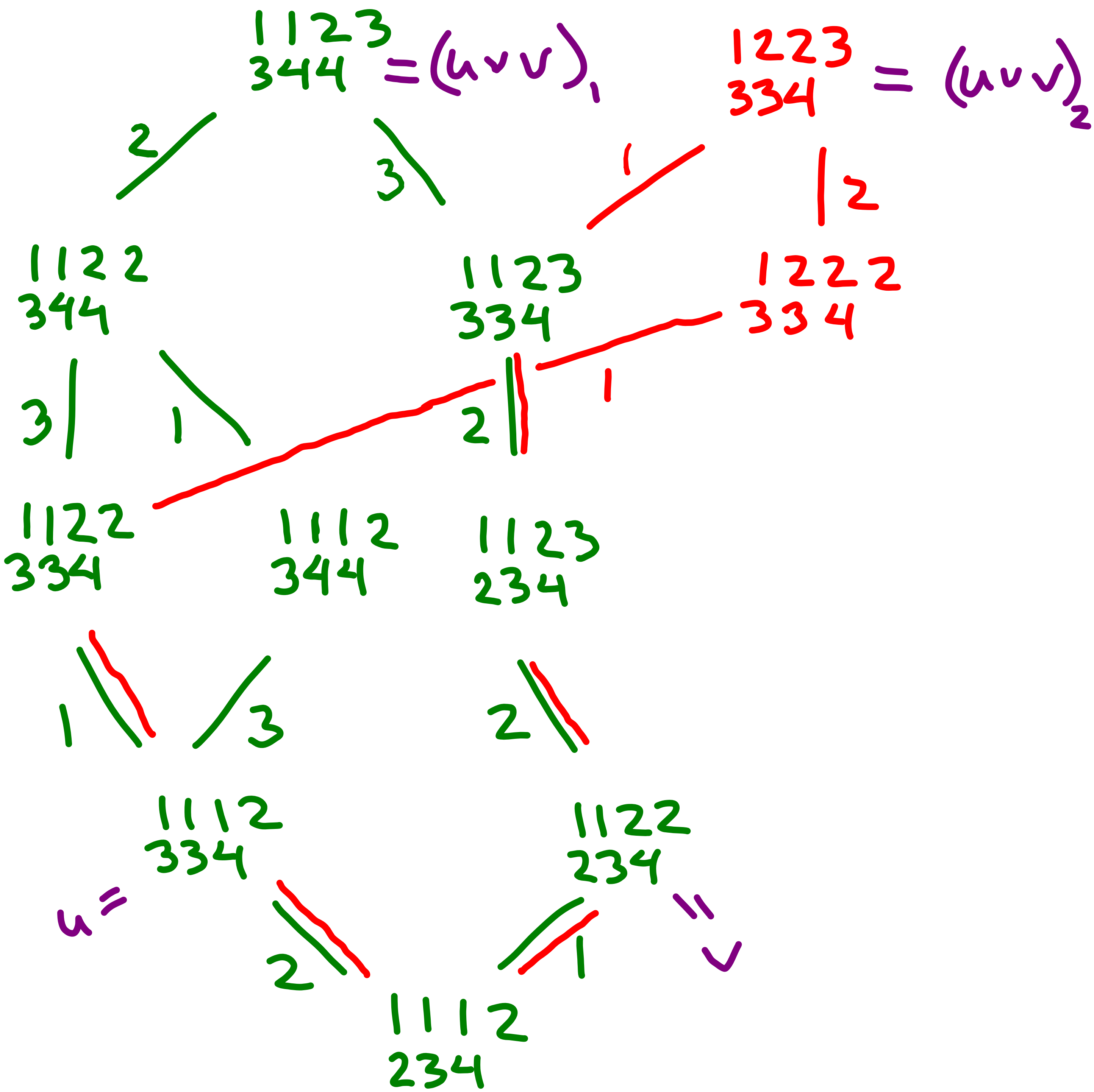
& $\varepsilon_i(y) = \varepsilon_i(x) - 1$

$\varphi_i(y) = \varphi_i(x) + 1$



(A2) $\varphi_i(x) - \varepsilon_i(x) = \langle wt(x), \alpha_i^\vee \rangle$

Non-Lattice Example:



Examples with $\mathcal{M}(u,v) = 2^j$

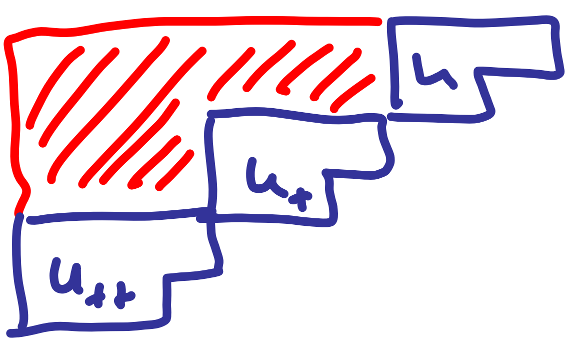
$j=1$: $\quad u = \begin{array}{l} 1112 \\ 234 \end{array} \qquad v = \begin{array}{l} 1123 \\ 344 \end{array}$

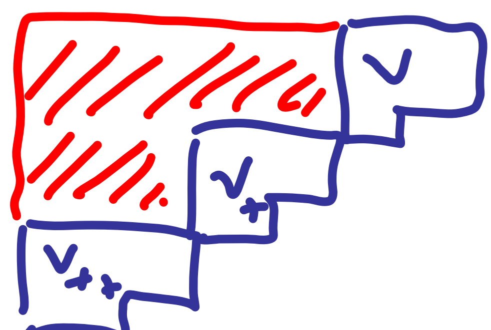
$j=2$: $\quad u^{(2)} = \begin{array}{ll} 1111 & \boxed{\begin{array}{l}1112\\234\end{array}} = u \\ 2222 & \\ \boxed{\begin{array}{l}6667\\789\end{array}} & \end{array} \qquad v^{(2)} = \begin{array}{ll} 1111 & \boxed{\begin{array}{l}1123\\344\end{array}} = v \\ 2222 & \\ \boxed{\begin{array}{l}6678\\899\end{array}} & \end{array}$

$u_+ := u+5 = \begin{array}{l} 6667 \\ 789 \end{array} \qquad v_+ := v+5 = \begin{array}{l} 6678 \\ 899 \end{array}$

$$[u^{(2)}, v^{(2)}] \cong [u,v] \times [u,v]$$

so $\mathcal{M}(u^{(2)}, v^{(2)}) = 2^2$

$u^{(3)} =$ 

$v^{(3)} =$ 

$$[u^{(k)}, v^{(k)}] \cong \underbrace{[u,v] \times \dots \times [u,v]}_{k\text{-fold}} \qquad \mathcal{M} = 2^k$$