Discrete Morse Theory from a Matching Theoretic Perspective

Patricia Hersh

North Carolina State University

Discrete Morse Theory (Robin Forman)

A discrete Morse function is a function $f: \Delta \to \mathbb{R}$ assigning real numbers to the faces of a simplicial complex or cells of (regular) cell complex s.t. for each $\sigma^{(p)}$ ← notation for p-dimensional cell

1. $\left| \{ \tau^{(p+1)} \mid \sigma^{(p)} \leq \tau^{(p+1)} \text{ \& } f(\sigma) \geq f(\tau) \} \right| \leq 1$

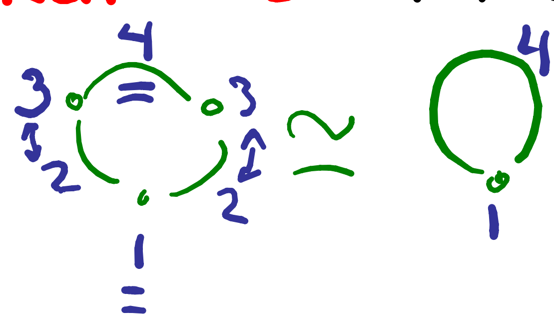
& 2. $\left| \{ \mu^{(p-1)} \mid \mu^{(p-1)} \leq \sigma^{(p)} \text{ \& } f(\mu) \geq f(\sigma) \} \right| \leq 1$

- for each $\sigma^{(p)}$ at most one of these cardinalities is positive
- $\sigma^{(p)}$ is **critical cell** when both cardinalities are 0

