Topological Combinatorics of Posets & Stratified Spaces

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Lecture 1: Möbius fns & Shellability

Lecture 2: Discrete Morse theory

Lecture 3: Stratified spaces and face posets

Discrete Morse theory

(as introduced by Robin Forman)

Defn: A discrete Morse function $f: \Delta \longrightarrow \mathbb{R}$ is a fn assigning a real number to each face of a simplicial complex (or more generally each cell of a (regular) CW complex s.t. for each $\sigma^{(p)}$ ← notation for p-dim'l cell

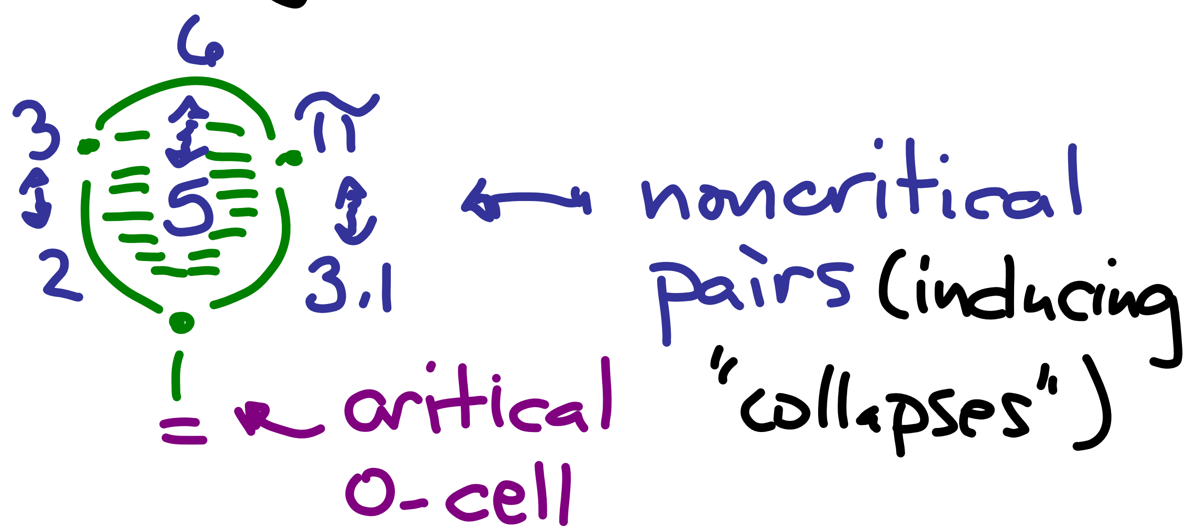
1. $|\{\tau^{(p+1)} \mid \sigma^{(p)} \subseteq \overline{\tau^{(p+1)}} \text{ \& } f(\sigma) \geq f(\tau)\}| \leq 1$

& 2. $|\{\mu^{(p-1)} \mid \mu^{(p-1)} \subseteq \overline{\sigma^{(p)}} \text{ \& } f(\sigma) \leq f(\mu)\}| \leq 1$

(so $\alpha \subseteq \overline{\beta} \Rightarrow f(\alpha) < f(\beta)$ for "almost all" pairs α, β)

Consequence: Each $\sigma^{(p)}$ either has both cardinalities 0 (in which case $\sigma^{(p)}$ is a critical cell) or has one cardinality 0 & other cardinality 1 (in which case $\sigma^{(p)}$ belongs to a noncritical pair)

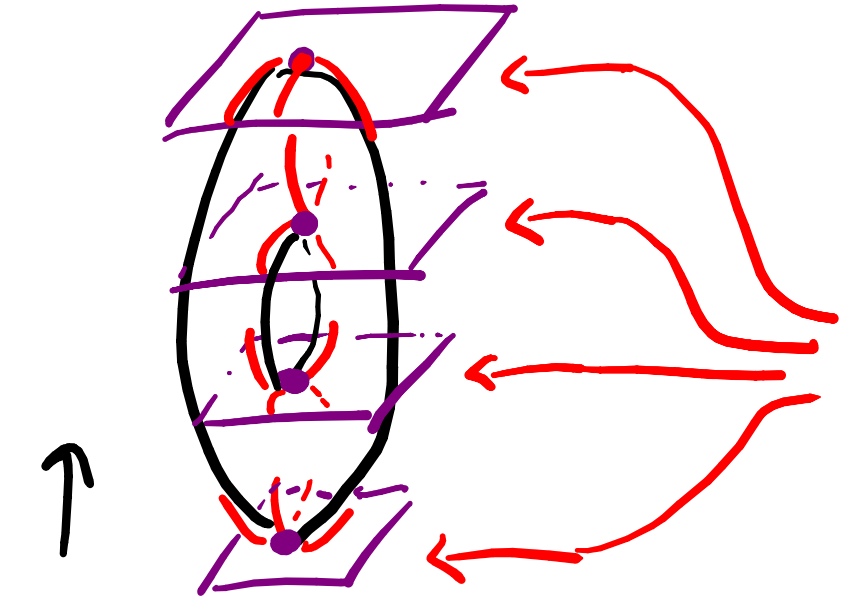
e.g.



