Shelling Face Posets of Stratified Spaces of Electrical Networks & of Phylogenetic Trees

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(slides at https://pages.uoregon.edu/plhersh/talks.html)
**CW Complexes & their Face Posets**

**Example:**

- Let $K$ be a ball.

$$F(K) = e_1.$$

- Let $K'$ be $\mathbb{RP}^2$.

$$F(K') = \text{"closure poset" or "face poset"}$$

$$(u \leq v \iff u \leq v)$$

**Recall:** A CW complex is comprised of open cells each homeomorphic to an open ball. A regular CW complex further has cell closures homeomorphic to closed balls. E.g. simplicial complexes
**Defn:** The order complex (or nerve) of a poset $P$ is the abstract simplicial complex $\Delta(P)$ whose $i$-dimensional faces are the $(i+1)$-"chains" $v_0 < \ldots < v_i$ in $P$.

**Example:**

$P = (a_1, b_1, c)$

$\Delta(P) = \{ a_1, b_1, c, a_1b_1, a_1c, b_1c \}$

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**Key Property (Hall; popularized by Rota):**

$$M_P(x, y) = \tilde{X}(\Delta_P(x, y)) = -1 + \# \text{vertices} - \# \text{edges} + \# 2-\text{faces} - \ldots$$

$$= -1 + \beta_0 - \beta_1 + \beta_2 - \ldots$$

- $(u, v) = \sum_{z \in P} \mid u < z < v \mid^3$
- $u < v$ means $u < v \land \exists z \text{ s.t. } u < z < v$
- Saturated chains up to $v := u < \ldots < v$
Background on Face Posets & CW Posets

- A graded poset with $\hat{0} \neq \hat{1}$ is **Eulerian** if $M(u,v) = (-1)^{rk(v) - rk(u)}$ for all $u \leq v$.

- A graded poset $P$ is a **CW poset** if
  1. $\hat{0} \in P$
  2. $P$ has at least one other element
  3. $\Delta(\hat{0}, u) \leq S^{rk(u)-2}$ for $u \neq \hat{0}$

**Thm (Björner)**: $P$ is CW poset $\iff$ there exists regular CW complex with $P$ as poset of closure relins

**Cor**: CW Poset $\Rightarrow$ Eulerian
Some CW Posets

• all graded, thin, shellable posets (Danaraj-Klee)
  "thin" $\iff \begin{tikzpicture}
  \draw (0,0) node (u) {};
  \draw (1,1) node (v) {};
  \draw (1,-1) node (w) {};
  \draw (0,0) -- (1,1) -- (1,0) -- (1,-1) -- (0,0);
\end{tikzpicture}$
  \[ \text{rk}(v) - \text{rk}(w) = 2 \]
  \[ |E(u,v)| = 4 \]

• Bruhat order (Björner-Wachs, Dyer)

• Face posets of stratified spaces of electrical networks
  - conjectured by Thomas Lam
  - proved by H.-Kenyon (a main topic for today's talk)
Face Poset for Electrical Networks

\[ \mathcal{P}_3 = \]

Diagram of a face poset with labeled elements and ordered by inclusion.
The Uncrossing Poset (Face Poset for Electrical Networks)

1. $\hat{1} :=$ wire diagram w/ all $\binom{n}{2}$ crossings of $n$ wires

2. $u <_v$ if $u$ obtained from $v$ uncrossing pair of wires without introducing double crossing

3. $\hat{0}$ adjoined below Catalan many atoms

4. $P_n :=$ uncrossing poset with $n$ wires
A Conjecture of Thomas Lam

Thm (Lam): The uncrossing poset is Eulerian.

Conjecture (Lam): The uncrossing poset is lexicographically shellable.

Thm (H.-Kenyon): Uncrossing posets are dual EC-shellable.

