Topological Combinatorics

of Posets and Stratified Spaces

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Lecture 1: Möbius fns & Shellability

Lecture 2: Discrete Morse theory

Lecture 3: Stratified spaces and face posets
Discrete Morse theory (as introduced by Robin Forman)

**Defn:** A discrete Morse function $f : \Delta \to \mathbb{R}$ is a fn assigning a real number to each face of a simplicial complex (or more generally each cell of a (regular) CW complex) s.t. for each $\sigma(p)$ notation for a cell

1. $\exists (p+1) \mid \sigma(p) \leq \sigma(p+1) \mid f(\sigma) \geq f(\sigma+1) \leq 1$

2. $\exists (p-1) \mid \sigma(p) \leq \sigma(p-1) \mid f(\sigma) \leq f(\sigma-1) \leq 1$

(So $\alpha \leq \beta \Rightarrow f(\alpha) < f(\beta)$ for "almost all" pairs $\alpha, \beta$)
Consequence: Each $\sigma(p)$ either has both cardinalities 0 (in which case $\sigma(p)$ is a critical cell) or has one cardinality 0 and other cardinality 1 (in which case $\sigma(p)$ belongs to a noncritical pair).

E.g.,

- critical 0-cell

- noncritical pairs (inducing "collapses")

Remarks:
- Critical i-dim'l cells analogous to critical pts of index i.
- Noncritical pairs can be eliminated by internal elem. collapses.
Traditional Morse Theory:

"critical point": where partial derivatives all 0

"index" of crit pt = max dim'l subspace of tangent space where f flows downward

(See John Milnor, "Morse Theory")
**CW Complexes & their Face Posets**

*Example:*

\[ K = \text{ball} \]

\[ K' = \mathbb{R}P^2 \]

**Recall:** A CW complex: cells \( e_d \subseteq \mathbb{R}^d \), characteristic maps \( f_d : B^{\dim(e_d)} \to \bigcup e \in \partial \)

attaching maps \( f_d |_{\partial B^{\dim(e_d)}} \)

- regular: \( f_d \)'s are homeomorphisms

**Face Poset**

\[ F(K) = e_1 \]

\[ F(K') = \vdash \]

"Closure poset" or "face poset"

\[ (u \leq v \iff u \subseteq v) \]
Face Poset Reformulation of Discrete Morse Theory (M. Chari)

Given any regular CW complex \( \Delta \), construct an acyclic matching (a.k.a. Morse matching) on its face poset, i.e.,

an edge orientation s.t. "up edges" give a matching and directed graph has no cycles.

Useful Fact for Proving Acyclicity:
Any directed cycle must alternate "up" \& "down" steps
Observations: 1. Discrete Morse fn on $\Delta$ induces acyclic matching w/ arrows in direction fn decreases
2. Every acyclic matching on face poset is induced by a nonempty set of discrete Morse fn's

$\Delta = \begin{array}{c}
\bullet 4 \\
\bullet e \\
\bullet 3 \\
\bullet i
\end{array}$

$\xrightarrow{f(\Delta)}$

$\begin{array}{c}
\bullet 4 \\
\bullet e \\
\bullet 3 \\
\bullet i
\end{array}$

Theorem (Forman): $\Delta \cong \Delta^M$ a CW complex comprised of the unmatched cells, called critical cells. e.g., $\bullet i \cong \bigcirc$

critical i-cells

critical pts of index i
First Examples

1. Boolean algebra of subsets of \( \xi_1, \xi_2, \ldots, \xi_3 \), face poset of simplex, matching \( S \times \xi_3 \) with \( S \times \xi_3 \) A S

\[
\begin{align*}
\text{Base pt} & \quad \text{critical} \\
\theta \quad \text{0-cell} & \\
\end{align*}
\]

matching edge in "reduced homology" version of discrete Morse theory
2. Any union of acyclic matchings on $F(\Delta_2 \setminus \Delta_1), F(\Delta_3 \setminus \Delta_2), \ldots$ for $\Delta_1 \subseteq \Delta_2 \subseteq \ldots \subseteq \Delta_k = \Delta$ a filtration of subcomplexes is an acyclic matching for $\Delta$

c.g. $\overline{F}_1 \subseteq \overline{F}_1 \cup \overline{F}_2 \subseteq \overline{F}_1 \cup \overline{F}_2 \cup \overline{F}_3$

