Sharp Representation Stability Bound via Young Symmetrizers & "Ribbon" Basis for Rank-Selected Homology of Partition Lattice

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Plan for Today

I. Background

II. Main Results

III. Sharp Uniform Representation Stability Bound $2\max S - |S| + 1$ for rank-selected homology β_S of Boolean lattice (also warm-up for more difficult $\beta_S(\Pi_n)$)

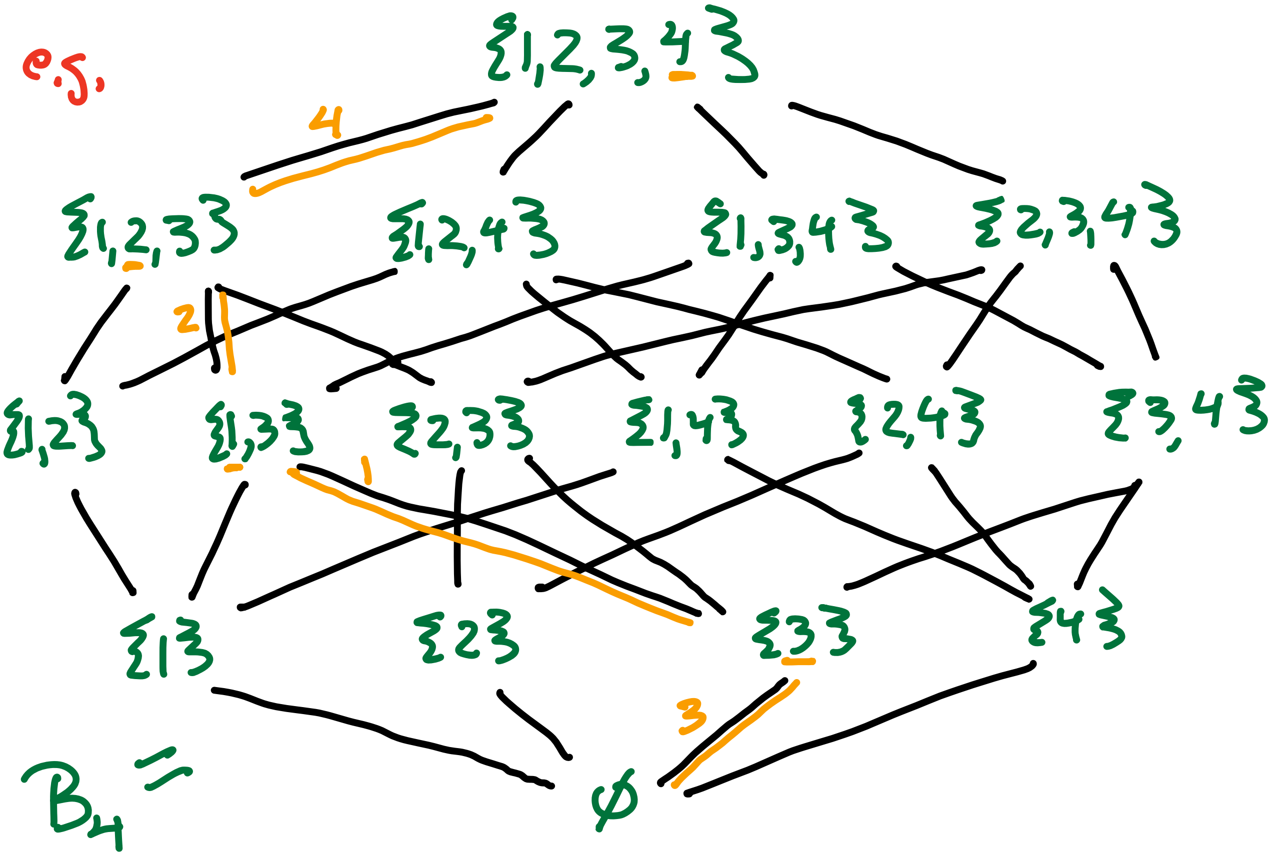
IV. New "ribbon basis" for rank-selected homology & Whitney homology of geometric lattices
- interacts well with Young-symmetrizers

V. Applic: sharp uniform representation stability bound $4\max S - |S| + 1$ (conjectured in H-Reiner) for $\beta_S(\Pi_n)$ & $WH_S(\Pi_n)$.

Boolean Lattice B_n and its S_n-representations

e.g.