Cor: They are CW posets.
Dual Graph for $i \in P_7$

Embedding Tree in Such Dual Graph (with leaves $l_1$ - $l_7$ and body nodes)
**Edge Product Space** $E_n$, of **Phylogenetic Trees with $n$ leaves**

**Step 1:** Given tree $T$ with (independent) edge probabilities $x_1, x_2, \ldots, x_m$; leaves $l_1, l_2, \ldots, l_n$, may calculate vector $m_T(x_1, \ldots, x_m) \in (0, 1)^n$ of probabilities for each pair of leaves that path between them is fully present.

**E.g.**

$\left( x_1 x_2, x_1 x_3 x_4, x_1 x_3 x_5, x_2 x_3 x_4, x_2 x_3 x_5, x_4 x_5 \right)$

$P_{l_2} \quad P_{l_3} \quad P_{l_4} \quad P_{l_3} \quad P_{l_4} \quad P_{l_3}$

**Step 2:** Define edge-product space $E_n$ for trees with $n$ leaves as

$U \text{ im } (m_T) \subseteq (0, 1)^n$

T trees

$\omega/\text{leaves } l_1, \ldots, l_n \text{ and nonleaves all deg } \geq 3$
Theorem (Gill-Linusson-Moulton-Steel): There exists a shelling for each interval in the face poset for \( E_n \), namely the Tuffley poset.

Corollary (Gill-Linusson-Moulton-Steel): The edge product space of phylogenetic trees is a regular CW complex.

Theorem (Stadnyk): The Tuffley poset in its entirety is not shellable.

Theorem (H.-Kenyon): Each interval in the Tuffley poset is isomorphic to an interval in the uncrossing poset, hence has an explicitly constructed shelling by virtue of the uncrossing poset shelling.
Another Example:

Image of \((x, y, z) \mapsto (xy, xz, yz)\)

(bounding surfaces \(m_1m_2 \leq m_3\), \(m_1m_3 \leq m_2\) in \([0,1]^3\))
Small Example of Edge Product Space of Phylog.

Trees (up to Decoming)
Motivations for Edge Product Space of Phylogenetic Trees

(1) Determining edge lengths from distances between leaves (data measurable from DNA)

(2) Determining tree type (i.e., which cell tree lives in)

Convention: 
\[ (0, 1] \xrightarrow{\text{homeom}} \mathbb{R}_+ \]
\[ p_1, p_2 \rightarrow d(p_1) \]
\[ p_1 \rightarrow -\ln(p) + d(p_2) \]

\[ \text{e.g., } d_3 \]

\[ \text{vs.} \]

\[ d_1 + d_2 \leq d_3 + d_4 \]

(1) + (2) \Rightarrow \text{likely phylogenetic history from “distances”}
Some Earlier Work on $E_n$

- Billera-Holmes-Vogtmann: uniqueness of geodesics
- Owen-Provan: efficient algorithm to calculate geodesics
- Moulton-Steel: CW complex
- Ardila-Klivans: homeomorphism (via subdivision) to order complex of partition lattice
- Basu-Gabrielov-Vorobjov: “toric cubes” are regular CW complexes

Note: $E_n$ is compactification of “free space” in BHU, OP, AK
Space of "Response Matrices" of a Planar Electrical Network

\[ I = \text{interior nodes} \quad N = \text{boundary nodes} \]

\[
\begin{align*}
\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix}
\begin{bmatrix}
v_I \\
v_N
\end{bmatrix} &=
\begin{bmatrix}
v \v_I \\
v_N
\end{bmatrix} =
\begin{bmatrix}
\text{vector of voltages} \\
\text{vector of currents}
\end{bmatrix} \\
(A - BC^{-1}B^T)v_N &= C_N
\end{align*}
\]

"response matrix" of the network
**A Goal:** Given a graph $G$, study the space of response matrices as image of
\[
  f: \{ \text{conductance} \} \rightarrow \{ \text{response} \}
\]
\[
  (1R_{2,0} \cup \{oo3\})^\mathbb{E}_1
\]