3. Shelling $\Rightarrow$ Discrete Morse fn with homology facets as critical cells (using a 2nd definition of shelling as total order $F_1, F_2, \ldots, F_k$ s.t. each $\overline{F}_j \setminus (\cup_{i < j} \overline{F}_i)$ has unique minimal face)
Explanation for $\Delta \simeq \Delta^m$: Matching edges specify (internal) elementary collapses preserving homotopy type (generalizing "elementary collapses" of simple homotopy theory - cf. Rourke & Sanderson Appendix including "torsion")

Consequences of $\Delta \simeq \Delta^m$:
1. If $F(\Delta)$ has complete acyclic matching ($\varnothing \in F(\Delta)$) then
   $\Delta$ is collapsible.

Recall: Some contractible complexes are not collapsible.

   e.g. dunæ cap

   [Diagram of dunæ cap]
2. If $F(\Delta)$ has acyclic matching with all unmatched elements at rank $j$, then $\Delta$ is homotopy equivalent to wedge of $j$-dimensional spheres.

3. If $F(\Delta)$ has acyclic matching with all unmatched elements at rank $\geq j$ for some $j \geq 1$, then $\pi_i(\Delta) = 0$ for $i < j$ (i.e. $(j-1)$-connected).
4. If $F(\Delta)$ has acyclic matching with only facets of $\Delta$ unmatched then $\Delta$ homotopy equiv. wedge of spheres (w/ $\#_i$-spheres $=$ $\#$ unmatched $i$-dim'f facets)

5. If $F(\Delta)$ has acyclic matching with all unmatched elements at even ranks then $\Delta$ has homology concentrated in even degrees
6. $\tilde{\chi}(\Delta) = \tilde{\chi}(\Delta^m)$

$$= -1 + \# 0\text{-cells} - \# 1\text{-cells}$$

$$+ \# 2\text{-cells} - \cdots$$

$$= -1 + \beta_0 - \beta_1 + \beta_2 - \cdots$$

**For Posets:** $M_p(x,y) = \tilde{\chi}(\Delta(x,y)) = \tilde{\chi}(\Delta^m(x,y))$

7. Morse Inequalities:

1. $\tilde{\beta}_i(\Delta) \leq \tilde{m}_i(\Delta) = \# \ i\text{-dim' critical cells}$

2. $\sum_{i \leq j} \tilde{\beta}_i(\Delta) \leq \sum_{i \leq j} \tilde{m}_i(\Delta)$

   (for each $j \leq \dim(\Delta)$)

**Rk:** "Greedy" matchings tend to satisfy acyclicity requirement.
Some Examples & Applications

1. Complex of not 2-connected graphs
   (Bobson-Björner-Linusson
    - Shareshian-Welker)
   - motivated by Vassiliev knot invariants
   (not 2-connected := disconnected after deleting a vertex)

2. More generally: any monotone graph property $P$ gives complex $\Delta_P$ with $G$-edges as vertices in $\Delta_P$
   (see Jakob Jonsson’s thesis & book)

3. Applies to persistent homology
   and strong analogy w/ birth (resp. death) of homology classes
   & critical cells creating (resp. destroying) homology
**Question (H.):** Is there a good way to "complete the square":

- lexicographic shelling

\[ \Rightarrow \quad ?? \]

\[ \Rightarrow \quad \text{shelling} \Rightarrow \text{discrete Morse function} \]

To understand posets that fail to be shellable (e.g. not wedge of spheres)?

