Topological Combinatorics of Posets & Stratified Spaces

Patricia Hersh

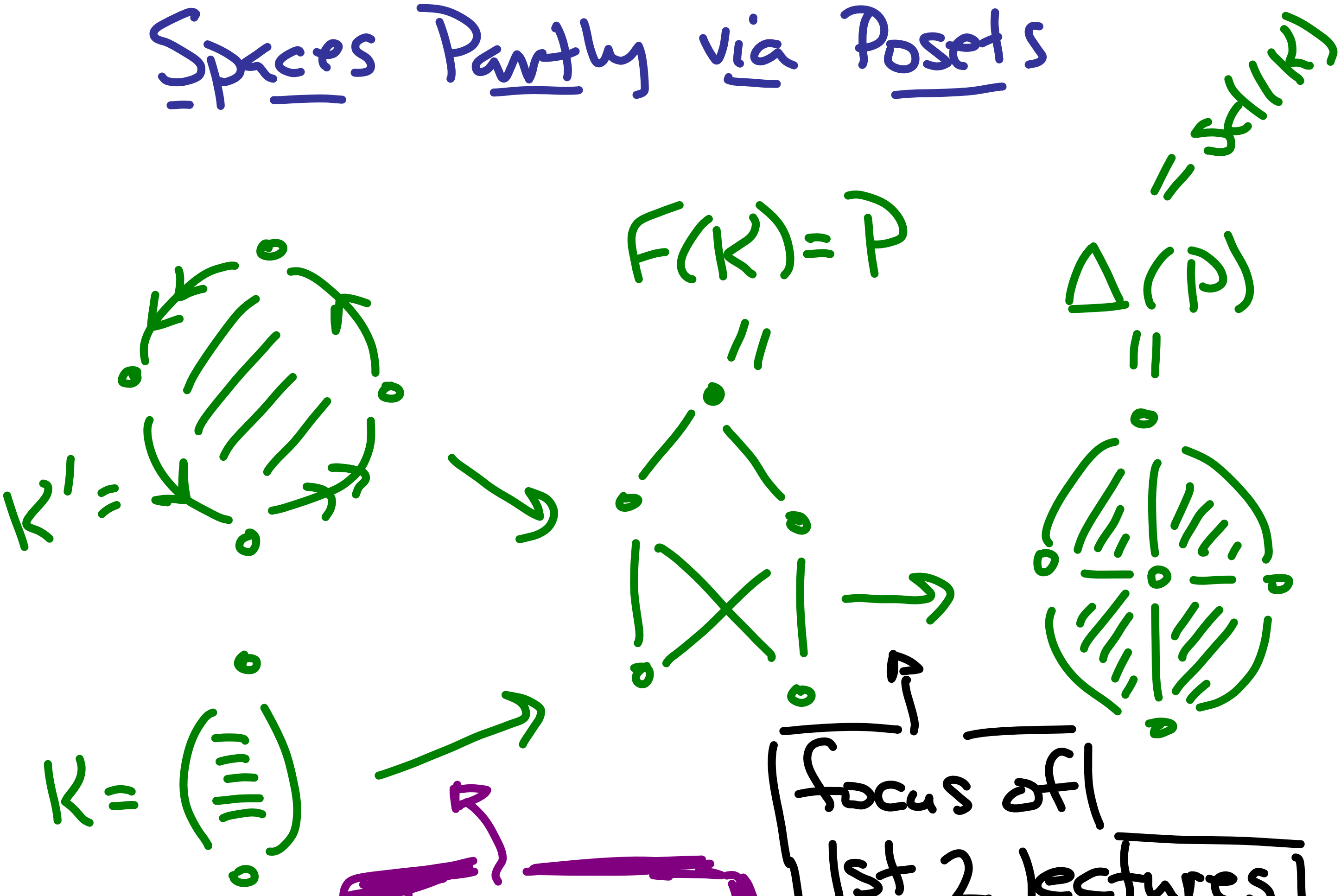
North Carolina State University

Lecture 1: Möbius fns & Shellability

Lecture 2: Discrete Morse theory

Lecture 3: Stratified spaces from rep'n theory & their closure posets

Studying Topology of Stratified Spaces Partly via Posets



focus of 1st 2 lectures

Today we want to determine K using F(K) + codim 1 info about attaching maps & study fibers of maps f via homeomorphism type analysis for K = im(f)

Bruhat Order

Closure poset $F(L)$ for Schubert cell decomposition L of flag variety $\mathcal{F}l_n = GL_n/B$ & "Schubert varieties" (over $\mathbb{C}$), namely for cell closures. Likewise for G/B in other types.

e.g. $$\overline{\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \\ 0 & * & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}} \supseteq \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right\}$$