$B_4 =$

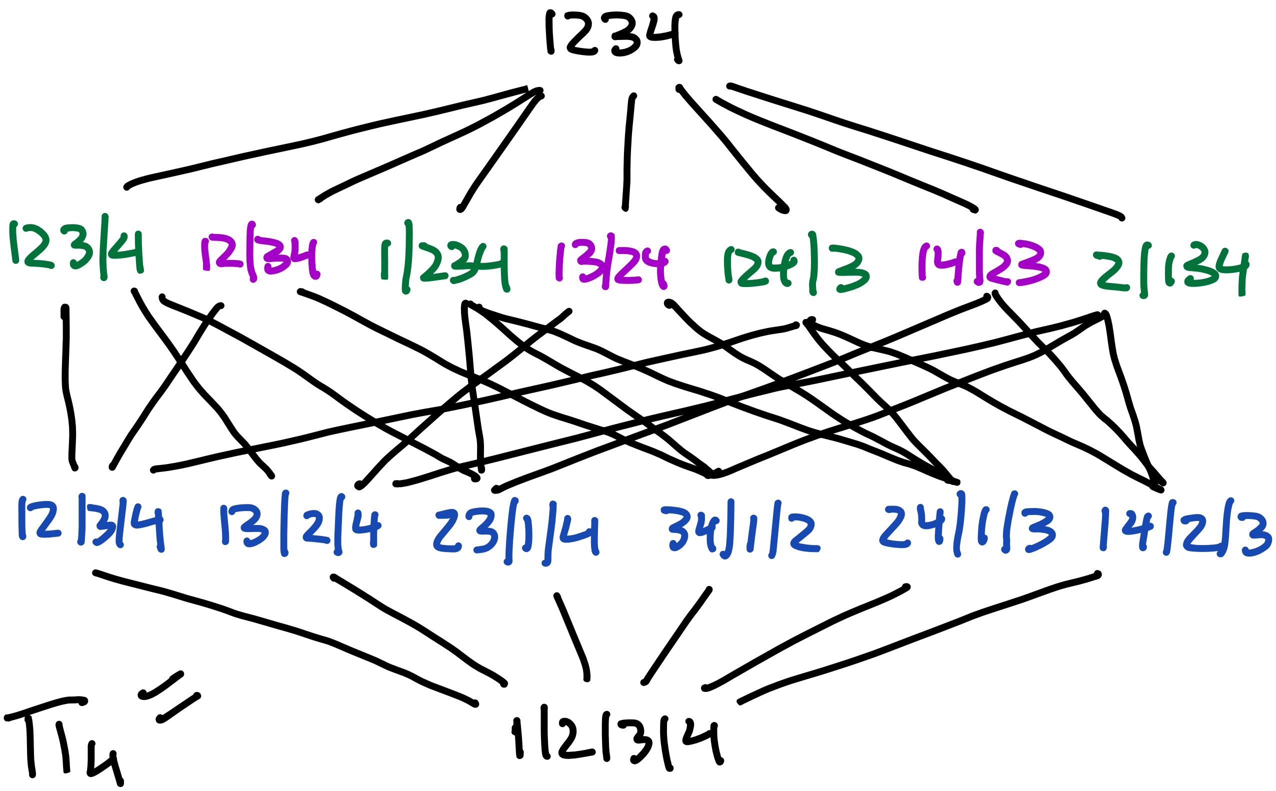


- S_n acts on B_n by permuting values
- action preserves $<$ so induces action on chains
- Maximal chains in B_n $\xleftrightarrow{\text{bijection}}$ permutations of $1,\ldots,n$ (label sequences)

$\hat{0} < \{3\} < \{1,3\} < \{1,2,3\} < \{1,2,3,4\} \longleftrightarrow \boxed{3}\boxed{1}\boxed{2}\boxed{4}$

Partition Lattice Π_n and its S_n-Representations

$\Pi_4 =$



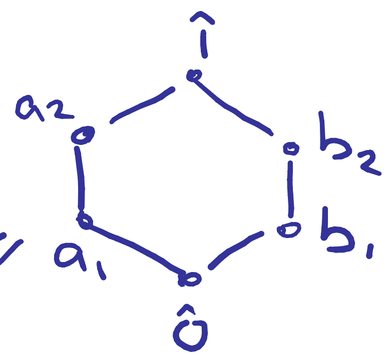
- S_n acts on Π_n by permuting values

  e.g. $(13)[12|3|45] = 32|1|45 \in \Pi_5$

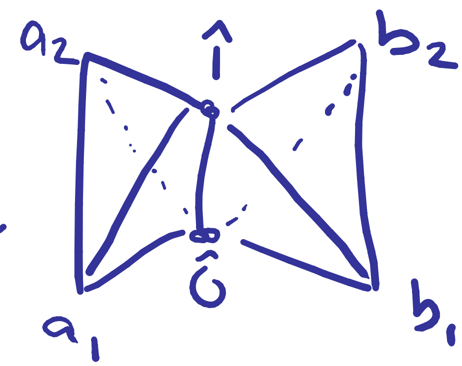
- Action preserves $<$ so permutes chains $u_1 < u_2 < \ldots < u_k$

Def'n: The **order complex** of a finite poset P is the simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains in P.

e.g. $P =$



$\Delta(P) =$



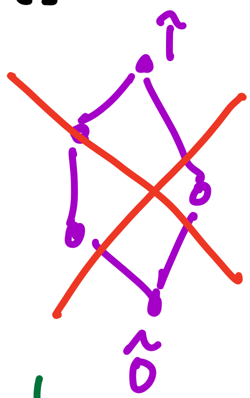
- Let $\overline{P} = P \setminus \{\hat{0}, \hat{1}\}$ e.g. for Π_n

Convention: When we speak of topological properties (homology, etc.) of poset P, we mean $\Delta(P)$ or $\Delta(\overline{P})$.