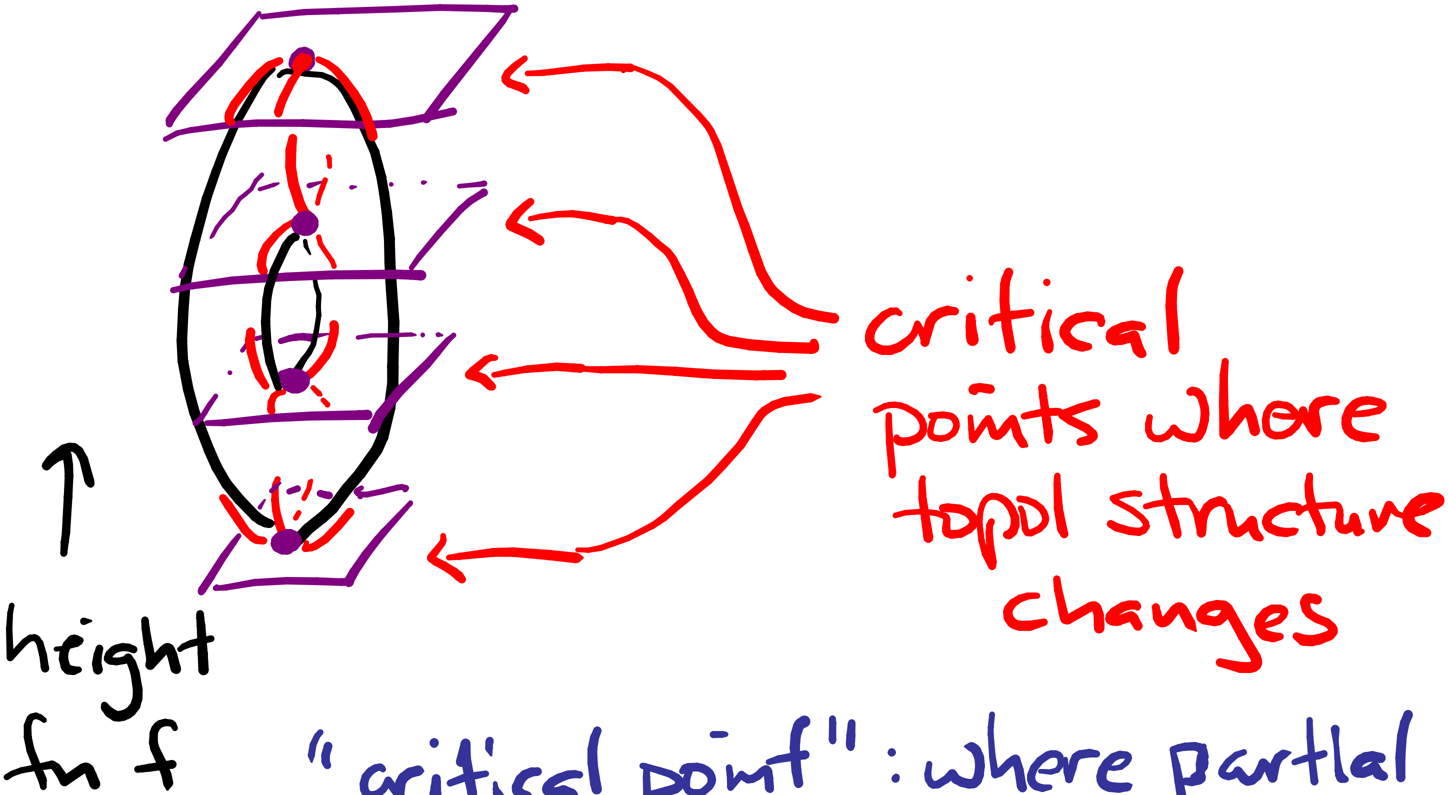
Key Theorem of Discrete Morse Theory

Theorem (Forman): Δ is homotopy equivalent to a CW complex Δ^M comprised of the **critical cells** of the discrete Morse function, i.e "unmatched" cells

e.g.



Traditional Morse Theory:



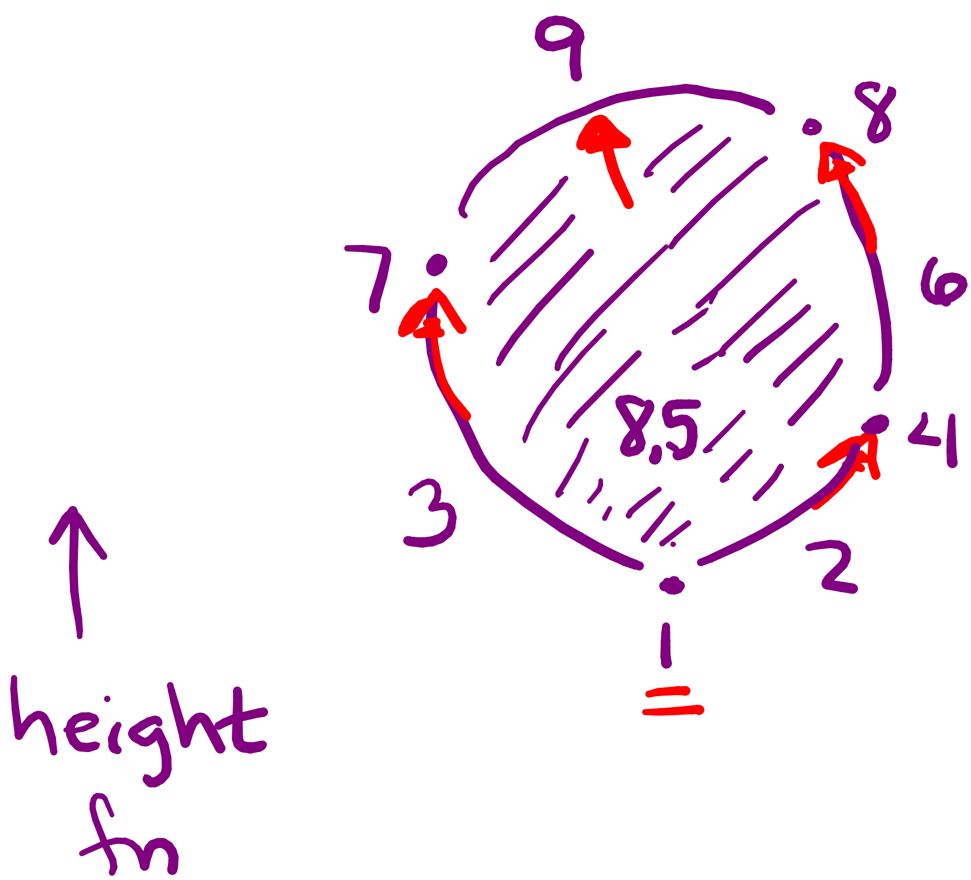
"critical point": where partial derivatives all 0