Remarks: • critical i-dim'l cells analogous to critical pts of index i

• noncritical pairs can be eliminated by internal elem. collapses

Traditional Morse Theory:



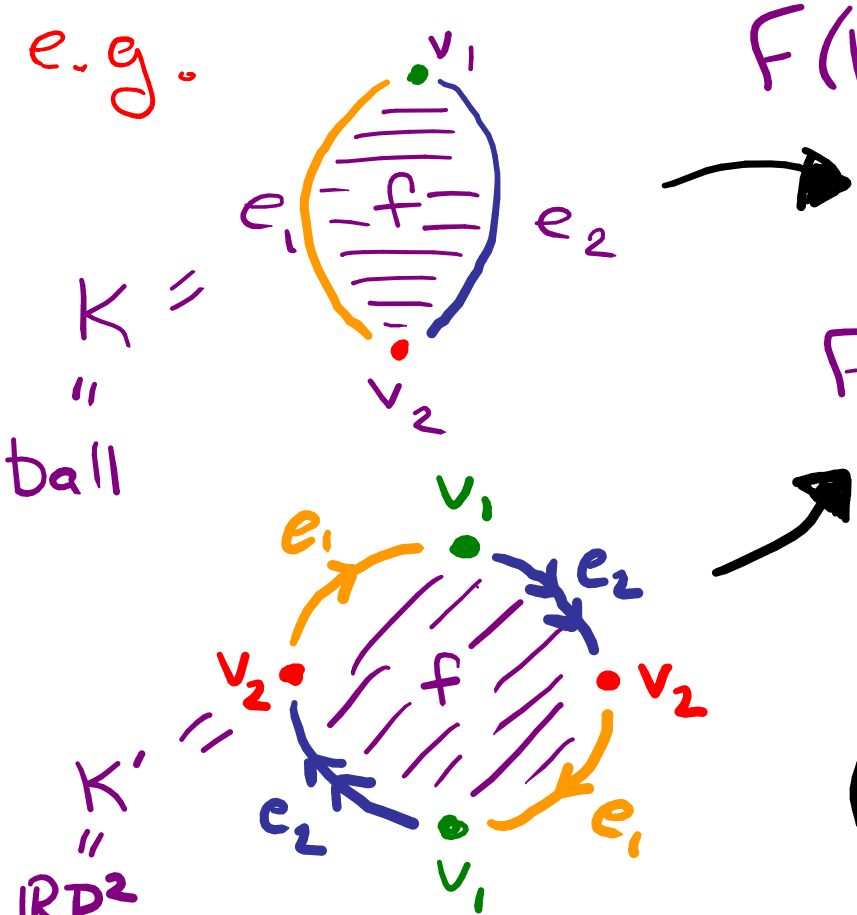
"critical point": where partial derivatives all 0

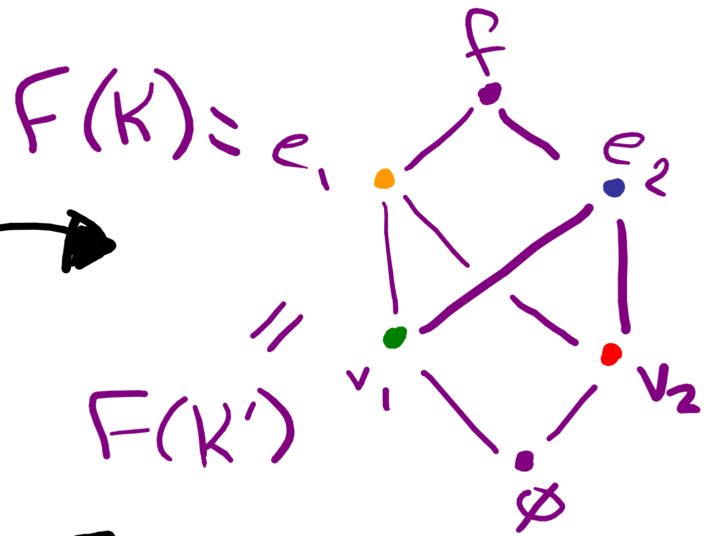
"index" of crit pt = max dim'l subspace of tangent space where f flows downward

(See John Milnor, "Morse Theory")

CW Complexes & their Face Posets

e.g.





"closure poset" or "face poset"

$(u \le v \iff u \le \bar{v})$

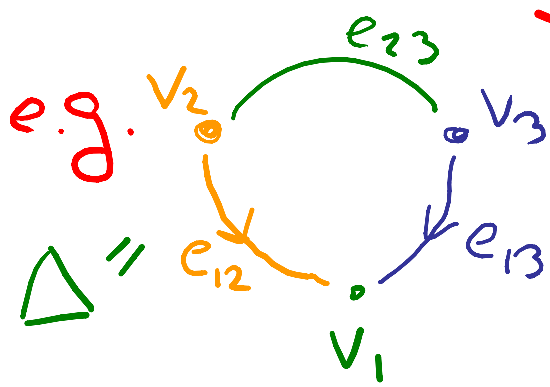
Recall: A **CW complex**: cells $e_\alpha \cong \mathbb{R}^{d(\alpha)}$, characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \to \bigcup_{e_\beta \le \bar{e}_\alpha} e_\beta$ & attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

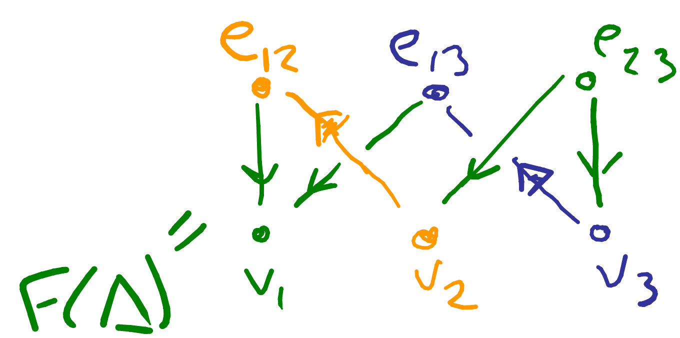
- **regular**: f_α's are homeomorphisms

Face Poset Reformulation of Discrete Morse Theory (M. Chari)

Given any regular CW complex Δ, construct an acyclic matching a.k.a. Morse matching on its face poset, i.e.,

e.g.





an edge orientation s.t. "up edges" give a matching and directed graph has no cycles.