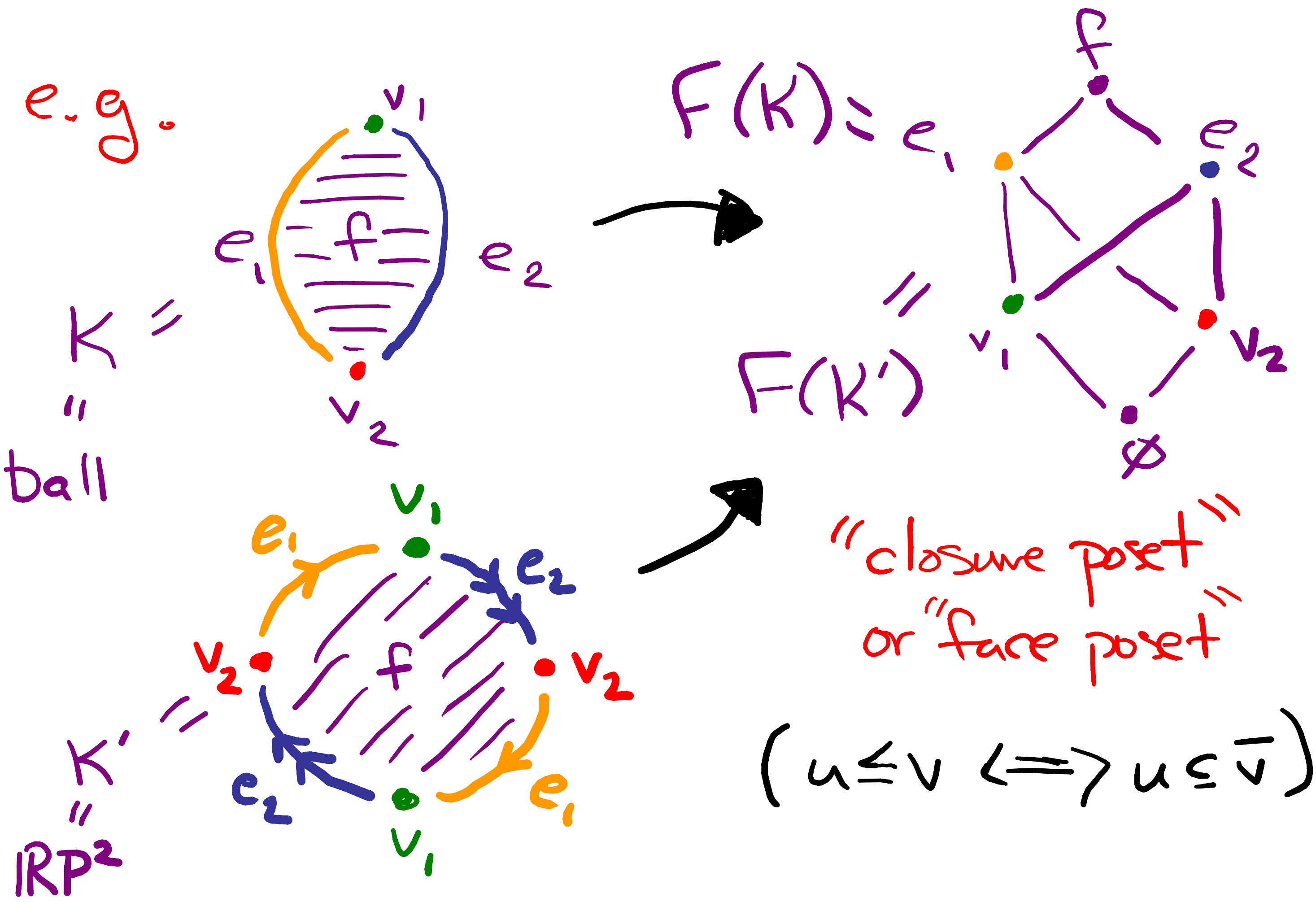
$\updownarrow$ $\qquad\qquad\qquad$ $\updownarrow$

$\mathbb{R}^6 = \mathbb{C}^3$ cell indexed by $1432 \in S_4$ $\qquad$ 1342-indexed cell $\cong \mathbb{C}^2 = \mathbb{R}^4$

$\updownarrow$

flags $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ * \\ * \\ 1 \end{pmatrix} \right\rangle \subseteq \cdots$

CW Complexes & their Face Posets

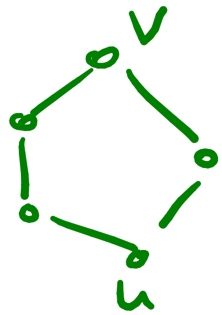


"closure poset" or "face poset"

$(u \leq v \iff u \leq \bar{v})$

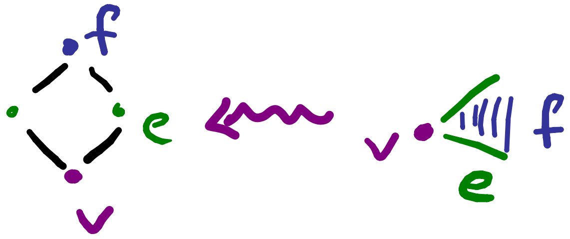
Recall: A **CW complex:** cells $e_\alpha \cong \mathbb{R}^{d(\alpha)}$, characteristic maps $f_\alpha : B^{\dim(e_\alpha)} \rightarrow \bigcup_{e_\beta \leq \bar{e}_\alpha} e_\beta$ & attaching maps $f_\alpha|_{\partial B^{\dim(e_\alpha)}}$

- A poset is **graded** if $u \leq v$ in P implies all minimal paths upward from u to v have same length (i.e. #edges)

e.g. 

is *not* graded

- A graded poset is **thin** if each rank 2 interval has exactly 4 elements.



Recall: A CW complex is **regular** if the attaching map f_α for each cell e_α is a homeomorphism, i.e. cell closures are closed balls.

Recall: 

Recall: CW Posets

- A graded poset P is a **CW poset** if
  - (1) $\hat{0} \in P$
  - (2) P has at least one other element
  - (3) $\Delta(\hat{0}, u) \cong S^{rk(u)-2}$ for $u \neq \hat{0}$

Thm (Björner): P is CW poset <=> there exists regular CW complex with P as poset of closure rel'ns

Consequence of Danaraj-Klee: P graded, thin & shellable w/ $\hat{0}$ $\Rightarrow$ CW poset

Thm (Björner & Wachs): Bruhat order is thin & shellable.

A New Tool: Regularity Criterion

Prop'n (H.) Let K be a finite CW complex w/ characteristic maps $\{f_\alpha\}$.

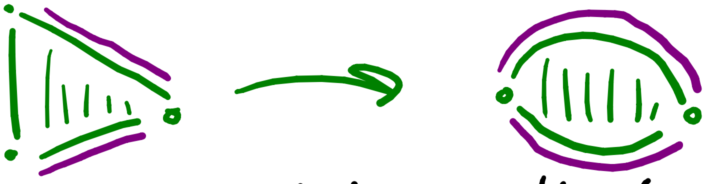
Suppose

(1) $\forall \alpha$, $f_\alpha(\partial B^{\dim \alpha})$ is a union of open cells (surjectivity)

Non-Example:

(2) $\forall f_\alpha$, the preimages of the open cells of codim. one in $\bar{e}_\alpha$ are dense in $\partial(B^{\dim \alpha})$

Non-Example:



Then F(K) is graded by cell dimension.

Remark: Next theorem "spreads around" injectivity requirement.