Poset rank := # steps from bottom

S_n-action on poset chains gives rise to S_n-representations on Homology Groups

Recall: P is graded if $u < v$ in P implies all maximal chains u to v have same length



- B_n graded of rank n
- Π_n graded of rank $n-1$

- S_n-action on B_n (resp. Π_n) preserves rank (distance from $\hat{0}$)
- Action on chains of P induces action on faces of $\Delta(P)$

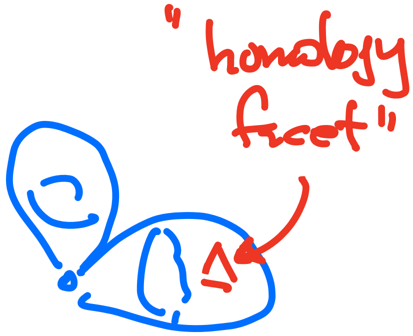
- S_n-action on chains of P commutes with bdry map for $\Delta(P)$

$$d(u_0<\cdots<u_r)=\sum_{i=0}^{r}(-1)^i(u_0<\cdots<\hat{u}_i<\cdots<u_r)$$

- Therefore, S_n-permutation rep'n on i^{th} chain gp of $\Delta(P)$ induces S_n-rep'n on i^{th} homology gp of $\Delta(P)$ (action sends cycles to cycles, bdry to bdry)

- But $\bar{B}_n$ & $\bar{\Pi}_n$ have homology concentrated in top degree

(due to shellability, implying wedge of top-dim'l spheres)



Def'n: Given a graded poset P of rank n with G-action preserving rank & $<$, for each $S \subseteq \{1,2,\ldots,n-1\}$, define:
$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\{s_1,s_2,\ldots,s_k\}}$

$\alpha_S(P) :=$ permutation rep'n of G on chains $c_1 < \cdots < c_k$ in P where $\mathrm{rank}(c_i) = s_i$ for $i=1,2,\ldots,k$

$$\beta_S(P) := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T(P) = \text{virtual } G\text{-rep'n}$$

Fact: If graded P with G-action is "Cohen-Macaulay" (e.g. shellable), then:

(a) $P^S = \{u \in P \mid rk(u) \in S\}$ has homology concentrated in top degree $\forall\, S \subseteq \{1,\ldots,rk P - 1\}$

& (b) $\beta_S(P)$ is an actual G-representation, namely rep'n on top homology of P^S coming from G-action on P^S.

Whitney Homology for Graded Posets

$$WH_S(\bar{P}) \cong \bigoplus_{rk(u)=\max S} \beta_{S\setminus \max S}(\overline{[\hat{0},u]})$$

- for $P=\Pi_n$ this decomposes into S_n-submodules:

$$WH_{S,\lambda}(\bar{\Pi}_n) \cong \bigoplus_{\substack{rk(u)=\max S \\ type(u)=\lambda}} \beta_{S\setminus \max S}(\overline{[\hat{0},u]})$$

e.g. $type(123|4|5) = (2,0,1,\dots)$

$= type(234|1|5)$

$\uparrow$ # parts of size 1 ; $\leftarrow$ # parts of size 3

$$WH_S(\bar{P}) \cong \beta_S(\bar{P}) \oplus \beta_{S\setminus \max S}(\bar{P})$$

Observation (H.-Reiner; essentially Orlik-Solomon)

$$WH_{\{1,2,\dots,i\}}(\bar{\Pi}_n) \cong_{S_n} H^i(PConf_n(\mathbb{R}^2))$$

Thm (H-Reiner):

$WH_{\{1,\dots,i\}}(\bar{\Pi}_n)$ "stabilizes" at $3i+1$

Rank-Selected Homology $\beta_S(\overline{\Pi}_n)$ of Partition Lattice Π_n

$S = \{1,2,\ldots,n-2\}$ case:

$$\beta_{\{1,2,\ldots,n-2\}}(\overline{\Pi}_n) \cong H_{top}(\overline{\Pi}_n)$$

Thm (Hanlon + Stanley):

$$\beta_{1,\ldots,n-2}(\overline{\Pi}_n) \cong_{S_n} \mathrm{sgn}_n \otimes \left(\varphi \uparrow_{C_n}^{S_n}\right)$$

where φ is linear rep'n sending generator for C_n to $e^{2\pi i/n}$

Rk: This is an $(n-1)!$-dim'l rep'n of S_n

Thm (Joyal): $\beta_{1,\ldots,n-2}(\overline{\Pi}_n) \cong_{S_n} \mathrm{sgn}_n \otimes \underbrace{\mathrm{lie}_n}_{\text{multilinear part free Lie alg.}}$

& nice pfs by Barcelo, Wachs

$\beta_S(\overline{\Pi}_n)$ for other rank sets S?

Qn (Stanley, '82): Understand $\beta_S(\overline{\Pi}_n)$ $\forall$ $S \subseteq \{1,\ldots,n-2\}$

- Lots of past work on $\langle \beta_S(\overline{\Pi}_n), \mathbb{1} \rangle$, but $\beta_S(\overline{\Pi}_n)$ not well understood (encapsulates hard questions on plethysm)

Much of our focus: "representation stability" for $\beta_S(\overline{\Pi}_n)$ for any fixed S (as n grows)