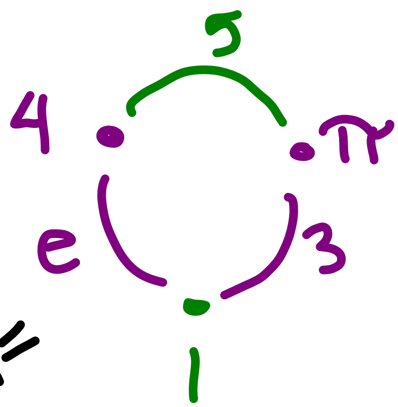
Useful Fact for Proving Acyclicity:

Any directed cycle must alternate "up" & "down" steps

Observations: 1. Discrete Morse fn on Δ induces acyclic matching w/ arrows in direction fn decreases

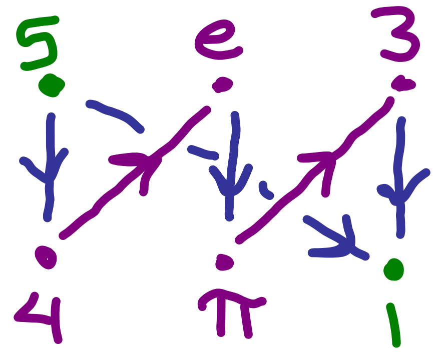
2. Every acyclic matching on face poset is induced by a nonempty set of discrete Morse fn's

$\Delta =$



$\rightsquigarrow$

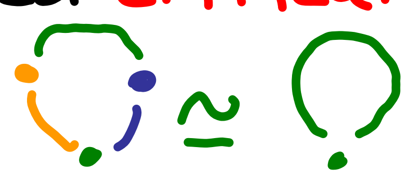
$F(\Delta) =$



Theorem (Forman): $\Delta \simeq \Delta^M$ a CW complex comprised of the unmatched cells, called critical cells.

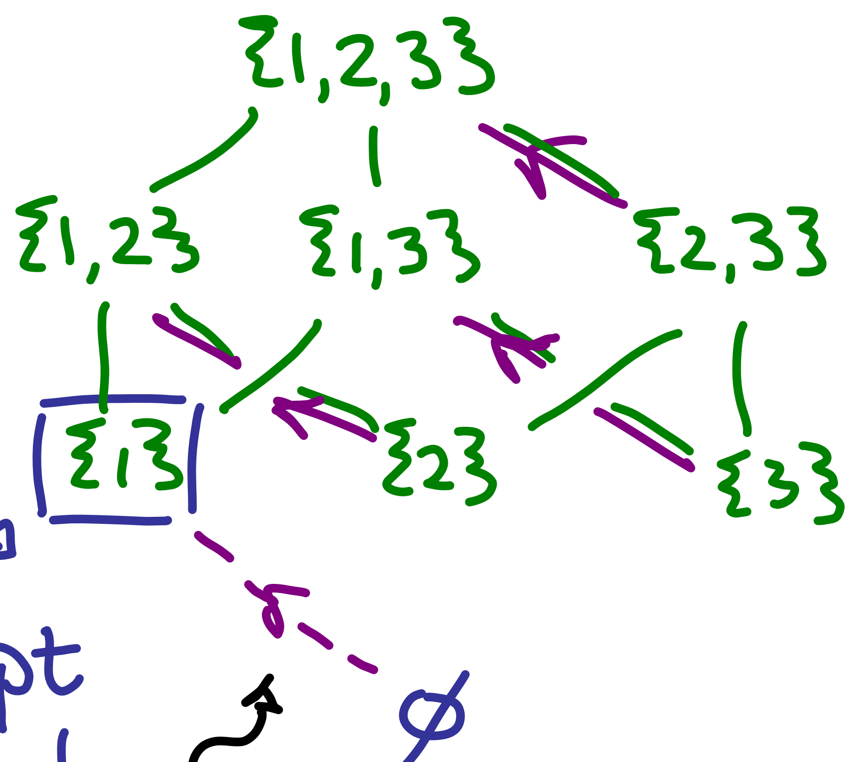
critical i-cells $\updownarrow$ critical pts of index i

e.g.



First Examples

1. Boolean algebra of subsets of $\{1,2,\ldots,n\}$, face poset of simplex, matching $S \setminus \{1\}$ with $S \cup \{1\}$ $\forall S$

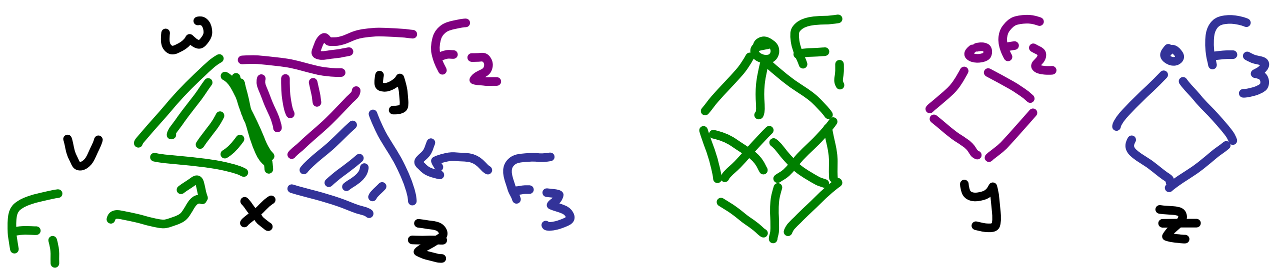


base pt critical 0-cell

matching edge in "reduced homology" version of discrete Morse theory

2. Any union of acyclic matchings on $F(\Delta_2 \setminus \Delta_1), F(\Delta_3 \setminus \Delta_2), \ldots$ for $\Delta_1 \subseteq \Delta_2 \subseteq \ldots \subseteq \Delta_k = \Delta$ a filtration of subcomplexes is an acyclic matching for Δ

e.g. $\overline{F_1} \subseteq \overline{F_1} \cup \overline{F_2} \subseteq \overline{F_1} \cup \overline{F_2} \cup \overline{F_3}$