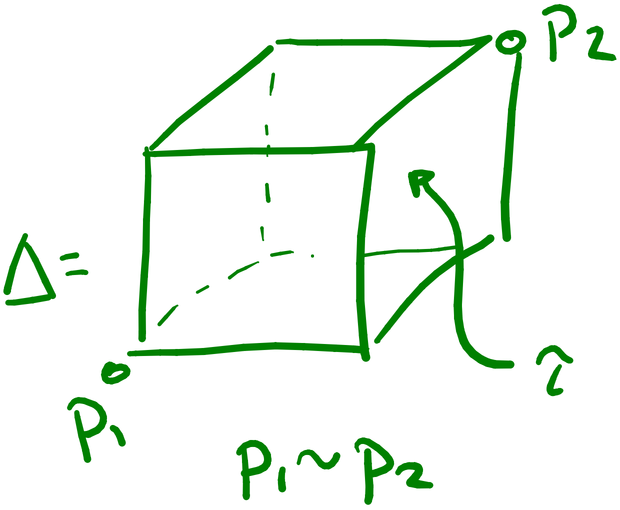
Thm (H.) Let K be finite CW complex w.r.t. characteristic maps $\{f_\alpha\}$. Then K is regular w.r.t. $\{f_\alpha\}$ <=>

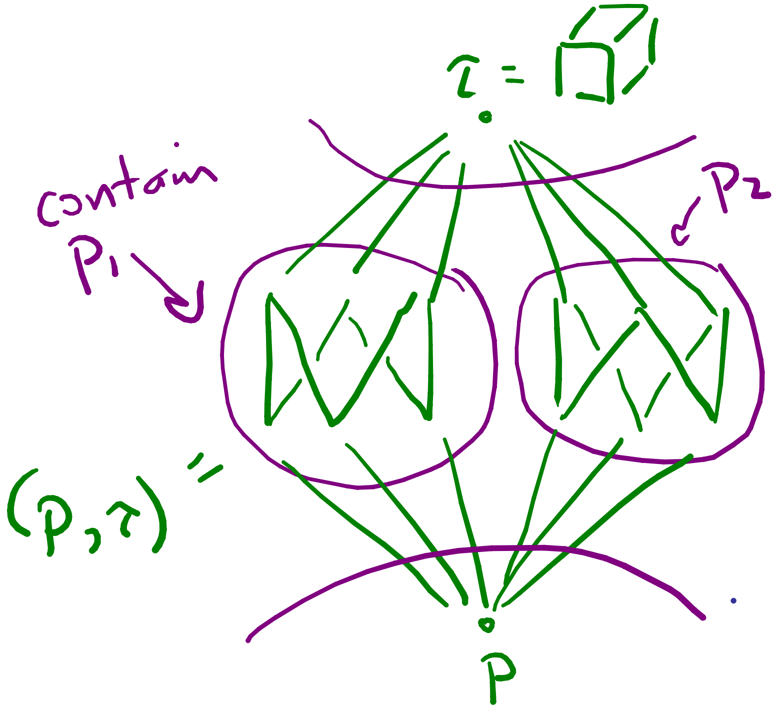
(1) K meets requirements of prop'n for $F(K)$ to be graded by cell dim.

(2) $F(K)$ is thin and each open interval (u,v) for $\dim(v)-\dim(u)>2$ is connected (as graph)

(combinatorial condition)

Non-Example





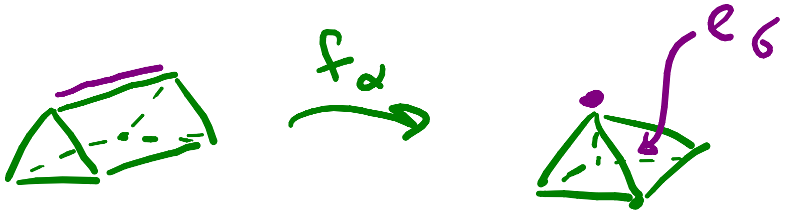
(3) For each α, the restriction of f_α to preimages of codim. one cells in $\overline{e_\alpha}$ is injective.

(topological condition)

Non-Example:

(4) $\forall\, e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as continuous inclusion $i: B^{\dim\sigma} \to B^{\dim\alpha}$ followed by f_α.

Non-Example:
(due to David Speyer)



Notably Absent: Injectivity requirement for $\{f_\alpha\}$ beyond codim. one

Bruhat Order Defined Combinatorially

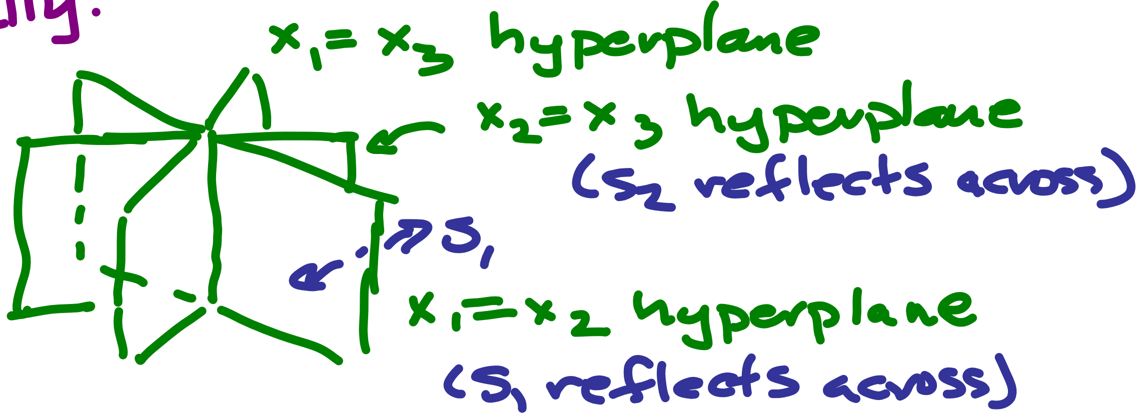
Coxeter group: generators $\{s_i \mid i \in S\}$

& rel'ns $s_i^2 = e$ & $(s_i s_j)^{m(i,j)} = e$

e.g. $\underset{\substack{\| \\ \text{symmetric} \\ \text{group}}}{S_3} = \langle \underset{\substack{\| \\ (1,2)}}{s_1}, \underset{\substack{\| \\ (2,3)}}{s_2} \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e \rangle$ (& other Weyl gps)

geometrically:

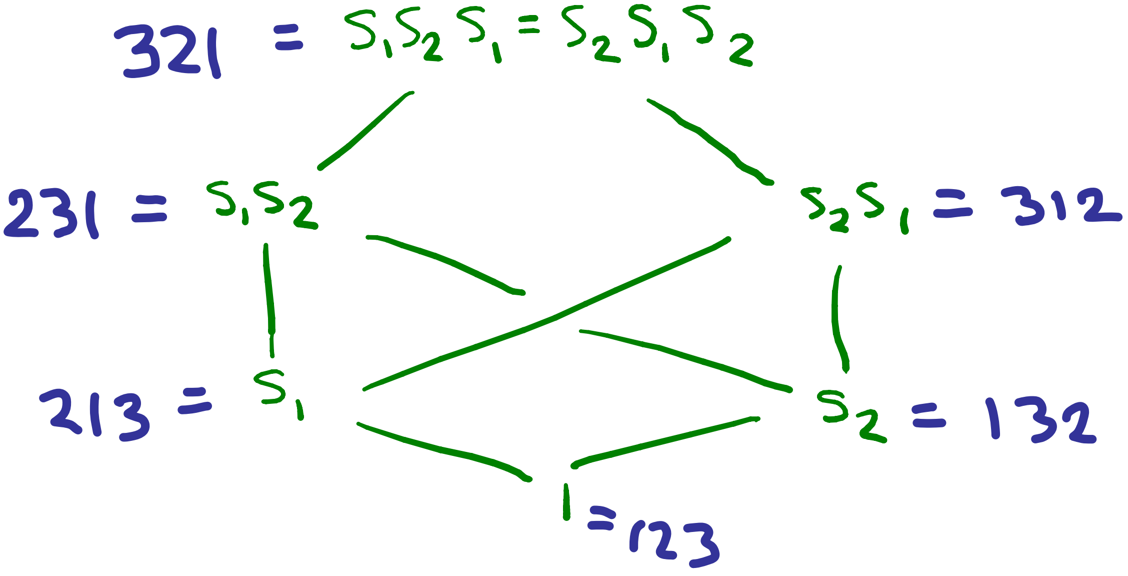
(as group of reflections)



A **reduced expression** for $w \in W$ is an expression of minimal "length" e.g. $s_1 s_2$ but not $s_1 s_2 s_1 s_2$ $(= s_2 s_1 s_2 s_2 = s_2 s_1)$

The **Bruhat order** is partial order on elements of Coxeter group W with $u \le v$ <=> there exist reduced expressions $r(u)$ and $r(v)$ for u & v, respectively, with $r(u)$ subexpression of $r(v)$.

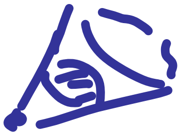
e.g. $W = S_3$ with generators $s_1 = (1,2)$
$s_2 = (2,3)$



- **reduced word** $(i_1, \ldots, i_d)$ for $s_{i_1} s_{i_2} \cdots s_{i_d}$

Reconciling w/ Previous Def'n: e.g. $\overline{\begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \supseteq \begin{pmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Qn (Björner & Bernstein): Since Bruhat order arises in rep'n theory & is CW poset, are there naturally arising regular CW complexes with Bruhat order as closure poset?

Conjectural Sol'n by Fomin & Shapiro: Link of ID  within totally nonnegative, real part of unipotent radical in semisimple, simply-connected algebraic group defined & split over $\mathbb{R}$ (space arising in Lusztig's work)

Aside: Totally positive & totally nonneg. matrices arise in statistics

Recall: A real matrix is totally nonnegative if all minors are nonnegative.

e.g. $$\left\{ \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| t_1, t_2, t_3 \geq 0 \right\}$$

Since $t_1+t_3 \geq 0$ $\quad t_2(t_1+t_3) - t_1t_2 \geq 0$

$t_2 \geq 0$

$t_1t_2 \geq 0$

also: $$\left\{ \begin{pmatrix} 1 & t_2' & t_2't_3' \\ 0 & 1 & t_1'+t_3' \\ 0 & 0 & 1 \end{pmatrix} \middle| t_1', t_2', t_3' \geq 0 \right\}$$

unipotent radical e.g. $$\left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Theorem (H.): Fomin-Shapiro Conjecture indeed holds.

Special Case (Running Example): Space of totally nonnegative upper triangular matrices with 1's on diagonal & entries just above diagonal summing to fixed, positive constant, stratified by which minors are positive and which are 0.

Concrete Realization: image of upcoming map from simplex given by products of certain elementary matrices, by results of Whitney & Lusztig.

The Totally Nonnegative Part of a Space of Matrices

column $i+1$

- $x_i(t) = I_n + tE_{i,i+1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & t & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ row i

$\shortparallel$

$\exp(te_i)$

(general finite type, expon'd Chevalley generator)

(type A)

- $f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \subseteq \mathbb{R}^{n^2}$

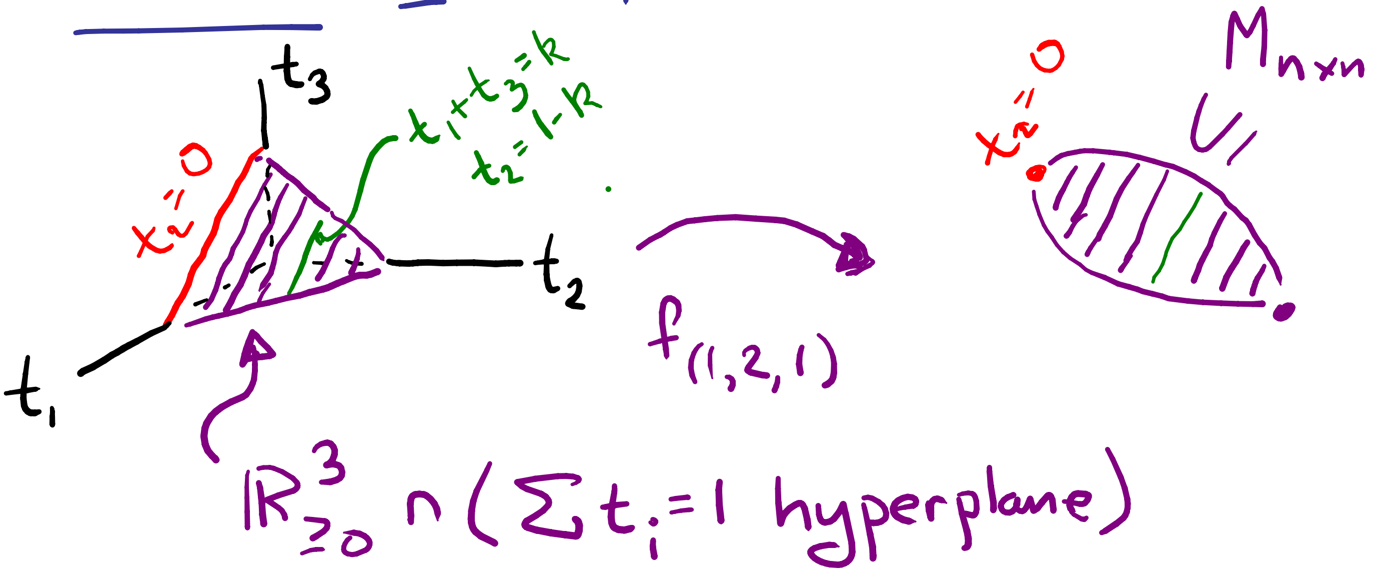
$$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

e.g. $f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

"Picture" of Map



$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\big\downarrow {\scriptstyle t_2=0}$$

$$x_1(t_1)\circ x_1(t_3)$$

$$f_{(1,2,1)}(t_1,0,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

Non-injectivity: results from "nil-moves"