Aside: For P geometric lattice, P^S is matroid-analogue of partial flag variety

Representation Theoretic Stability

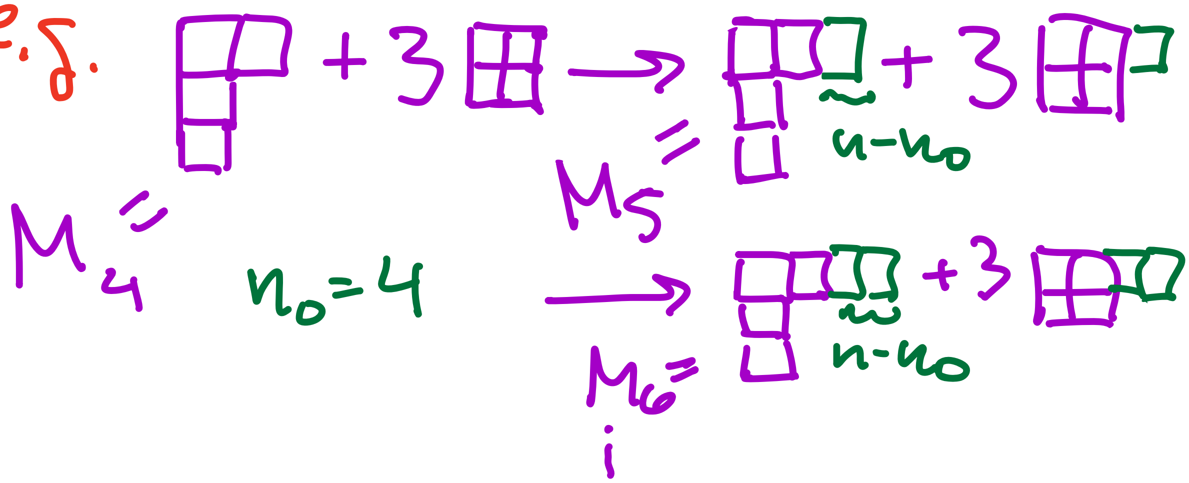
Def'n (Church–Farb): A series of S_n-modules $M_1, M_2, M_3, \ldots$ for $n = 1, 2, 3, \ldots$ stabilizes at $n = n_0$ if

$$M_{n_0} = \bigoplus_{\lambda \vdash n_0} c_\lambda S^\lambda \Rightarrow$$

$$M_n = \bigoplus_{\lambda \vdash n_0} c_\lambda S^{(\lambda_1 + n - n_0, \lambda_2, \lambda_3, \ldots)}$$

(for $n \geq n_0$)

e.g.



How Representation Stability Arises in WHs (but NOT Bs)

- Have finite # irreps S^λ s.t.

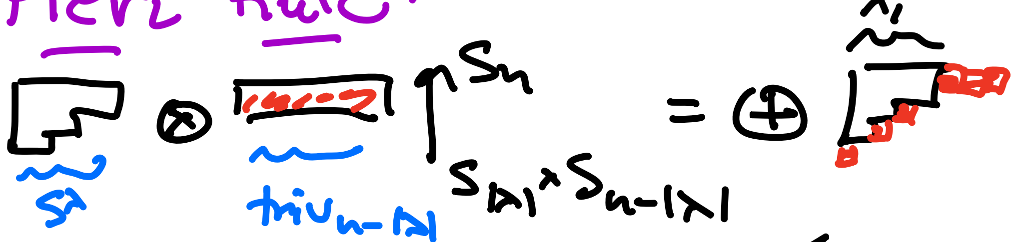
(1) S^λ 1st appears in $M_{|\lambda|}$

(2) each M_n for $n \geq |\lambda|$ has

$$S^\lambda \otimes \mathbb{1}_{n-|\lambda|} \uparrow_{S_{|\lambda|} \times S_{n-|\lambda|}}^{S_n} \quad \text{component}$$

(3) All S^μ are in some $S^\lambda \otimes \mathbb{1}_{n-|\lambda|} \uparrow_{S_{|\lambda|} \times S_{n-|\lambda|}}^{S_n}$

- Pieri Rule:



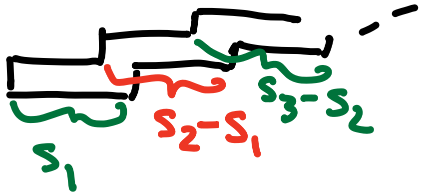
implies stability at $\max\{|\lambda| + \lambda_1\}$

- $\hat{M} = \bigoplus S^\lambda$ as above, "essential part" of M

S_n- Module Structure for $\beta_S(\bar{B}_n)$

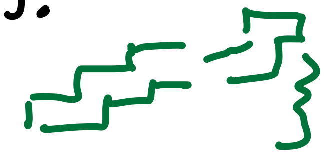
Thm (Solomon): $\beta_S(\bar{B}_n) \cong_{S_n} S$ 
for $S=\{s_1,\ldots,s_k\}$

Specht module of ribbon shape

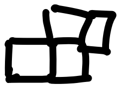


Non-standard Explanation (that foreshadows results for geometric lattices)

Step 1: Consider well-known Specht module basis (polytabloid basis):

$\mathcal{B}=\{v_T \mid T$ std of shape  $\}$

for S .

e.g. $n=4$, $S=\{2\}$: $\mathcal{B}=\{\underbrace{v_{\begin{smallmatrix} & 2 & 4 \\ 1 & 3 & \end{smallmatrix}},\ v_{\begin{smallmatrix} & 1 & 4 \\ 2 & 3 & \end{smallmatrix}},\ldots}_{5\text{ vectors}}\}$ for 

with $v_{\begin{smallmatrix} & 2 & 4 \\ 1 & 3 & \end{smallmatrix}} = \begin{smallmatrix} \overline{2\ 4} \\ \overline{1\ 3} \end{smallmatrix} - \begin{smallmatrix} \overline{3\ 4} \\ \overline{1\ 2} \end{smallmatrix}$ (alt. sum of tabloids)