**Proposed Answer:** "lexicographic discrete Morse functions"
**Lexicographic Discrete Morse Functions: A General Construction**

Partly joint work with E. Bobbson

**Step 1:** Any edge labeling on poset $P$ induces lexicographic order $F_1,F_2,\ldots,F_m$ on maximal faces (facets) of $\Delta(P)$

**Example:**

$P = \begin{array}{c}
\text{u}_1 & \text{u}_2 & \text{v}_1 & \text{v}_2
\end{array}$

$\begin{array}{c}
\text{F}_1 = 135 \\
\text{F}_2 = 147 \\
\text{F}_3 = 297
\end{array}$

(Usually not EL-labeling!)
Step 2: Morse matching on each $\overline{F_j} \setminus (u, \overline{F_i})$ s.t.

1. Each $\overline{F_j} \setminus (u, \overline{F_i})$ has 0 or 1 unmatched (critical) cells
2. Union of these matchings is Morse matching for $\Delta(P)$

Theorem (Babson-H.) Any edge labeling on any finite poset gives rise to a lexicographic discrete Morse fn s.t. critical cells $\sim$ facets whose attachment changes the homotopy type of complex.
Description of Critical Cells

"interval system"

I \rightarrow J

\text{critical cell}
\text{lowest element of each (truncated) interval}

\{ \text{Faces in } F_j \} \rightarrow \{ \text{Subsets of ranks in } F_j \text{ that "hit" all intervals in } I\text{-system} \}

\{ F_j - (\bigcup_{i < j} F_i) \} \rightarrow \{ \text{Subsets of ranks in } F_j \text{ that "hit" all intervals in } I\text{-system} \}

\bullet \text{ No critical cell unless truncated interval system } J \text{ fully covers } F_j
Truncation Algorithm

Start with interval system $I$ and initialize truncated system $J$ to $\emptyset$.

E.g. $I = \{[1,2], [2,3], [3,4]\}$ \(\Rightarrow\) $J = \emptyset$

Repeatedly:
1. Move \(\text{min}(I)\) to truncated system $J$ after truncating all other elements of $I$ to eliminate overlap with \(\text{min}(I)\).
2. Throw away elements of $I$ no longer minimal.

E.g.

$I = \{[1,2], [2,3], [3,4]\}$

\(\Rightarrow\) $\{[3]\}$

\(\Rightarrow\) $\emptyset$

\(\Rightarrow\) $\{[1,2]\}$

\(\Rightarrow\) $\{[1,2], [3]\}$

(4 uncovered, so no critical cell)
Remarks: (1) Lexicographic shellability is a special case with all intervals in $I$ of size one.

(2) Saturated chain does not contribute critical cell unless fully covered by $J$-system.

(3) Critical cell dimension is $1|J|-1$ since it consists of $\{i \mid i = \min(j) \text{ for some } j \in J\}$.

(4) Upper bd on interval size for all $f_j \Rightarrow$ lower bd on connectivity of $\Delta(\Delta)$.

(5) Match based on uncovered elt or lowest $J$-interval differing from critical cell.
Using lex. discrete Morse-funs in Practice:

Use "natural" labelings enabling characterization of types of intervals appearing in interval systems

* e.g. weak Bruhat order (not shellable)

\[ S_1 \xrightarrow{321} S_2 \xrightarrow{S_1} 231 \xrightarrow{S_2} 312 \xrightarrow{S_1} 123 \xrightarrow{S_2} 132 \]

- proceeding down across $S_i$ swaps positions $i,i+1$ sorting letters
- interval system governed by braid relns!
  * e.g. $S_2 S_1 S_2 \Rightarrow S_1 S_2 S_1$

Rk: Especially well suited to posets from algebra w/ interval system \( \sim \) relns
Example: Semigroup Ring $k[\Lambda]$ and Associated (Infinite) Monoid Poset $\Lambda$

e.g. $k[z_1^2, z_1z_2, z_2^2] \cong k[x_1, x_2, x_3]/<x_1x_3-x_2^2>$

Toric ideal = $I_\Lambda$ of syzygies

Partial order $\Delta$: Order equivalence classes of monomials in $k[x_1, \ldots, x_n]/I_\Delta$ by "divisibility".

e.g. Submonoid of $\text{IN}^d$ generated by multidegrees $(2,0), (1,1), (0,2)$
Laundel-Sletsjøe:
\[ \widehat{H}_i(\emptyset, \lambda) \cong \text{Tor}_{i+2}^{k[\Lambda]}(k, k) \]

Idea: Bar resolution of \( k \) as \( k[\Lambda] \)-module after tensoring with \( k \) translates to order complex of "monoid poset."