3. Shelling $\Rightarrow$ Discrete Morse fn with homology facets as critical cells

(using a 2nd definition of shelling as total order $F_1, F_2, \ldots, F_k$ s.t. each $\overline{F_j} \setminus (\bigcup_{i<j} \overline{F_i})$ has unique minimal face)

Explanation for $\Delta \simeq \Delta^M$: Matching edges specify (internal) elementary collapses preserving homotopy type (generalizing "elementary collapses" of simple homotopy theory - cf. Rourke & Sanderson Appendix including "torsion")

Consequences of $\Delta \simeq \Delta^M$:

1. If $F(\Delta)$ has complete acyclic matching (w/ $\emptyset \in F(\Delta)$) then Δ is collapsible.

Recall: Some contractible complexes are not collapsible.

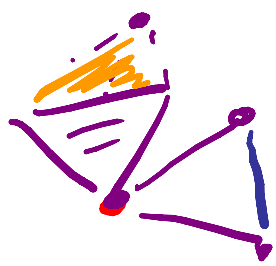
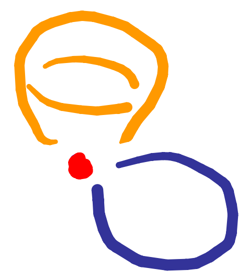
e.g. dunce cap 

2. If $F(\Delta)$ has acyclic matching with all unmatched elements at rank j, then Δ is homotopy equivalent to wedge of j-dimensional spheres.

3. If $F(\Delta)$ has acyclic matching with all unmatched elements at rank $\geq j$ for some $j \geq 1$, then $\pi_i(\Delta) = 0$ for $i < j$ (i.e. $(j-1)$-connected)

4. If $F(\Delta)$ has acyclic matching with only facets of Δ unmatched then Δ homotopy equiv. wedge of spheres (w/ # i-spheres = # unmatched i-dim'l facets)

e.g.

$\Delta =$  $\longrightarrow \Delta^M =$ 

5. If $F(\Delta)$ has acyclic matching with all unmatched elements at even ranks then Δ has homology concentrated in even degrees

6. $\tilde{\chi}(\Delta) = \tilde{\chi}(\Delta^M)$

$= -1 + \#\, 0\text{-cells} - \#\, 1\text{-cells}$

$+ \#\, 2\text{-cells} - \cdots$

$= -1 + \beta_0 - \beta_1 + \beta_2 - \cdots$

For Posets: $\mu_P(x,y) = \tilde{\chi}(\Delta(x,y)) = \tilde{\chi}(\Delta^M(x,y))$

7. Morse Inequalities:

1. $\tilde{\beta}_i(\Delta) \leq \tilde{m}_i(\Delta) = \#$ i-dim'l critical cells

2. $\sum_{i \leq j} (-1)^{j-i} \tilde{\beta}_i(\Delta) \leq \sum_{i \leq j} (-1)^{j-i} \tilde{m}_i(\Delta)$

(for each $j \leq \dim(\Delta)$)

Rk: "Greedy" matchings tend to satisfy acyclicity requirement.

Some Examples & Applications

1. Complex of not 2-connected graphs
   (Babson-Björner-Linusson
   -Shareshian-Welker)
   - motivated by Vassiliev knot inv't's

(not 2-connected := disconnected after deleting a vertex)

2. More generally: any monotone graph property P gives complex Δ_P with G-edges as vertices in Δ_P
   (see Jakob Jonsson's thesis & book)

3. Applic's to persistent homology and strong analogy w/ birth (resp. death) of homology classes ⇝ critical cells creating (resp. destroying) homology

Question (H.): Is there a good way to "complete the square":

$$
\begin{tikzcd}
\text{lexicographic shelling} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & ?? \arrow[d, Rightarrow] \\
\text{shelling} \arrow[r, Rightarrow] & \text{discrete Morse function}
\end{tikzcd}
$$

to understand posets that fail to be shellable (e.g. not wedge of spheres)?

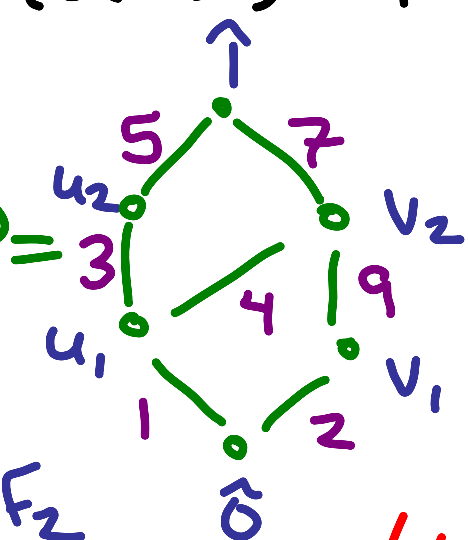
Proposed Answer: "lexicographic discrete Morse functions"