$x_i(u)x_i(v) = x_i(u+v)$ & "long braid moves"

First: A Motivation for Nonneg. Real Part of Unipotent Radical

e.g. $$(t_1,t_2,t_3)\mapsto\left(\underset{\overset{\parallel}{t_1'}}{\frac{t_2t_3}{t_1+t_3}},\ \underset{\overset{\parallel}{t_2'}}{t_1+t_3},\ \underset{\overset{\parallel}{t_3'}}{\frac{t_1t_2}{t_1+t_3}}\right)$$

("simply laced" case)

- Tropicalizes to change-of-basis map for Lusztig's "canonical bases":

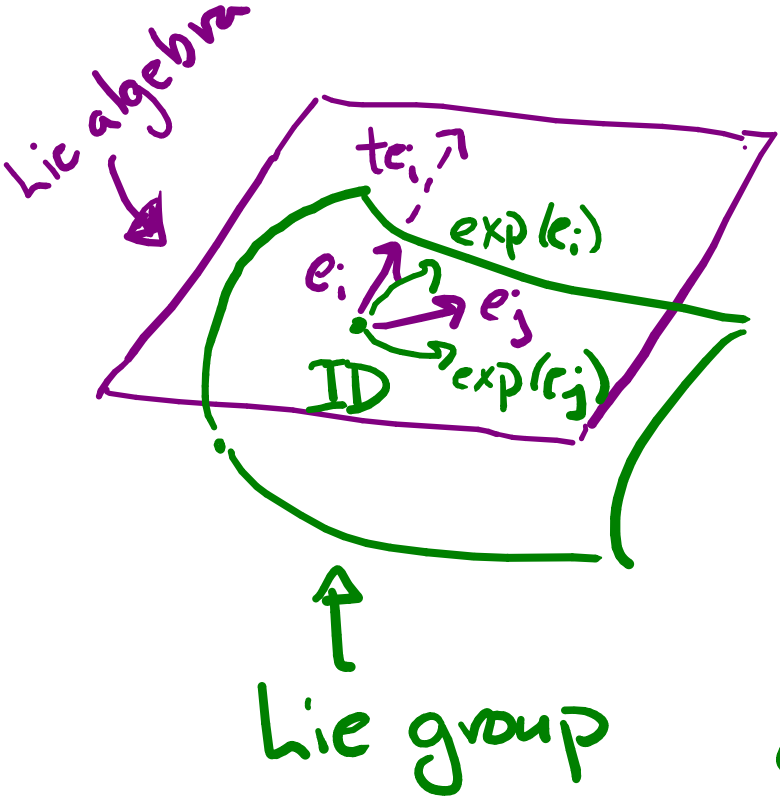
$$(a,b,c)\mapsto(b+c-\min(a,c),\min(a,c),a+b-\min(a,c))$$

(applying braid move to reduced expression for w_0 w.r.t. which canonical basis is defined)

- Given quantized env. alg. $U=U^-\otimes_{\mathbb{Q}(v)}U^0\otimes_{\mathbb{Q}(v)}U^+$

then canonical basis is a basis B for U^- such that highest weight module with highest weight vector v_λ has basis $\{v_\lambda b \mid v_\lambda b\neq 0\}$ for each λ.

Another Motivation: Understanding Relations Among (Exponentiated) Chevalley Generators



Chevalley generators

$$t_1 e_i \circ t_2 e_j \in \mathcal{U}(\mathfrak{g})$$

$$\Big\downarrow \text{ exp\_ map}$$

$$\begin{aligned} &\exp(t_1 e_i \circ t_2 e_j) \\ &\quad\| \\ &\exp(t_1 e_i)\exp(t_2 e_j) \\ &\quad\| \\ &x_i(t_1)\, x_j(t_2) \end{aligned}$$

$$\exp(te_i) = \boxed{ID + te_i} + \underbrace{\frac{t^2 e_i^2}{2}}_{0=} + \underbrace{\frac{t^3 e_i^3}{6}}_{0=} + \dots$$

Proof Strategy (Phrased for Possible Future Applic's too)

Set-up: Continuous, surjective fn

$f : P \longrightarrow Y$

from convex polytope P (e.g. Δ_n) s.t. f maps $\mathrm{int}(P)$ homeomorphically to $\mathrm{int}(Y)$.

Step 1: Perform "collapses" on ∂P, each preserving regularity and homeomorphism type, via continuous, surjective collapsing functions $P \to P$ yielding $P/\sim$ with fewer cells s.t. $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$

(to accomplish IDs we know are needed by collapsing all "non-reduced" faces)

Step 2: Prove $\overline{f} : P/\sim \longrightarrow Y$ is homeomorphism by new regularity criterion

(to prove no further IDs are needed)

1st Key Idea to Proof of F-S Conjecture

0-Hecke Algebra Captures which Simplex Faces have Same Image under $f_{(i_1,\dots,i_d)}$

(1) $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

"nil-move"

$\big\downarrow$ suppress parameters

$x_i^2 = x_i$ (0-Hecke alg. reln, up to sign)

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}\left(\frac{t_2t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1t_2}{t_1+t_3}\right)$

$\big\downarrow$ (type A)

assuming $t_1+t_3>0$

$x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}$

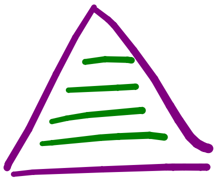
"long braid move" with enrichment from parameters

(similar relation holds outside type A)

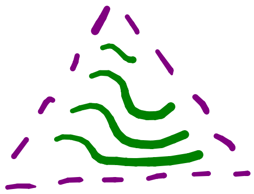
Fibers as Curves:

$x_i^2 \to x_i$

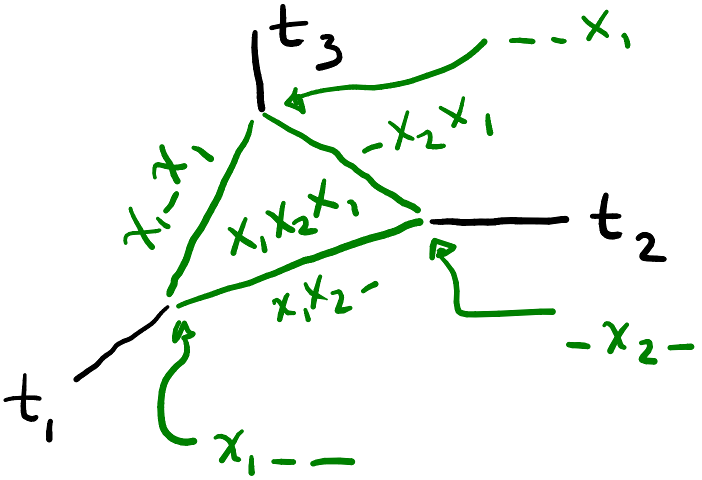
$t_1+t_2=k$



OR after a braid move:



Indexing Faces of Preimage of $f_{(i_1,\dots,i_d)}$ by Words in 0-Hecke Algebra



Key Observation About $f_{(i_1,\dots,i_d)}$:

$$\mathrm{im}(F_1)=\mathrm{im}(F_2) \iff \underbrace{x(F_1)=x(F_2)}_{\substack{\text{equal as}\\ \text{0-Hecke algebra elements}}}$$

Thm (Lusztig): If $(i_1,\dots,i_d)$ is reduced, then $f_{(i_1,\dots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

Observation: "non-reduced" subwords give redundant faces covered by curves, each in single fiber of $f_{(i_1,\dots,i_d)}$

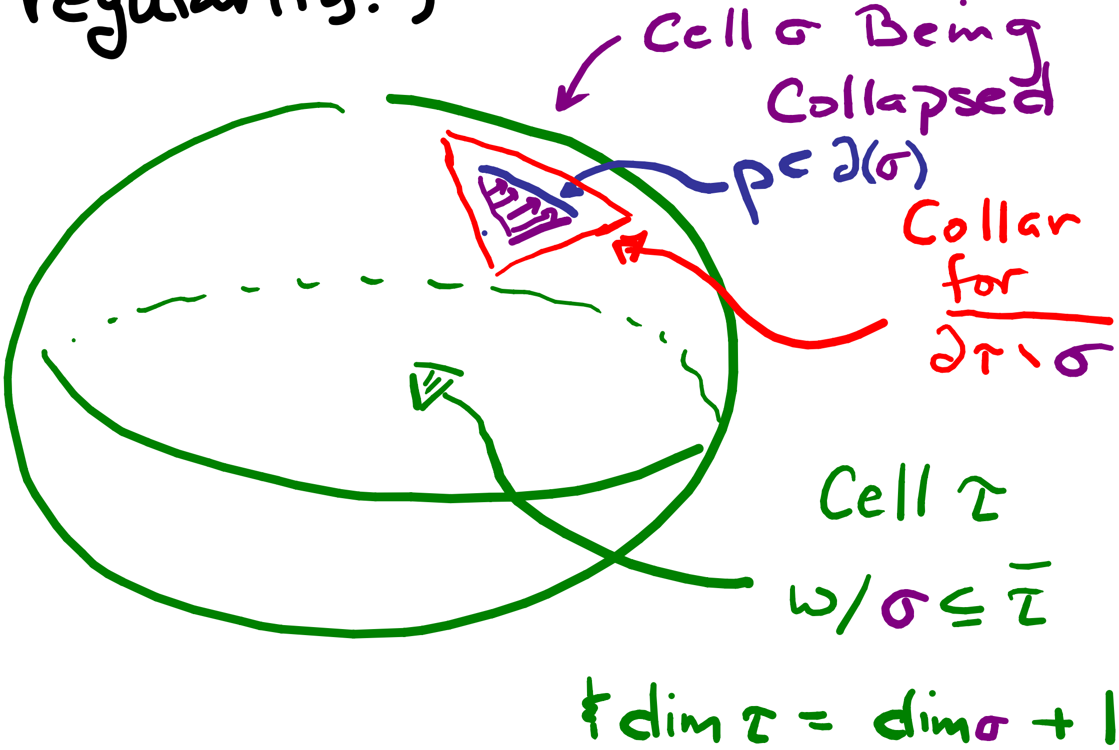
We'll Prove: Only obvious $\exp(te_i)$ reln's occur

Step 1 Collapsing Cell σ onto Cell $\rho \subseteq \bar{\sigma}$ within $\partial\tau$

Thm (M. Brown; Connelly): Any topological manifold with boundary ∂M has a collar (i.e. a nbh'd homeomorphic to $\partial M \times [0,1]$).

e.g. $\overline{\partial\tau \setminus \sigma} =$ 

Plan: Collapse $\bar{\sigma}$ onto $\bar{\rho} \subseteq \partial\sigma$, stretching collar for $\overline{\partial\tau \setminus \sigma}$ to cover $\bar{\sigma} \setminus \bar{\rho}$, preserving top'l manifold structure (& homeom. type & regularity!)

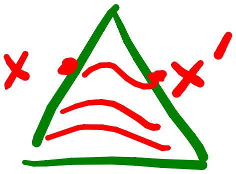


(Mainly Combinatorial) Conditions Allowing Such Face Collapses Across Curves

$$\begin{tikzcd} & \Delta \arrow[dl, phantom, "\cong"] & & & \\ K_0 \arrow[r, "g_1"] \arrow[d, phantom, "\parallel"] & K_1 \arrow[r, "g_2"] \arrow[d, phantom, "\cong"] & K_2 \arrow[r] \arrow[d, phantom, "\cong"] & \cdots \arrow[r] & K_i \\ \Delta & \Delta/\sim_1 & \Delta/\sim_2 & & \end{tikzcd}$$

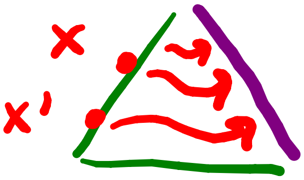
polytope!