"index" of crit pt = max dim'l subspace of tangent space where f flows upward

(See John Milnor, "Morse Theory")

Discrete Morse Theory as Discretization of Morse Theory

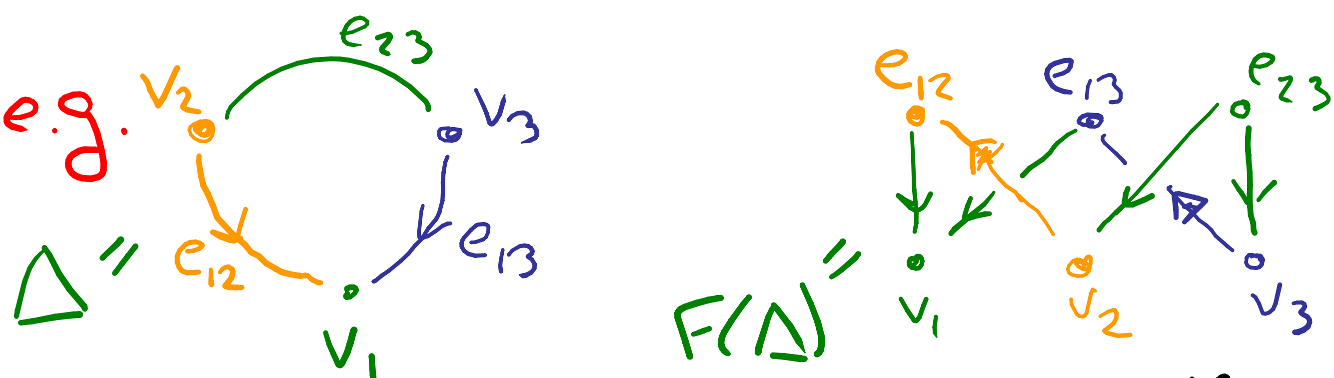
- discrete Morse fns behave like Morse fn / height fn



Chari's Combinatorial Formulation for Discrete Morse Theory

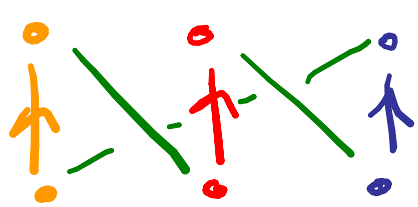
Given simplicial complex Δ, construct an "acyclic matching" aka "Morse matching" on its face poset

e.g.



Hasse diagram s.t. directed graph with matching edges oriented upward & all others downward has no directed cycles.