Lexicographic Discrete Morse Functions: A General Construction

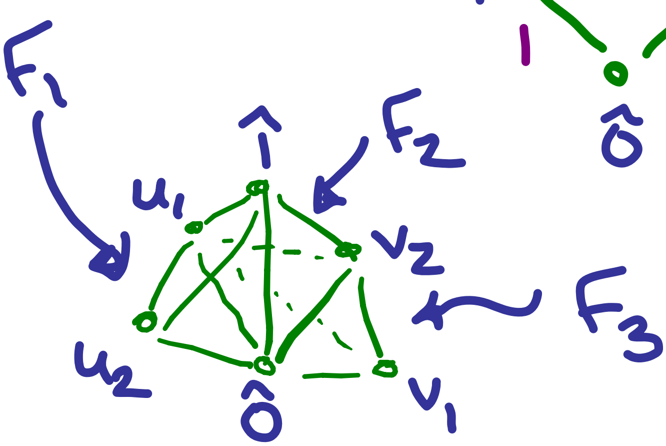
(partly joint work with E. Babson)

Step 1: Any edge labeling on poset P induces lexicographic order $F_1, F_2, \dots, F_m$ on maximal faces (facets) of $\Delta(P)$

e.g.

$P =$ 

$F_1 = 135$
$F_2 = 147$
$F_3 = 297$

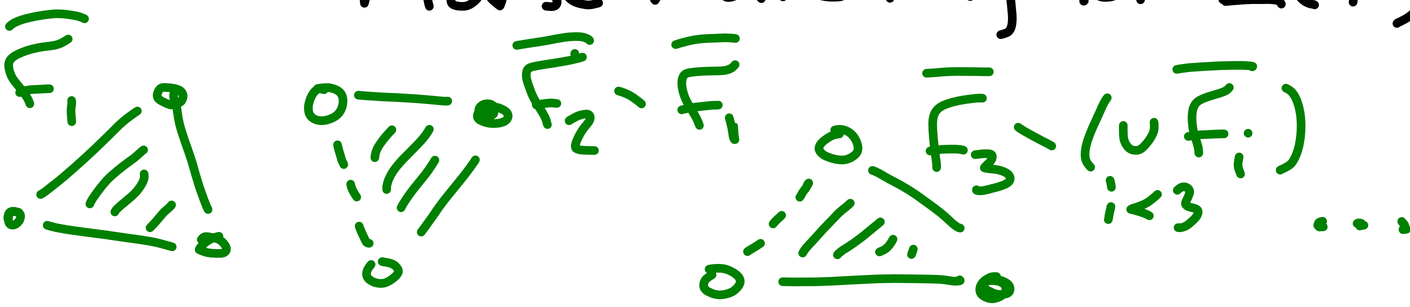


(Usually not EL-labeling!)

Step 2: Morse matching on each $\overline{F_j} \setminus (\cup_{i<j} \overline{F_i})$ s.t.

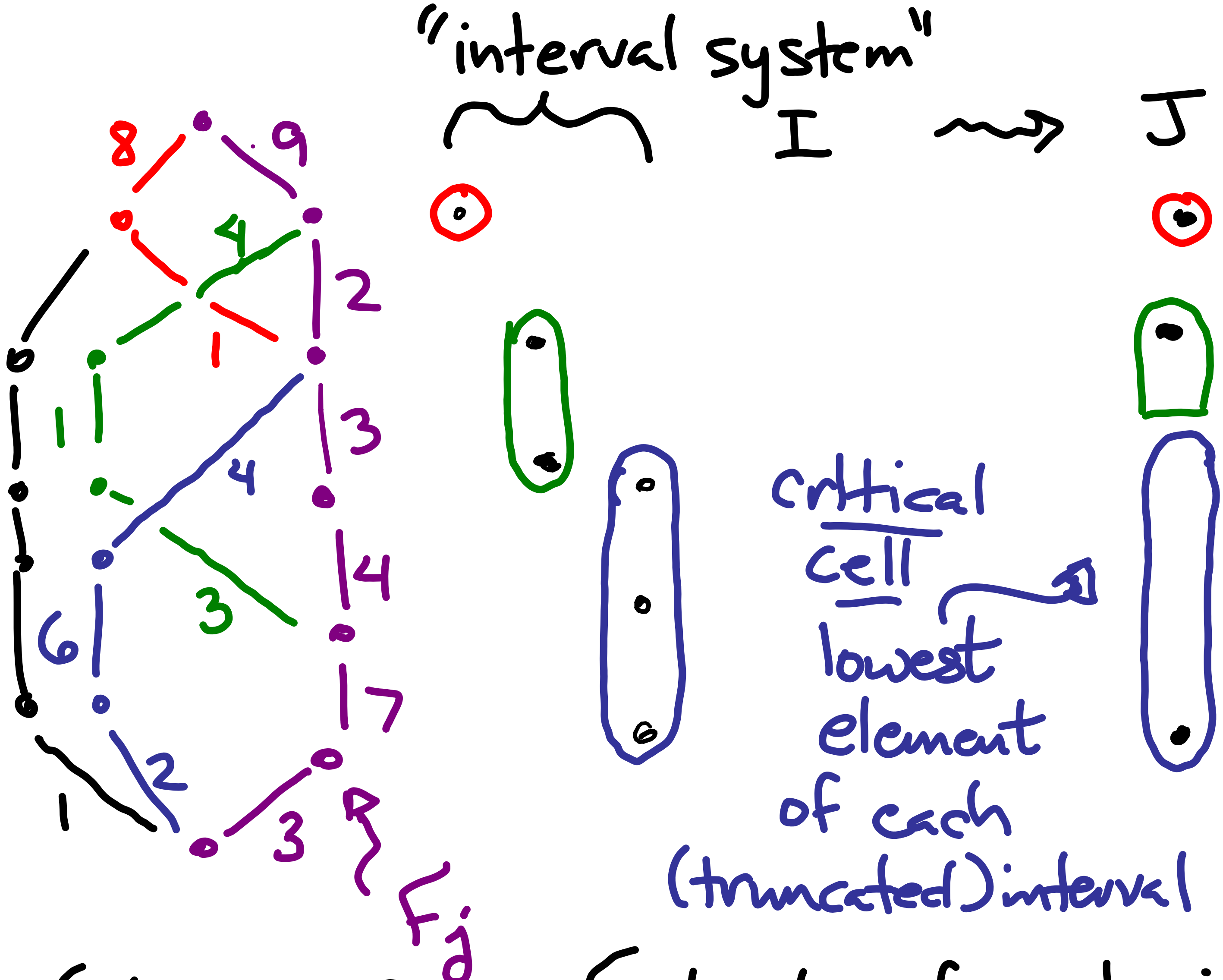
(1) Each $\overline{F_j} \setminus (\cup_{i<j} \overline{F_i})$ has 0 or 1 unmatched (critical) cells