Step 2: Apply S_n-equivariant quotient map "$\bar{f}_{chain}$" to $\mathcal{B}$ to get "ribbon" basis for $\beta_S(\bar{B}_n)$

- $f_{chain}: \left\{\begin{matrix}\text{ribbon}\\ \text{fillings}\end{matrix}\right\} \xrightarrow[\text{bijection}]{} \left\{\begin{matrix}\text{max chains}\\ \text{in } B_n\end{matrix}\right\}$

$\Downarrow$ quotient

- $\bar{f}_{chain}: \left\{\begin{matrix}\text{ribbon}\\ \text{tabloids}\end{matrix}\right\} \xrightarrow[\text{bijection}]{} \left\{\begin{matrix}\text{max chains}\\ \text{in } B_n^S\end{matrix}\right\}$

e.g. $S = \{2\}$, $n = 5$

$$f_{chain}\left(\begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 2 & 4 & 5 \\ \cline{1-4} 1 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}\right) = \emptyset < \{1\} < \{1,3\} < \{1,3,2\} < \ldots$$

$$\bar{f}_{chain}\Bigg(\underbrace{\begin{array}{cccc} \cline{2-4} & 2 & 4 & 5 \\ \cline{1-4} 1 & 3 & & \\ \cline{1-2} \end{array}}_{\text{tabloid}}\Bigg) = \underbrace{\emptyset < \{1,3\} < \{1,2,3,4,5\}}_{\text{add entire rows in single steps}}$$

Step 3: Use shellability theory to prove $\bar{f}_{chain}(\mathcal{B})$ is indeed basis for $\beta_S(\bar{B}_n)$.

e.g. $\bar{f}_{chain}\left(\begin{array}{cccc} \cline{2-4} & 2 & 4 & 5 \\ \cline{1-4} 1 & 3 & & \\ \cline{1-2} \end{array} - \begin{array}{cccc} \cline{2-4} & 3 & 4 & 5 \\ \cline{1-4} 1 & 2 & & \\ \cline{1-2} \end{array}\right)$

$$= \{1,3\} - \{1,2\} \in \beta_{\{2\}}(\bar{B}_5)$$

(notation: omit $\emptyset$ & $\{1,2,3,4,5\}$)

II. Main Results of Talk

1. Sharp Uniform Representation Stability Bound for $\beta_S(\bar{B}_n)$ as S held fixed and n grows.

2. New "ribbon basis" for $\beta_S(\bar{P})$ and $WH_S(\bar{P})$ for any geometric lattice P.

3. Proof of H.-Reiner Conjecture, i.e. sharp uniform representation stability bound for $\beta_S(\bar{\Pi}_n)$ & $WH_S(\bar{\Pi}_n)$ for fixed S as n grows

Technique for Stability Bound: Apply Young symmetrizer to ribbon basis for V to show $\langle S^\lambda, V\rangle = 0$ for $\lambda_1 >$ bound $- |\lambda|$

III. Stability Bound for $\mathcal{B}_S(B^n)$

Thm (H.-Sundaram): The S_n-modules $\mathcal{B}_S(\bar{B}^n)$ for fixed $S = \{s_1, s_2, \ldots, s_k\} \subseteq \{1, \ldots, n-1\}$ stabilize sharply at $2\max S - |S| + 1$.

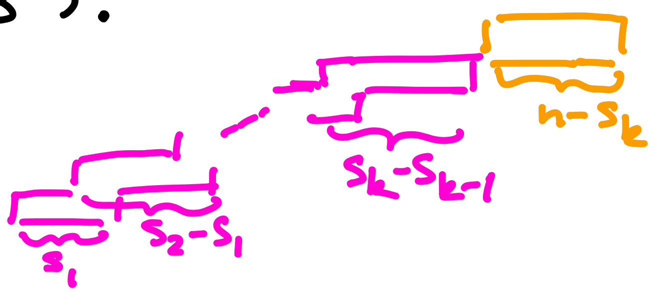
Idea: • Prove $|\lambda| + \lambda_1 \leq 2\max S - |S| + 1$ for each $S^\lambda \in \underbrace{\widehat{WH}_S(\bar{B}^n)}$ (that's enough!)

"essential part" of $WH_S(\bar{B}^n)$

• Def'n of $\widehat{WH}_S(\bar{B}^n) \Rightarrow |\lambda| = \max S$ for $S^\lambda \in \widehat{WH}_S(\bar{B}_n)$

• Passing to $\widehat{WH}_S(\bar{B}^n)$:

$$\widehat{WH}_S \otimes \mathbb{1}_{n-s_k} \Big\uparrow_{S_{s_k}\times S_{n-s_k}}^{S_n} \cong_{S_n}$$



• Use ribbon basis $f_{\text{chain}}(B)$ for $\widehat{WH}_S$ to prove $\lambda_1 \leq \max S - |S| + 1$

Showing $\lambda_1 \le \max S - |S| + 1$:

e.g. $n=7$ $\quad S=\{2,3\}$ $\qquad \begin{smallmatrix} & & 2 & 3 & 6 & 7 \\ & 4 & & & & \\ 1 & 5 & & & & \end{smallmatrix}$