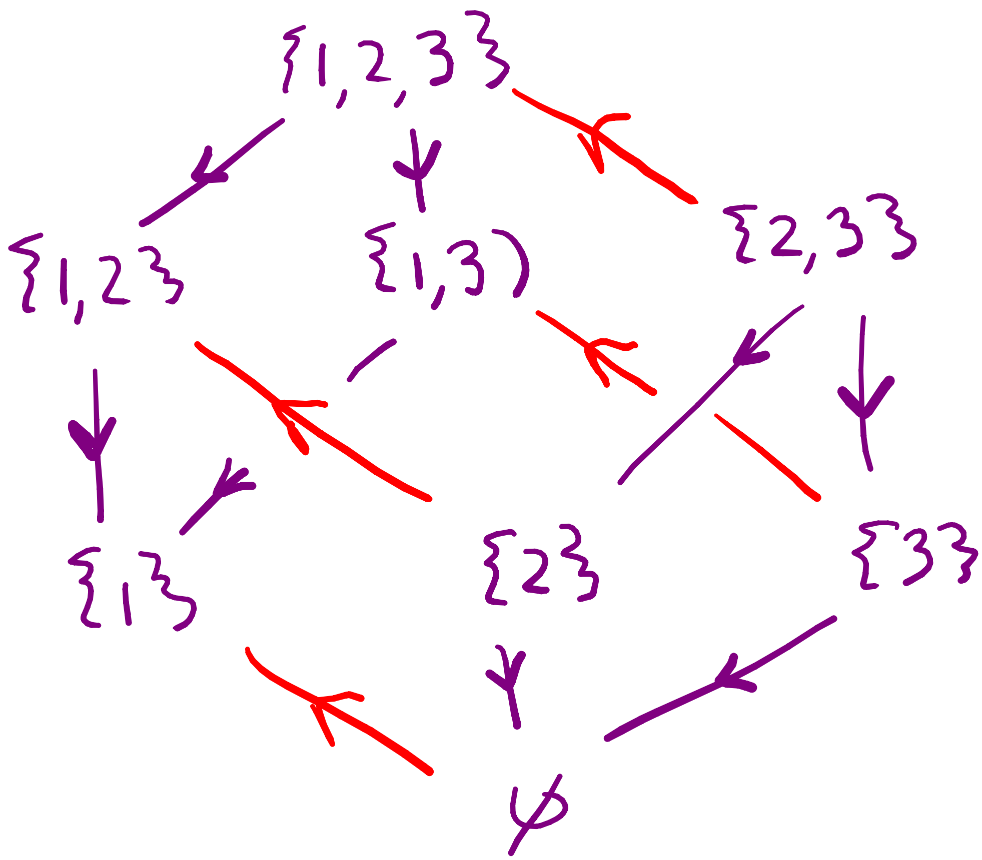
Critical cells = unmatched elements

Nonexample:  not acyclic

First Example

Δ = simplex

$F(\Delta)$ = Boolean algebra



- Match $S \setminus \{1\}$ with $S \cup \{1\}$ for all $S \subseteq [n] := \{1, \ldots, n\}$

Consequences of $\Delta \simeq \Delta^M$:

1. If $F(\Delta)$ has complete acyclic matching (w/ $\emptyset \in F(\Delta)$) then Δ is collapsible.

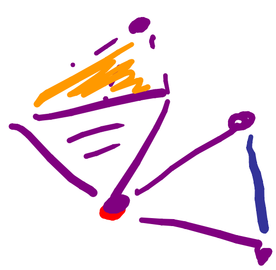
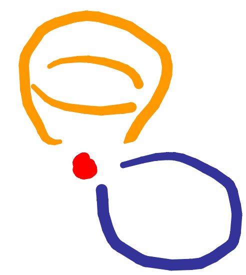
Warning: Some contractible complexes are not collapsible.

e.g. dunce cap 

2. If $F(\Delta)$ has acyclic matching with all unmatched elements at rank j, then Δ is homotopy equivalent to wedge of j-dimensional spheres

3. If $F(\Delta)$ has acyclic matching with all unmatched elements at rank $\geq j$ then Δ is simply connected & $H_i(\Delta, \mathbb{Z}) = 0$ for $i < j$

4. If $F(\Delta)$ has acyclic matching with only facets of Δ unmatched then Δ homotopy equiv. wedge of spheres (w/ # i-spheres = # unmatched i-dim'l facets)

e.g. $\Delta =$  $\longrightarrow$ $\Delta^M =$ 

5. If $F(\Delta)$ has acyclic matching with all unmatched elements at even ranks then Δ has its homology concentrated in even degrees

6. $\tilde{\chi}(\Delta) = \tilde{\chi}(\Delta^M)$

$= -1 + \#\,0\text{-cells} - \#\,1\text{-cells}$

$+ \#\,2\text{-cells} - \ldots$

$= -1 + \beta_0 - \beta_1 + \beta_2 - \ldots$

which can be easier to compute for Δ^M

For Posets: $\mu_P(x,y) = \tilde{\chi}(\Delta(x,y)) = \tilde{\chi}(\Delta^M(x,y))$

order complex of P (discussed later...)

7. Morse Inequalities...

Rk: "Greedy" matchings tend to satisfy acyclicity requirement.