(2) Union of these matchings is Morse matching for $\Delta(P)$



Theorem (Babson-H.) Any edge labeling on any finite poset gives rise to a lexicographic discrete Morse fn s.t. critical cells $\leftrightarrow$ facets whose attachment changes the homotopy type of complex.

Description of Critical Cells



- $\left\{ \text{Faces in } \overline{F_j} \setminus \left(\bigcup_{i<j} \overline{F_i} \right) \right\} \leftrightsquigarrow \left\{ \text{subsets of ranks in } F_j \text{ that "hit" all intervals in } I\text{-system} \right\}$
- No critical cell unless truncated interval system J fully covers F_j

Truncation Algorithm

Start with interval system I & initialize truncated system J to $\emptyset$

e.g. $I = \{[1,2],[2,3],[3,4]\}$ & $J = \emptyset$

Repeatedly: (1) move $\min(I)$ to truncated system J after truncating all other elements of I to eliminate overlap w/ $\min(I)$

(2) throw away elements of I no longer minimal

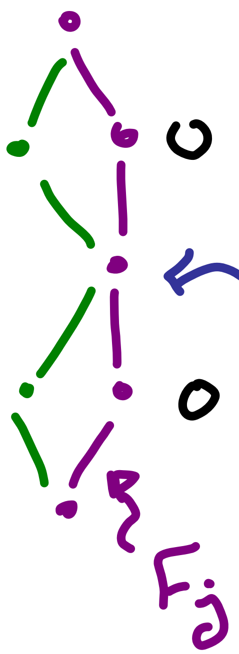
e.g.

$$I = \begin{Bmatrix} {[1,2]} \\ {[2,3]} \\ {[3,4]} \end{Bmatrix} \xrightarrow[(1)]{} \begin{Bmatrix} \\ {[3]} \\ {[3,4]} \end{Bmatrix} \xrightarrow[(2)]{} \begin{Bmatrix} {[3]} \end{Bmatrix} \xrightarrow[(1)\,\&\,(2)]{} \begin{Bmatrix} \ \end{Bmatrix}$$

$$J = \emptyset \longrightarrow \{[1,2]\} \longrightarrow \begin{Bmatrix} {[1,2]} \\ {[3]} \end{Bmatrix}$$

(4 uncovered, so no critical cell)

Remarks: (1) Lexicographic shellability is a special case with all intervals in I of size one



(2) Saturated chain does not contribute critical cell unless fully covered by J-system

match by include/exclude uncovered rank

(3) Critical cell dimension is $|J|-1$, since it consists of $\{i \mid i = \min(j) \text{ for some } j \in J\}$

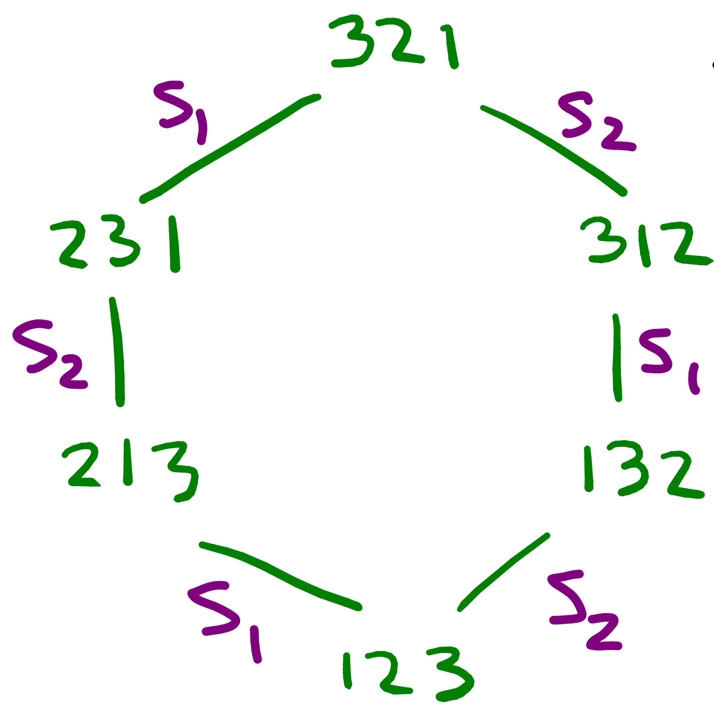
(4) Upper bd on interval size for all $F_j \Rightarrow$ lower bd on connectivity of $\Delta(\bar{P})$