**Note:**
- Contracting an edge $\leftrightarrow$ Sending conductance to $\infty$ (i.e., resistance to 0)
- Deleting an edge $\leftrightarrow$ Sending conductance to 0 (resistance to $\infty$)

**Secondary Goal:** Study fibers of $f$
Face Poset for Resp. Matias
Electrical Networks for Resp. Matias
Correspondence: From Graphs to Uncrossing Wire Diagrams "Medial graphs"

$\Gamma = \ldots$

$G(\Gamma) = \ldots = \text{wire diagram}$

Uncrossing:

deletion

contraction
Turning to Fibers via "Electrical Equivalence"

1. $a \sim \begin{array}{c} a \\ b \end{array}$
2. $\emptyset \sim \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{array}$
3. $a \sim \begin{array}{c} \frac{ab}{a+b} \\ b \end{array}$
4. $\begin{array}{c} a \\ b \end{array} \sim \begin{array}{c} a+b \\ a \end{array}$
5. "Y-\Delta moves"

\[ \begin{array}{c} \text{Y-\Delta} \\ \text{moves} \end{array} \sim \begin{array}{c} \frac{cc}{a+b+c} \\ \frac{ab}{a+b+c} \\ \frac{bc}{a+b+c} \end{array} \]
Shellability

- Simplicial complex is pure of dim. d if all maximal faces ("facets") are d-dimensional.
- Simplicial complex is shellable if there is total order $F_1, F_2, \ldots, F_k$, a shelling, on facets s.t. $F_j \cap (\cup_{i \leq j} F_i)$ is pure, codimension one subcomplex of $F_j$ for each $j > 1$ (hence is $\partial F_j$ or has a cone point).

- Each facet attachment preserves homotopy type or closes off a new sphere.
**Lexicographic Shellability for** \( \Delta(P) \) (Björner, Björner-Weather)

A poset is **dual EL-shellable** if it admits edge labeling \( \lambda \) (called **dual EL-labeling**) of cover relations \( x \prec y \) s.t. for each \( u \prec v \) in \( P \):

1. \( \exists! \) Saturated chain
   \[ u \prec u_1 \prec \ldots \prec u_k \prec v \] s.t.
   \[ \lambda(u_k, v) \leq \lambda(u_k, u_i) \leq \ldots \leq \lambda(u_1, u) \]
   
   and

2. \( (\lambda(u_k, v), \ldots, \lambda(u, u_1)) \) is smaller in dictionary order than all other label sequences from \( v \) to \( u \).
Face Poset for Electrical Networks
**Edge Labeling**

**Step 1:** Define word of wire diagram $D$, denoted $w(D)$, as sequence of $2n$ wire endpoints encountered clockwise starting with "root" 1.

**Step 2:** Label $D \circ D'$ for $i < j$ as
- $(i, j)$ if $ijij$ in $w(D)$ becomes $ijji$ in $w(D')$
- $(ji)$ if $ijij$ becomes $ijji$
Label Ordering

- $\Delta \{ (i, j) \mid 1 \leq i < j \leq n \} \cup \Sigma L^3$
  $\cup \Sigma (j, i) \mid 1 \leq i < j \leq n^3$
- $(i, j) < L < (r, s)$ for all $i < j$
  and all $r > s$
- $(1, 2) < (1, 3) < (1, 4) < \ldots < (1, n) < (2, 3)$
  $< (2, 4) < \ldots < (2, n) < (3, 4) < \ldots < (3, n)$
  $< \ldots < (n - 1, n)$
- $(n, n - 1) < (n, n - 2) < (n - 1, n - 2) < (n, n - 3)$
  $< (n - 1, n - 3) < (n - 2, n - 3) < \ldots$
  $< (2, 1)$

Rk: finite type A reflection order $\Delta \{ (i, j) \}$
then $L$, then reversal for $\Sigma (j, i)$
"Start Sets" and Connection to Type A Bruhat Order

The start set of $D$, denoted $S(D)$, is the subset of $\mathbb{I}, 2, \ldots, 3n^3$ of positions in $\omega(D)$ where 1st copies of letters occur.