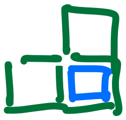
$\{1,5\} < \{1,5,4\} \longleftrightarrow \begin{smallmatrix} & 4 \\ 1 & 5 \end{smallmatrix}$

- $WH_{\{2,3\}}(\bar{B}^7) \cong \beta_{\{2\}}(\bar{B}^3) \otimes 1\!\!1_{S_4} \big\uparrow_{S_3 \times S_4}^{S_7}$

$\cong \underbrace{S^{\begin{smallmatrix} & \square \\ \square & \square \end{smallmatrix}}}_{\text{essential part}} \otimes S^{\square\square\square\square} \big\uparrow_{S_3 \times S_4}^{S_7}$

- Ribbon for essential part has exactly $\max S - |S| + 1$ columns:

e.g. $\quad \max S = 3 = \#$ boxes



$|S| - 1 = 1 = \#$ ribbon boxes having box directly above them

$\overset{\shortparallel}{\#\text{rows} - 1}$

$\max S - (|S|-1) = 3 - 1 = \#$ columns in ribbon

Young Symmetrizer Approach

Given filling T, e.g. $T = \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}$

Young symmetrizer: $b_T a_T = \Big(\sum_{\sigma \in C(T)} \operatorname{sgn} \sigma \cdot \sigma\Big)\Big(\sum_{\underbrace{\tau \in R(T)}_{\text{row symm.}}} \tau\Big)$

e.g. $(e - (12))(e + (13))$

Idea: If $\lambda_1 >$ # columns in $\underbrace{T'}_{\text{ribbon}}$, then

$$\underbrace{(b_T a_T)}_{\text{Young symmetrizer}} \underbrace{v_{T'}}_{\text{any basis element for } S^{\text{ribbon}}} = 0$$

because there exist two entries in 1st row of T and same column of T'

e.g. $T = \boxed{1}\boxed{2}\boxed{3}\boxed{4}$ ← of shape λ (so $\lambda_1 = 4$)

$T' = \begin{array}{|c|c|c|} \hline & 2 & 4 \\ \hline 1 & 3 & \\ \hline \end{array}$ ← ribbon shape (w/ # columns < 4)

$a_T = \sum_{\sigma \in R(T)} \sigma$ has factor $e + (23)$ with $v_{T'}$

$$(e + (23))\left(\overbrace{\begin{array}{ccc} & 2 & 4 \\ 1 & 3 & \end{array} - \begin{array}{ccc} & 3 & 4 \\ 1 & 2 & \end{array}}^{v_{T'}} \right) =$$

$$\left(\begin{array}{ccc} & 2 & 4 \\ 1 & 3 & \end{array} + \begin{array}{ccc} & 3 & 4 \\ 1 & 2 & \end{array} \right) - \left(\begin{array}{ccc} & 3 & 4 \\ 1 & 2 & \end{array} + \begin{array}{ccc} & 2 & 4 \\ 1 & 3 & \end{array} \right) = 0$$

Thus, $\underbrace{b_T a_T}_{\text{Young symmetrizer for } T} v_{T'} = b_T (\dots)\underbrace{(e+(23))v_{T'}}_{0} = 0$

- We show $b_T a_T v_{T'} = 0$ for all $v_{T'}$ in ribbon basis for V and all SYT T of shape λ w/ $\lambda_1 > \max S - |S| + 1$
- But $b_T a_T V = 0$ $\forall$ T of shape λ (w/ $\lambda_1 > \max S - |S| + 1$)
  $\Rightarrow \langle S^\lambda, V \rangle = 0$
- Thus we rule out all S^λ with $|\lambda| + \lambda_1 > 2\max S - |S| + 1$, yielding stability bound $2 \max S - |S| + 1$. $\square$

IV. Homology Basis for $\beta_S(\bar{P})$ & $WH_S(\bar{P})$ for all Geometric Lattices P

- totally order the atoms $a_1, a_2, \ldots, a_n$ in geometric lattice P

e.g. $12 < 13 < 23 < 14 < 24 < 34 < \cdots$ in Π_n

(recall: "atoms" over $\hat{0}$)

- label $u \lessdot v$ with $\lambda(u,v) = \min(A(v) \setminus A(u))$ where $A(x) = \{\text{atoms} \le x\}$

e.g. $\lambda(12|\underbrace{34|5}\lessdot 12|345) = 35$ (since $35 < 45$)

- label sequences on maximal chains of P are the ordered "NBC$^+$" bases of matroid
- this gives map f_{rib} sending max chains of P to fillings of ribbon $Rib(s_1, s_2 - s_1, s_3 - s_2, \ldots)$ with NBC$^+$ bases

row lengths

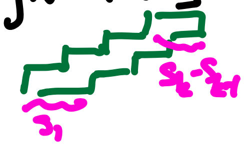
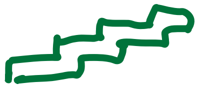
e.g. $S = \{2\}$ $\hat{0} < 12|3|4 < 12|34 < 1234 \rightarrow$

$$\begin{array}{cc} & \boxed{13} \\ \boxed{12} & \boxed{34} \end{array}$$

- (G-equiv't) "inverse" map

f_{chain}: matroid basis fillings $\longrightarrow$ max chains P

$\boxed{a_1}\boxed{a_2}\boxed{a_3} \longrightarrow a_1 \lessdot a_1 \vee a_2 \lessdot a_1 \vee a_2 \vee a_3$