Some Examples & Applications

1. Complex of not 2-connected graphs
   (Babson-Björner-Linusson
   -Shareshian-Welker)
   - motivated by Vassiliev knot invt's

(not 2-connected := disconnected after deleting a vertex)

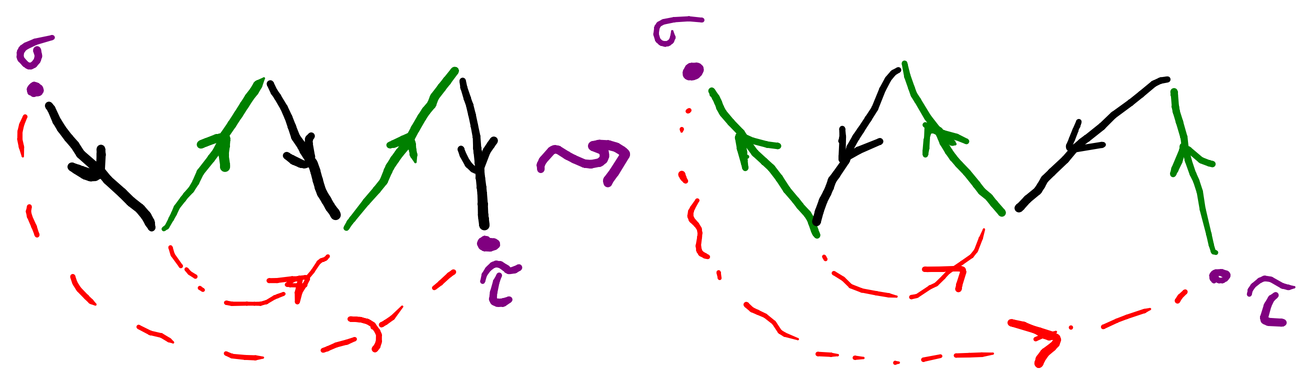
2. More generally: any monotone graph property P gives complex Δ_P with G-edges as vertices in Δ_P
   (see Jakob Jonsson's thesis & book)

3. Applic's to persistent homology & applied topology on point cloud data

Cancelling Pairs of Critical Cells by "Gradient Path Reversal"

If acyclic matching digraph has unique directed path from critical cell $\sigma^{(p+1)}$ to critical cell $\tau^{(p)}$, then reversing path gives new acyclic matching with σ, τ incorporated into the matching.

e.g.

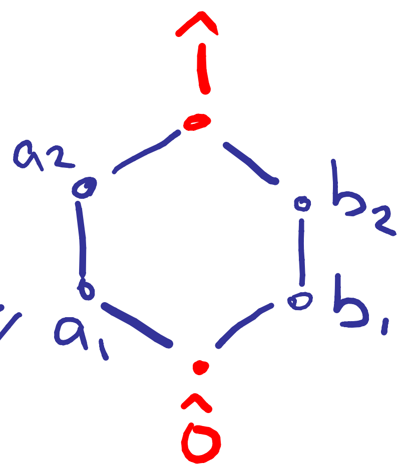


Discrete Morse Theory for Poset Order Complexes

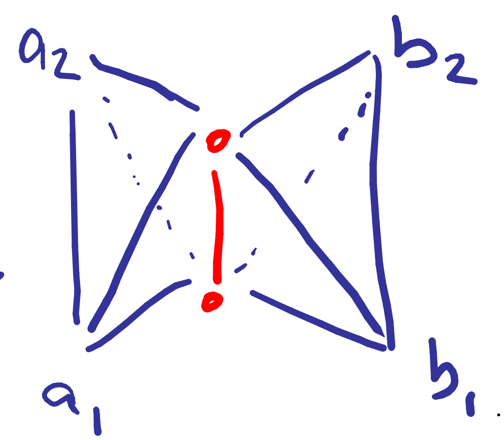
Def'n: The order complex (or nerve) of a poset P is the simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains $v_0 < \dots < v_i$ in P

e.g.