- collapse face in K_i across images of parallel line segments in K_0 satisfying:
  - Distinct endpoints condition (DE):



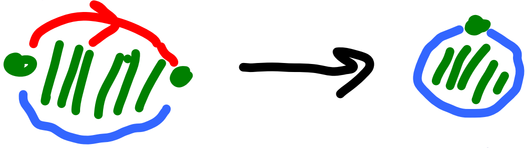
$x \neq x'$

  - Distinct initial points condition (DIP):



$x \neq x'$

  - Least upper bound condition (LUB)

to preclude 

(conditions checkable via 0-Hecke algebra)

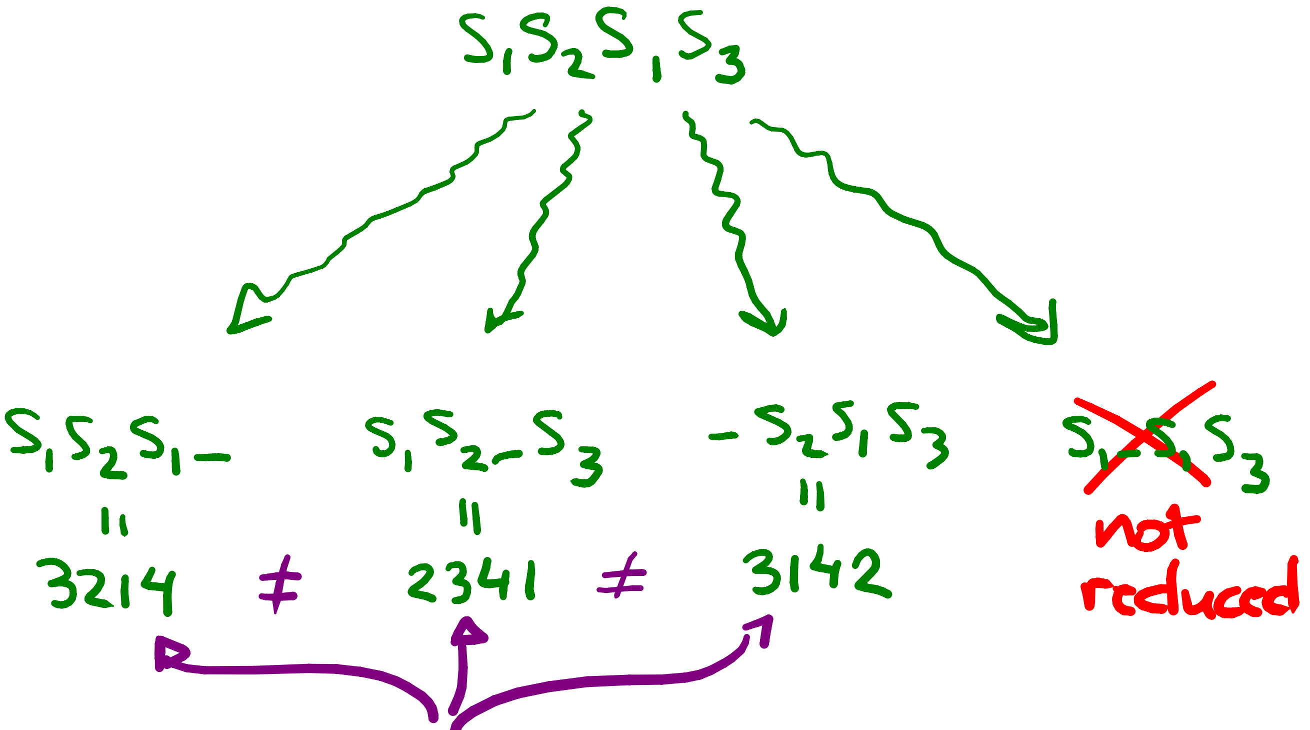
Step 2: Proving that induced map $\bar{f}_{(i_1,\dots,i_d)} : \Delta_n/\sim \;\longrightarrow \left\{\begin{matrix}\text{space}\\ \text{of}\\ \text{matrices}\end{matrix}\right\}$ on quotient space is a homeomorphism.

identifications from collapsing non-reduced faces

Method: use new regularity criterion, with shellability & thinness of Bruhat order giving its combinatorial requirement.

Summary: Eliminate known non-injectivity in step 1, then prove this is all the non-injectivity via regularity criterion.

Injectivity of Attaching Maps in Codimension One via Coxeter group strong exchange axiom



reduced subexpressions of reduced expression obtained by deleting one letter give distinct Coxeter group elements.

In contrast: fails in higher codimension.

Description of fibers via Flow (Based on Collapses) to Base Point

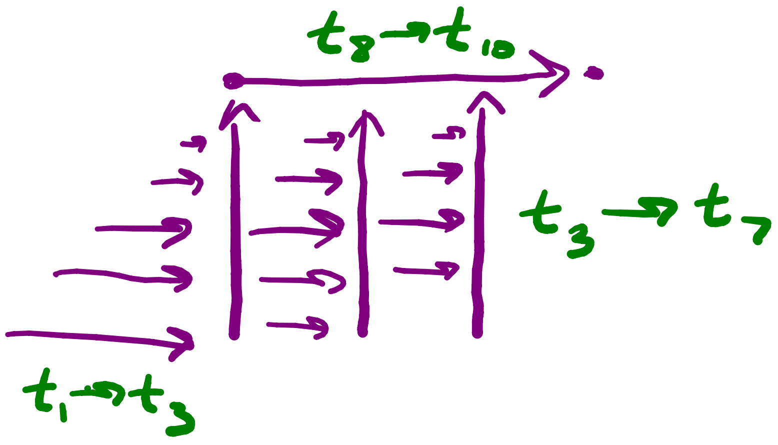
e.g.

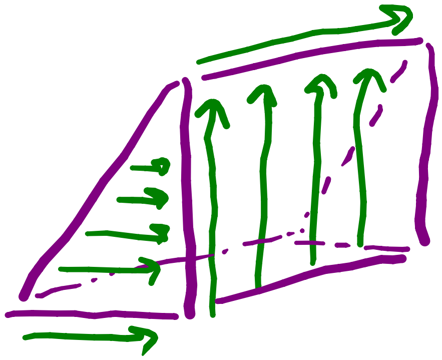
$$\begin{tikzcd}[column sep=tiny] x_1 \arrow[rr, bend left, "1"] & - & x_1 \arrow[rrr, bend left, "2"] & x_2 & x_1 & - & x_2 & x_3 \arrow[rr, bend left, "3"] & - & x_3 \end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny] x_1(t_1) \arrow[r, bend left] & x_1(t_3) \arrow[rrr, bend left] & x_2(t_4) & x_1(t_5) & x_2(t_7) & x_3(t_8) \arrow[r, bend left] & x_3(t_{10}) \end{tikzcd}$$

$$\underbrace{x_1(t_1)\,x_1(t_3)}_{x_1(t_1+t_3)}\,x_2(t_4)\,x_1(t_5)\,x_2(t_7)\,\underbrace{x_3(t_8)\,x_3(t_{10})}_{x_3(t_8+t_{10})}$$