(5) Match based on uncovered elt or lowest J-interval differing from critical cell

Using lex. discrete Morse fns in Practice:

Use "natural" labelings enabling characterization of types of intervals appearing in interval systems

e.g. weak Bruhat order (not shellable)



- proceeding down across s_i swaps positions $i, i+1$ sorting letters
- interval system governed by braid rel'ns!

e.g. $s_2 s_1 s_2 \to s_1 s_2 s_1$

Rk: Especially well suited to posets from algebra w/ interval system $\leftrightarrow$ rel'ns

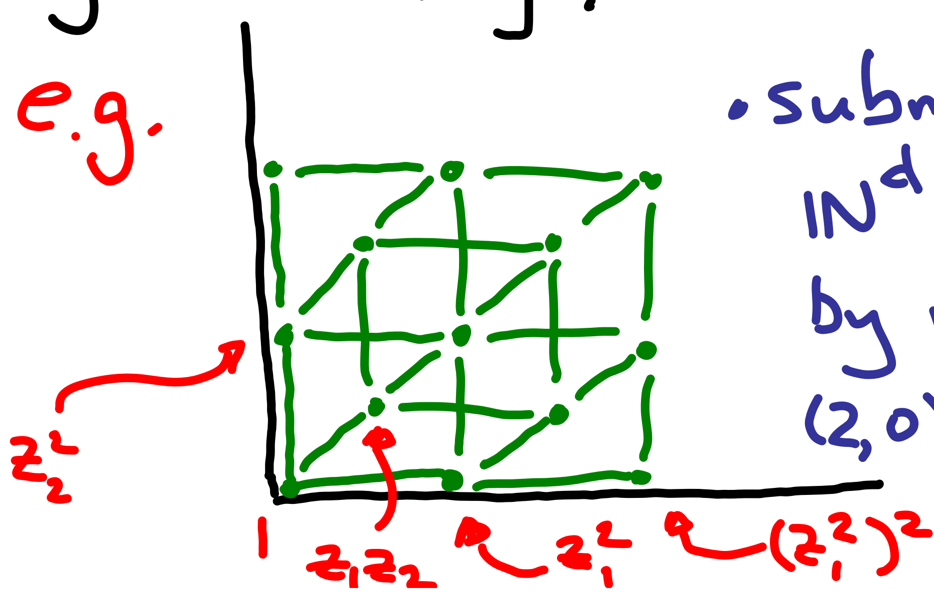
Example: Semigroup Ring $k[\Lambda]$ and Associated (Infinite) Monoid Poset Λ

e.g. $\underbrace{k[z_1^2, z_1z_2, z_2^2]}_{k[\Lambda]} \cong k[x_1,x_2,x_3] / \underbrace{(x_1x_3 - x_2^2)}$

toric ideal of syzygies $= I_\Lambda$

Partial order Λ: Order equivalence classes of monomials in $k[x_1,\dots,x_n]/I_\Lambda$ by "divisibility".

e.g.



- submonoid of $\mathbb{N}^d$ generated by multidegrees $(2,0)$, $(1,1)$, $(0,2)$

Landal-Sletsjøe:

$$\tilde{H}_i(\hat{0}, \lambda) \cong \mathrm{Tor}_{i+2}^{k[\Lambda]}(k,k)_\lambda$$

Idea: Bar resolution of k as $k[\Lambda]$-module after tensoring with k translates to order complex of "monoid poset".

(see also Peeva-Reiner-Sturmfels; Herzog-Reiner-Welker)

H.-Welker: Use discrete Morse theory for posets to explain

$\mathrm{Tor}_{i+2}^{k[\Lambda]}(k,k)_\lambda = 0$ for $i < -1 + \frac{\deg(\lambda)-1-d}{d-1}$

for d = degree of Gröbner basis for I_Λ for Λ graded (non-shellable if $d>2$)

Bar Resolution & Simplicial Homology of Posets

• B: $\cdots B_i \to B_{i-1} \to \cdots \to B_0 \to k \to 0$

where B_i has basis

$\{[m_1|m_2|\cdots|m_i]\}$ for $m_1, m_2, \ldots > 1$

• Tensoring with k yields:

$k\otimes B: \cdots \to k\otimes B_i \to \cdots \to k\otimes B_0 = k$

with bdry map

$$d_i[m_1|\cdots|m_i] = \sum_{1\le j\le i-1} (-1)^j [m_1|\cdots|m_{j-1}|m_j m_{j+1}|m_{j+2}|\cdots]$$