$P =$



$\Delta(P) =$

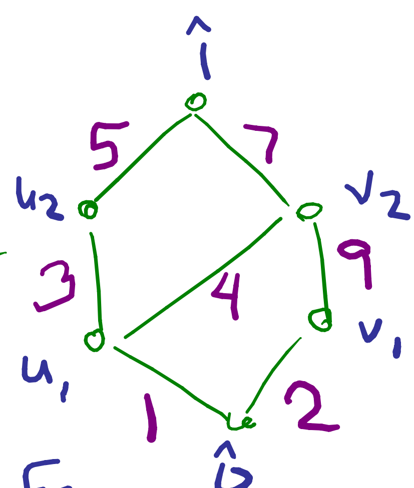


A Discrete Morse Theory Approach to Poset Order Complexes

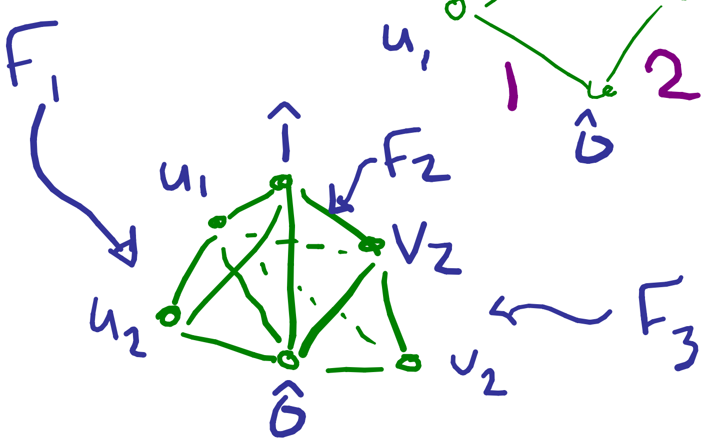
(partly joint work with Eric Babson)

Step 1: Any edge labeling (or chain labeling) on poset P induces lexicographic order $F_1, \ldots, F_m$ on maximal faces (facets) of $\Delta(P)$

e.g.

$P =$ 

$F_1 = 135$
$F_2 = 147$
$F_3 = 297$



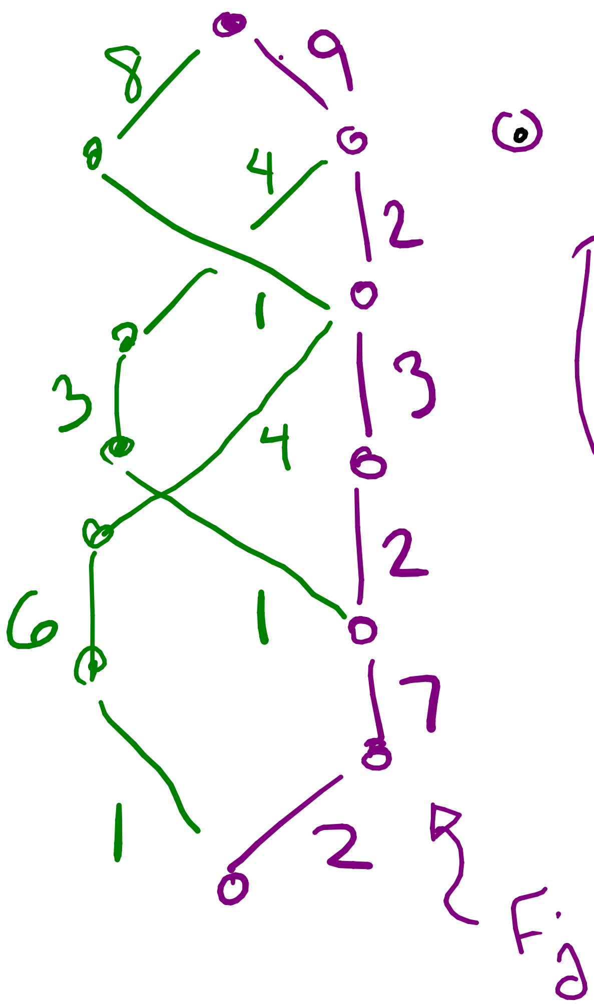
Step 2: Morse matching on each $\overline{F_j} \setminus (\cup_{i<j} \overline{F_i})$ s.t.