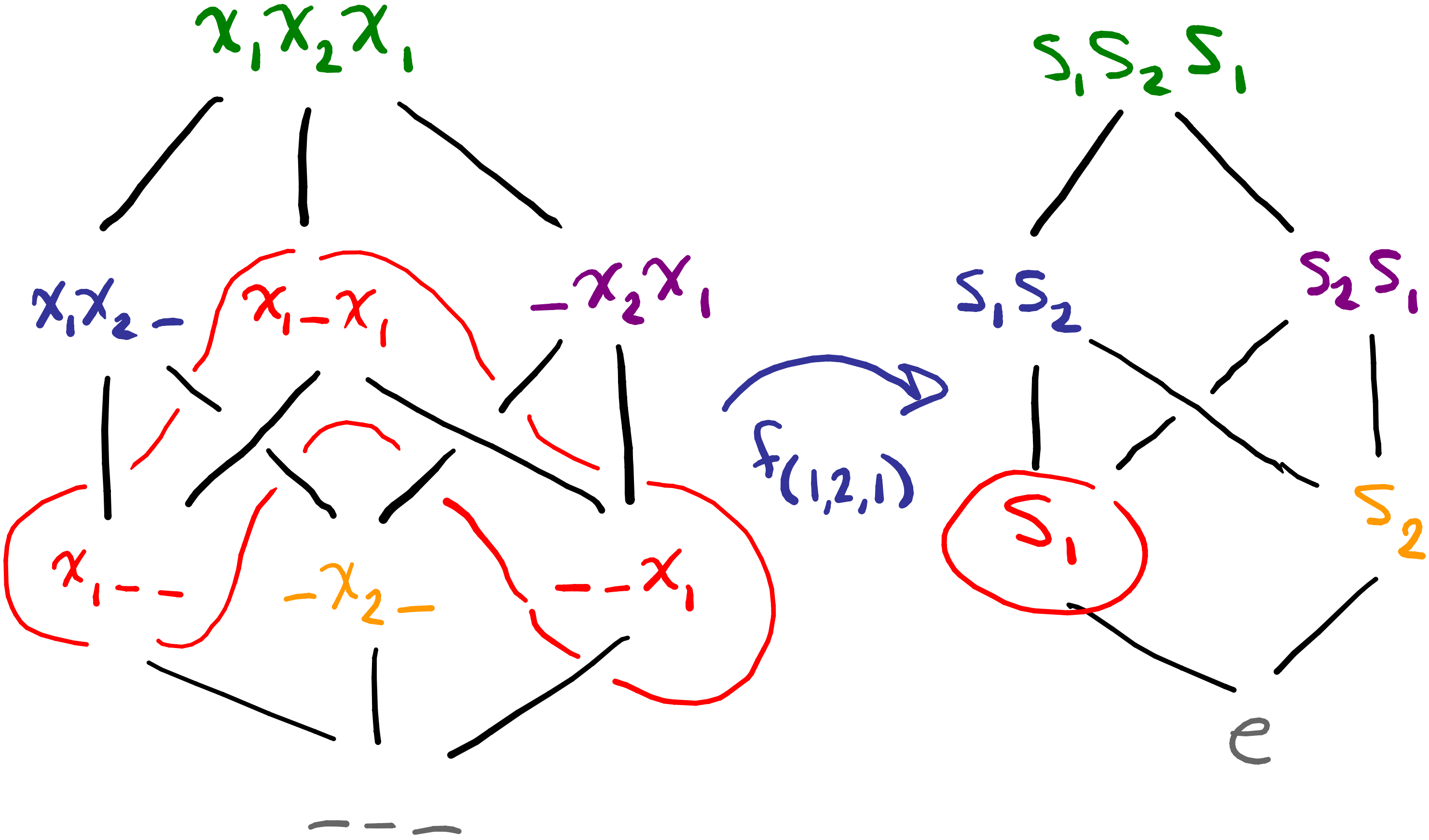
$$\underbrace{x_1(t_1+t_3)\,x_2(t_4)\,x_1(t_5)\,x_2(t_7)}_{x_2(t_1')\,x_1(t_2')\,x_2(t_3'+t_7)}$$





Remark: Our proof factors $f_{(i_1,\ldots,l(d))}$ as product of "nice" maps (as in CE-Approx Thm

A Poset Map (on Face Posets) induced by $f_{(i_1,\ldots,i_d)}$



Boolean Algebra B_n Bruhat Order

The subposets $\{x \in B_n \mid f_{(i_1,\ldots,i_d)}(x) \geq u\}$ are "dual" (upside-down) face posets of so-called "subword complexes"

Other Stratified Spaces with Seemingly Similar Features

1. Totally nonnegative part of Grassmannian

   Postnikov: polytope of plabic graphs w/ "measurement map" to $Gr_{\geq 0}$ + elaborate theory of plabic graphs

   Postnikov-Speyer-Williams: $Gr_{\geq 0}$ is CW complex (via attaching maps that are not homeomorphisms)

2. Closed cells in totally nonnegative part of loop group

   Lam-Pylyavskyy: developed theory of these spaces

3. Totally nonnegative part of flag variety

Rietsch: poset of closure reln's

Marsh-Rietsch: parametrization

Williams: poset is CW poset

Rietsch-Williams: CW complex w/ attaching maps via canonical bases.

Note: our spaces arise as links of cells

4. Stratified spaces of electrical networks

Kenyon-Propp-Wilson, Lam, Curtis-Ingerman-Morrow, Kenyon,...

Open Qn: homeomorphism type & other topol. structure for these spaces?

Intriguing (not-well-understood) Connection to Matrix Schubert Varieties

Observation (Armstrong-H.)

The fibers $f_{\geq}^{-1}(u) = \{x \in B_n \mid f(x) \geq u\}$ are dual to face posets of subword complexes – proven to be shellable balls by Allen Knutson & Ezra Miller (by technique called vertex decomposability).

Subword complexes previously arose as:

Stanley-Reisner complexes for Gröbner degeneration of matrix Schubert variety ideals (Knutson and Miller)