- Lexicographic order on label sequences gives EL-shelling for $\Delta(\bar{P})$ which induces shelling for $\Delta(P^S)$
- Homology facets in $\Delta(P^S)$ shelling given by standard atom fillings of ribbon  with NBC^+ indep. sets of size rank P,
- Get basis for $\beta_S(\bar{P})$ by sending each NBC^+ filling T' of  to $v_{T'} = \sum_{\sigma \in Col(T)} sgn(\sigma) \cdot \underbrace{\{\sigma T'\}}_{\text{tabloid}}$

  then applying $\bar{f}_{chain} : \{\text{tabloids}\} \to \left\{\begin{array}{c}\text{max}\\\text{chains}\\\text{in } P^S\end{array}\right\}$

Thm: $\left\{ \bar{f}_{chain}(v_{T'}) \;\middle|\; \begin{array}{l} T' \text{ is std atom filling of} \\ \text{ribbon with } NBC^+ \text{ basis} \end{array} \right\}$

is homology basis for $\beta_S(\bar{P})$

Thm: Ribbon basis for $WH_S(\bar{P})$ is:

$$\bigcup_{\{u \mid rk(u) = \max S\}} \{\text{ribbon basis for } \beta_{S \setminus \{\max S\}}([\hat{0}, u])\}$$

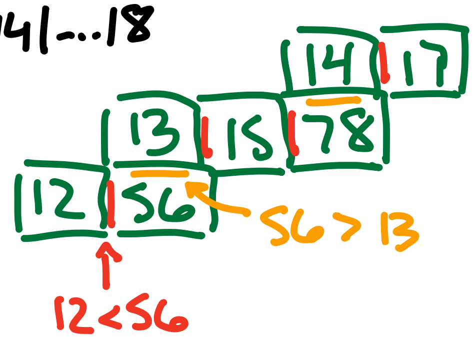
e.g. $n = 8$
$S = \{2,5\}$ $\rightsquigarrow$ ribbon row lengths $2, 5-2, 8-5$

Total order on atoms of Π_n:

$$12 < 13 < 23 < 14 < 24 < 34 < 15 < 25 < \cdots$$

$\updownarrow$

$12|3|4|\ldots|8$

$T' =$ 

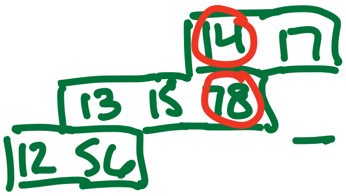
= "standard" Young tableau whose entries are NBC^+ basis

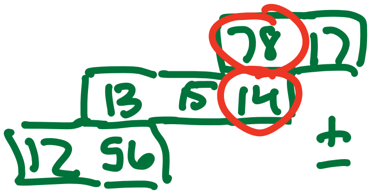
$$\bar{f}_{chain}(\{T'\}) = e_{12} \vee e_{56} < e_{12} \vee e_{56} \vee e_{13} \vee e_{15} \vee e_{78}$$

$\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad\qquad \|$

$\qquad 12|56|3|4|7|8 \qquad\qquad 12356|78|4$

max chain in Π_n^S for $S = \{2,5\}$

$v_{T'} = \bar{f}_{chain}\Big($ 

$-$ 

$\pm \ldots \Big)$

$$= (12 \vee 56 < 12 \vee 56 \vee 13 \vee 15 \vee 78) - (12 \vee 56 < 12 \vee \ldots) \pm \ldots$$

Basis for Whitney Homology WH_S

e.g. $WH_{\{2,5\}}(\overline{\Pi}_8)$

- choose u of rank S

e.g. $u = 12356 \mid 78 \mid 4$

type $u = (1, 1, 0, 0, 1, 0, \text{—})$

$\uparrow$ # parts size 1 $\quad \uparrow$ # parts size 2 $\quad$ ---

- choose NBC$^+$ standard filling for geometric lattice $[\hat{0}, u] \cong \Pi_{12356} \times \Pi_{78} \times \Pi_{\{4\}}$

e.g. $T' = \begin{array}{cccc} & \boxed{13} & \boxed{15} & \boxed{78} \\ \boxed{12} & \boxed{56} & & \end{array}$

- again calculate $v_{T'}$, for all T' for all u of rank max S

Useful Fact: $WH_S(\overline{\Pi}_n)$ decomposes into S_n-submodules based on "type" of u where
$$WH_S(\overline{\Pi}_n) := \bigoplus_{rk(u) = \max S} \beta_{S - \max S}(\hat{0}, u)$$

Young Symmetrizers "Behave Well" on Ribbon Basis for $\beta_S(\Pi_n)$

- Given S_n-module V, well-known that

$$\langle S^\lambda, V\rangle = 0 \iff \underbrace{b_T a_T}_{\text{Young symmetrizer of shape } \lambda \text{ given by filling } T} V = 0 \quad \forall T$$

$$\underbrace{\Big(\sum_{\tau \in Col(T)} sgn(\tau)\tau\Big)}_{b_T}\underbrace{\Big(\sum_{\sigma \in Row(T)} \sigma\Big)}_{a_T} =:$$

$$\iff b_T a_T v = 0 \quad \forall T, \; \forall v \text{ in basis for } V$$

Prop'n (H.-Sundaram): for ribbon basis for Π_n, $b_T a_T v_{T'} = 0$ whenever $\exists\, c,d,e,f$ in 1st row of T s.t. $\{c,d\}$ & $\{e,f\}$ in same column of T' where c,d,e,f not in any other box labels of T'

e.g.