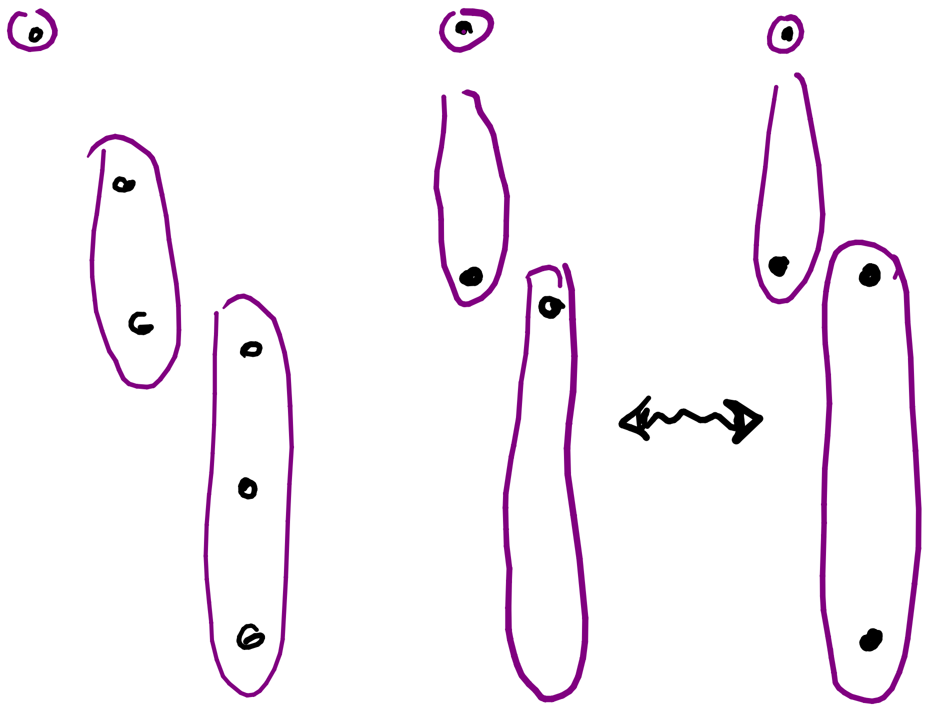
(1) Each $\overline{F_j} \setminus (\cup_{i<j} \overline{F_i})$ has 0 or 1 unmatched (critical) faces

(2) Union of matchings is Morse matching for $\Delta(P)$

Theorem (Babson-H, 2005) Every edge labeling on any finite poset gives rise to lexicographic discrete Morse function with "few" critical cells, i.e. 0 or 1 per facet attachment depending whether homotopy type changes with that facet attachment.

Acyclic matching idea





$\{3,5\} \longleftrightarrow \{1,3,5\}$

Faces in $\overline{F_j} \setminus \bigcup_{i<j} \overline{F_i}$ $\longleftrightarrow$ subsets of ranks $\{1,2,\ldots,5\}$ hitting $\{1,2,3\}$ & $\{3,4\}$ & $\{5\}$

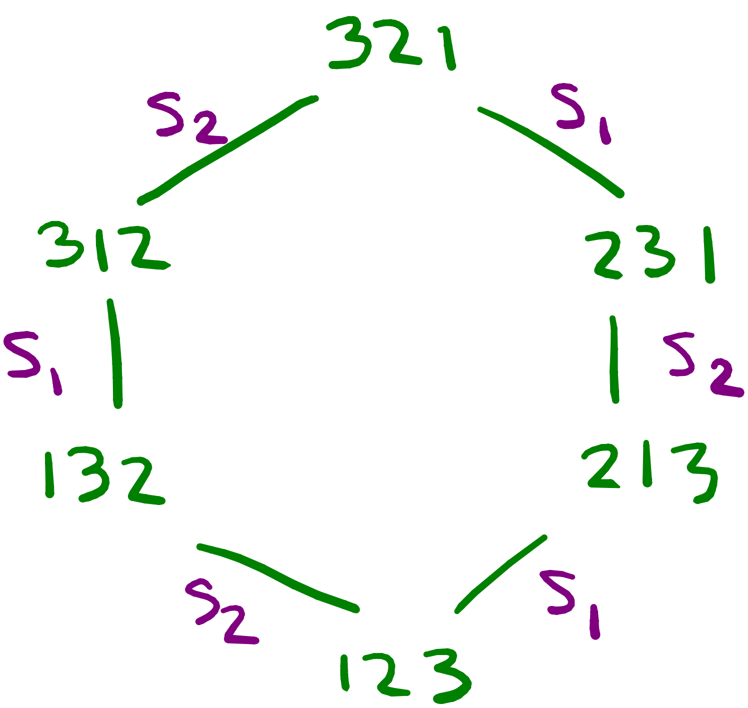
Remark: lexicographic shellability of Björner & Wachs is special case with intervals all of height one.