This <u>is</u> simplicial homology of $\Delta(\Lambda)$

i.e. $[1, m^\lambda] \leftrightsquigarrow$ multidegree λ

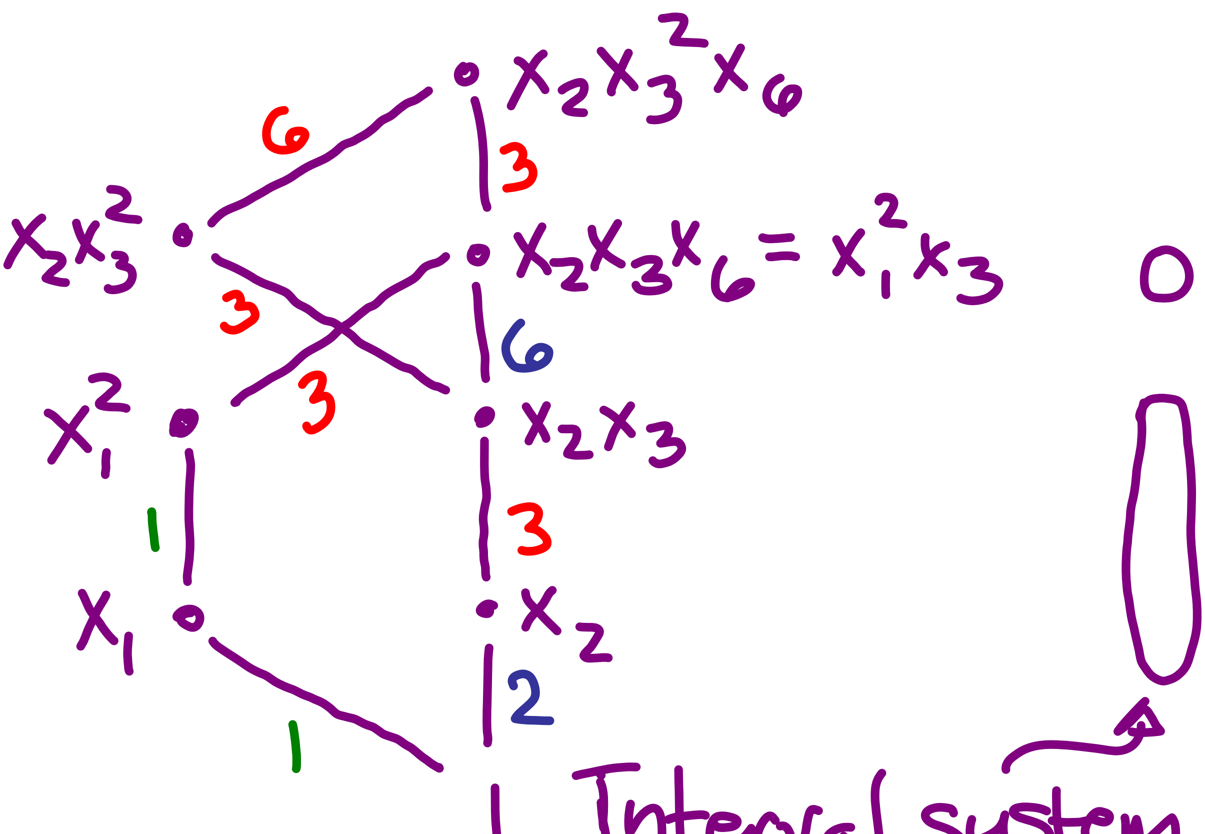
in $\mathrm{Tor}_\bullet^{k[\Lambda]}(k,k)$

Natural Labeling & Lexic. Discrete Morse Fn for Monoid Posets

e.g. $k[x_1, x_2, x_3, x_4, x_5, x_6] / (\underline{x_2 x_6} - x_1^2)$

$\cong$

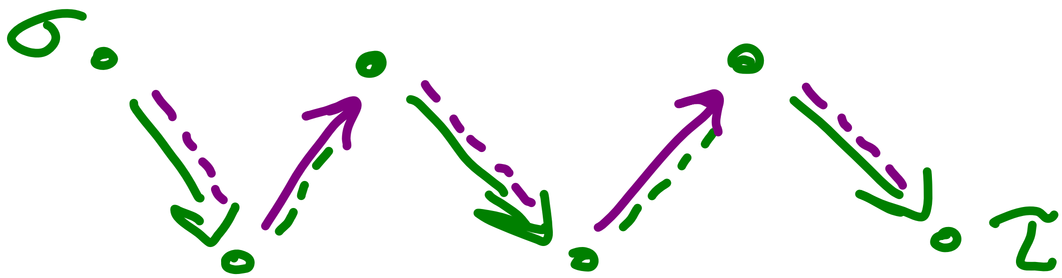
$k[ab, a^2, c, d, e, b^2]$



Interval system given by:

1. "descents" such as $x_6x_3 = x_3x_6$
2. "syzygies" such as $x_3(\underline{x_2x_6} - x_1^2) = 0$

- Use gradient path reversal



yielding "better" discrete Morse fn s.t. critical cells all come from saturated chains fully covered by intervals of average height $\leq d-1$.

- Upshot: connectivity lower bd $=$ min dim. of critical cell $\approx \dfrac{rk(v)-rk(u)}{d-1}$

Combinatorially "explains" Eisenbud-Reeves-Totaro result

Note: nonshellable complexes arise often in commutative algebra

Algebraic Version of Discrete Morse Theory

(see Jöllenbeck-Welker, Memoirs AMS)

- yields homological results e.g. w/ coefficients in field of finite char.

Rough Idea: Given an algebraic chain complex & generators of chain groups, build poset on generators with $u \lessdot v \iff \partial v|_u = \text{unit}$. Acyclic matching implies chain homotopy to chain complex gen'd by critical cells (unmatched elements)

"Quillen Fiber Lemma" (or "Quillen Theorem A")

If $f: P \to Q$ is a poset map, i.e. $u \leq v \Rightarrow f(u) \leq f(v)$, and if $\Delta(\underbrace{f^{-1}_{\leq q}}_{\{p \in P \mid f(p) \leq q\}})$ is contractible for each $q \in Q$, then $\Delta(P) \simeq \Delta(Q)$.

Application: Homotopy type of complex of not 2-connected graphs via poset map whose fibers are proven contractible by discrete Morse theory s.t. map image is shellable poset.

Application (H.-Lenart): Prove there are arbitrarily high degree reln's amongst "crystal operators" not implied by lower degree reln's via "SB-labelings" (notion of H-Mészáros) + poset map to Boolean algebra

Upshot: Matsumoto's Thm has analogue when applying crystal operators to highest weight vector in highest weight rep'n of Kac-Moody algebra in simply laced case, but fails arbitrarily badly for vectors other than highest weight vector.