$c,d = 4,5$

$e,f = 6,7$

$$T = \boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{7}\boxed{8}$$

$$T' = \begin{array}{ccc} & \boxed{4,5} & \boxed{1,8} \\ \boxed{1,2} & \boxed{6,7} & \end{array}$$

no copies of 4,5,6 or 7 in any other box labels

$b_T a_T v_{T'} = 0$ since

$$a_T = (\ldots)\cdot\left[e+(46)(57)\right]$$

& $$v_{T'} = \begin{array}{ccc} & \boxed{4,5} & \boxed{1,8} \\ \boxed{1,2} & \boxed{6,7} & \end{array} - \begin{array}{ccc} & \boxed{6,7} & \boxed{1,8} \\ \boxed{1,2} & \boxed{4,5} & \end{array}$$

- We later show S^λ with "large" λ_1 has $\langle S^\lambda, B_S(\pi_n)\rangle = 0$ by using Young symmetrizers. Big λ_1 forces pairs $\{c,d\}$ & $\{e,f\}$ as above in same column.

e.g. $T' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \end{array} \implies v_{T'} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array}$

$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$

Key fact:

$a_T v_{T'} = 0$ because

$$a_T = (\underbrace{e + (14) + (24) + (34) + (54)}_{\text{inserts } 4})\Big(\underbrace{\sum_{\sigma \in S_{\{1,2,3,5\}}} \sigma}_{\text{further factors as}}\Big)$$

$$H = \langle (13)(25) \rangle \qquad \Big(\sum_{\sigma \in S_{\{1,2,3,5\}}/H} \sigma\Big)\big(e + (13)(25)\big)$$

where $(\underbrace{e + (13)(25)}_{\text{exchanges boxes of } \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \end{array}}) v_{T'} = 0$

implying $b_T a_T v_{T'} = 0$ $\forall v_{T'} \in$ ribbon basis
$\forall$ fillings T of λ

Thus, $\langle S^\lambda, \beta_S(\overline{\Pi}_n) \rangle = 0$

Using Ribbon Bases to Reprove

$WH_{\{1,\dots,i\}}(P) \cong$ ith-graded piece of Orlik-Solomon algebra

- uses ribbon shape  $\}\, i$ where NBC^+ std fillings are exactly the NBC bases w/ atoms arranged in descending order
- map $e_{a_1}e_{a_2}\cdots e_{a_r} \mapsto \bar{f}_{\text{chain}}(\vee_{F(a_1,a_2,\dots)})$ gives G-equivariant isomorphism
- O.S. rel'ns translate to interesting rel'ns in Whitney homology!

V. Proof of H-Reiner Conjecture: Sharp Stability Bound $4\max S - |S| + 1$ for $\beta_S(\Pi_n)$:

- Stability Bd for $WH_S(\overline{\Pi}_n) \Rightarrow$ Stability Bd for $\beta_S(\overline{\Pi}_n)$
- Decompose $WH_S(\overline{\Pi}_n) = \bigoplus_{\mathrm{rk}\, u = \max S} \beta_{S \setminus \max S}(\hat{0}, u)$ into submodules based on partition type of u (i.e. sizes of blocks of u)

e.g. $u = 24|5,7,9,10|1|3|6|8$

type $m_1 = 4 = \#$ parts size 1

$m_2 = 1$

$m_3 = 0$

$m_4 = 1$

- Show component of $WH_S(\overline{\Pi}_n)$ of type $2^i, 1^{n-2i}$ stabilizes sharply at $4\max S - |S| + 1$

via $\widehat{WH}_{S,2^i,1^{n-2i}} \cong_{S_n} \beta_S(\bar{B}^i)[\mathbb{1}_2]$

$$\cong$$

$$S[\mathbb{1}_2]$$

- Show all other components of $WH_S(\overline{\Pi}_n)$ also satisfy $4\max S - |S| + 1$ stability bd due to all S^λ in essential part $\widehat{WH}_S$ satisfying $|\lambda| + \lambda_1 \leq 4\max S - |S| + 1$

- To this end, note that

$$|\lambda| = \sum_{j\geq 2} j \cdot m_j = \text{sum of part sizes of } \mu \text{ of size} > 1$$
$$\leq 2 \max S$$

- Use Young symmetrizers to prove $\lambda_1 > 4\max S - |S| + 1 - |\lambda|$ implies $b_T a_T v_{T'} = 0 \;\; \forall \; v_{T'}$ in ribbon basis (due to large 1st row in λ forcing $\boxed{b,c}$ & $\boxed{d,e}$ in same column of ribbon, all in 1st row of T)

- Conclude $\langle s^\lambda, \widehat{\omega H}_s(\Pi_n)\rangle = 0$ for $|\lambda| + \lambda_1 > 4\max S - |S| + 1$ $\blacksquare$

Thanks!