\[ \omega(D) = 121233 \]
\[ S(D) = 31, 2, 53 \]
\[ \pi(D) = 123 \in S_3 \]

**Propn:** If $D' < D$ and $S(C(D)) = S(D)$, then $[D', D] = [\pi(D'), \pi(D)]$

- Subwords of $\omega(D') \neq \omega(D)$
- Uncrossing order labeling coincides
  - Here w/ Dyer’s Bruhat order labeling
Concluding Remarks

1. Galashin-Karp-Lam recently announced proof that stratified space of response matrices is regular CW complex.
2. Analogy w/ my past work tot. norm, pos. part unip. radical of Borel?
3. Work in progress: graphical analogue of subword complexes to describe fibers.

Thanks for your attention!
Bruhat Order of a (Finite) Reflection Group / Coxeter Group & Dyer's EL-Labeling

e.g.

$S_i := (i, i+1)$

$321 = s_1 s_2 s_1 = s_2 s_1 s_2$

$231 = s_1 s_2$

$S_1$

$S_2$

$S_2 s_1 = 312$

$(s_1 s_2)^{s_1} = s_2$

$213 = s_1$

$S_1$

$S_1 s_2 s_1$

$S_1$

$S_2$

$S_2 = 132$

$S_2$

EL-labeling: $u \cdot v = u t$

$\lambda(u, v) := u^t v = t$
Dyer's EL-labeling (cont)

- Use any “reflection order” to totally order edge labels
- Dyer proved these exist and induce EL-labelings

**Defn:** A total order on positive roots (and associated reflections) is a **reflection order** if $\alpha < c_1\alpha + c_2\beta < \beta$ or $\beta < c_1\alpha + c_2\beta < \alpha$ for each such triple of positive roots $\omega/c_1, c_2 > 0$

E.g. $(1,2) < (1,3) < (2,3)$ or $(2,3) < (1,3) < (1,2)$ in type A
Stratification into "Cells" for Space of Response Matrices

e.g.

\[
\begin{align*}
&l_2 \quad b \quad c \quad l_3 \\
&5' \quad b' \quad c'
\end{align*}
\]
Minors of Response Matrix via "Gours"

\( \Pi(G) := \) set partition of bdry graph nodes into connected components

e.g. \( \Pi \left( \begin{array}{c}
\begin{array}{c}
\text{v}_1 \\
\text{v}_2 \\
\text{v}_4 \\
\text{v}_3
\end{array}
\end{array} \right) = 12 \begin{array}{c}
3/4
\end{array} \)

Thm (Special case of Next Result):

\[
L \cdot J(G) = \sum_{G' \leq G} \frac{\text{wt}(G')}{\Pi(G') = ij \text{ singletons}}
\]

\( \text{wt}(G') = \) product of edge weights (i.e. conductances)
Lam's Dual Embedding of Uncrossing Order into $\tilde{S}_{2n}$
Bnwhat Order

- $D_e = \text{fully crossed } \Rightarrow 1D \in \tilde{S}_{2n}$
  diagram
  $g_2 \text{ s.t. } i \rightarrow i+n$

- $D < \cdot D'$ where $D$ $\Rightarrow$ $g_D$
  obtained from $D'$ by $(i,j)g_{D'}(i,j)$
  $i, j$ wire endpoint swap

\[ e.g. \quad D' \quad D'' \quad g_D = (1,2)g_{D'}(1,2) \]
\[ = (4,5)g_{D'}(4,5) \]
Thm (Björner): EL-labeling $\Rightarrow$ Shelling

Idea: Lexicographic order on maximal chains (breaking ties arbitrarily) induces shelling order on corresponding facets of $\Delta(P)$.

"descents in labeling" $\leftrightarrow$ overlap of facets

"descending" $\leftrightarrow$ facets attaching along entire boly $\leftrightarrow$ spheres

$M_P(u,v) = \pm \# \text{descending chains } u \to v$ (for $P$ graded)