(see also Peeva-Reiner-Sturmfels; Herzog-H.-Welker; use discrete Reiner-Welker)

Morse theory for posets to explain
\[ \text{Tor}_{i+2}^{k[\Lambda]}(k, k) = 0 \text{ for } i < -1 + \frac{\deg(\lambda) - 1}{d - 1} \]
for \( d = \text{degree of Gröbner basis for } I_{\Lambda} \)
for \( \Lambda \) graded (non-shellable if \( d > 2 \))
Bar Resolution & Simplicial Homology of Posets

- $\mathbf{B} : \ldots \rightarrow B_i \rightarrow B_{i-1} \rightarrow \ldots \rightarrow B_0 \rightarrow R \rightarrow 0$

  where $B_i$ has basis

  $\{[m_1, m_2, \ldots, m_i]\}$ for $m_1, m_2, \ldots > 1$

- Tensoring with $k$ yields:

  $k \otimes B : \ldots \rightarrow k \otimes B_i \rightarrow \ldots \rightarrow k \otimes B_0 = R$

  with boundary map

  $d : [m_1, \ldots, m_i] = \sum_{1 \leq j \leq i-1} [m_1, \ldots, m_j, m_j m_{j+1}, m_{j+2}, \ldots]$

  This is simplicial homology of $\Delta(\Lambda)$

  i.e. $[1, m^\lambda] \mapsto$ multidegree $\lambda$

  in $\text{Tor}^k(k[\Lambda], k)$
Natural Labeling & Lexic. Discrete Morse Fn for Monoidal Posets

e.g. $R[x_1, x_2, x_3, x_4, x_5, x_6] / (x_2 x_6 - x_1^2)$

$r[ab, a^2, c, d, e, b^2]$

1. "descents" such as $x_0 x_3 = x_3 x_0$
2. "syzygies" such as $x_3 (x_2 x_6 - x_1^2) = 0$

Interval system given by:
Use gradient path reversal

yielding “better” discrete Morse fn s.t. critical cells all come from saturated chains fully covered by intervals of average height \( \leq d - 1 \).

Upshot: connectivity lower bd

\[
= \min \text{ dim. of } \cap r_k(u) - r_k(u)_{\text{critical cell}} \\
\overset{d - 1}{\text{d-1}}
\]

Combinatorially “explains” Eisenbud-Reeves - Totaro result

Note: nonshellable complexes arise often in commutative algebra
Algebraic Version of Discrete Morse Theory

(see Jöllenbeck-Welker, Memoirs AMS)

-yields homological results e.g. w/ coefficients in field of finite char.

Rough Idea: Given an algebraic chain complex & generators of chain groups, build poset on generators with 
\( u \prec v \iff \exists v_1 \in u = \text{unit} \). Acyclic matching implies chain homotopy to chain complex gener'd by critical cells (unmatched elements)
"Quillen Fiber Lemma" (or "Quillen Theorem A")

If \( f: P \to Q \) is a poset map, i.e. \( u \leq v \Rightarrow f(u) \leq f(v) \), and if \( \Delta(f^{-1}(g)) \) is contractible for each \( g \in Q \),

\[ \exists p \in P \mid f(p) \leq g \]

then \( \Delta(P) \cong \Delta(Q) \).

**Application:** Homotopy type of complex of non 2-connected graphs via poset map whose fibers are proven contractible by discrete Morse theory s.t. map image is shellable poset.
Application (H.-Lemait): Prove there are arbitrarily high degree reln's amongst "crystal operators" not implied by lower degree reln's via "SB-labelings" (notion of H-Mészáros) + poset map to Boolean algebra

Upshot: Matsumoto's Thm has analogue when applying crystal operators to highest weight vector in highest weight rep'n of Kac-Moody algebra in simply laced case, but fails arbitrarily badly for vectors other than highest weight vector.