• if all "minimal skipped intervals" are small, then large # to cover chain ∴ connectivity lower bound

Using this in Practice:

Use "natural" labelings enabling characterization of types of intervals in its interval systems

e.g. weak Bruhat order (not shellable)



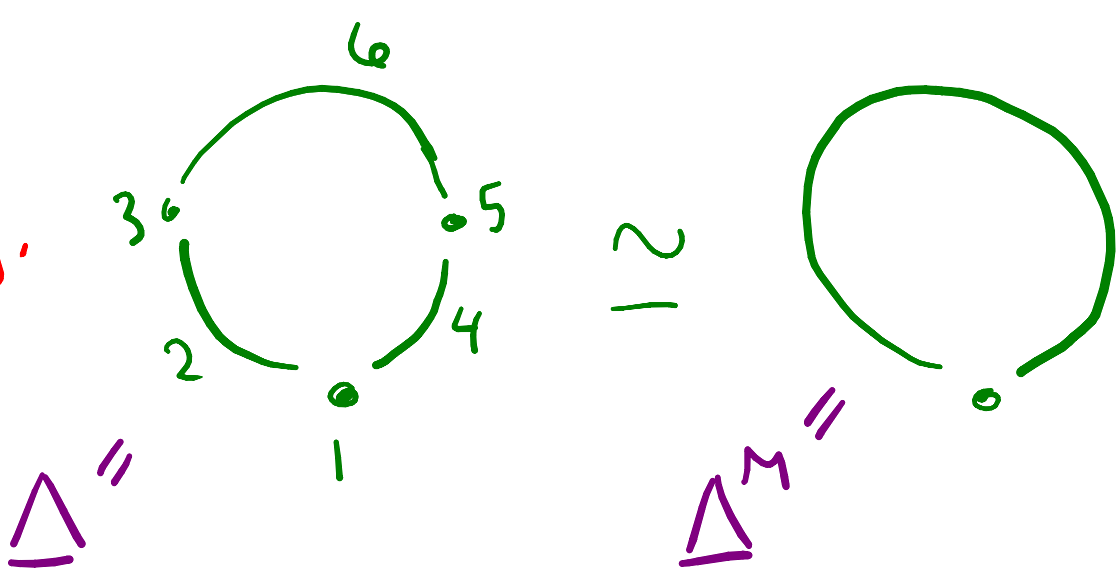
Rk: Especially well suited to posets from algebra

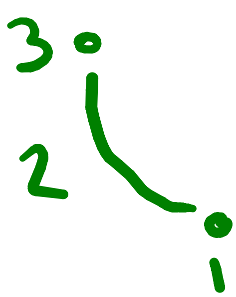
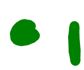
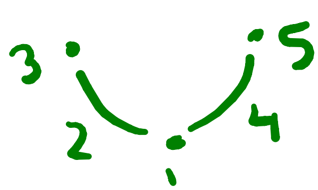
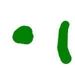
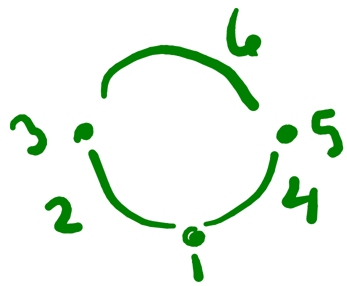
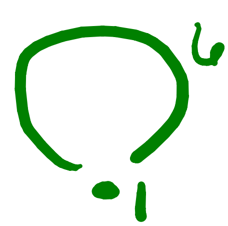
Why $\Delta \simeq \Delta^M$ (& how to construct Δ^M)

Idea: Build Δ & Δ^M showing homotopy equivalence of partial complexes preserved thru process.

- discrete Morse fn specifies cell insertion order

e.g.



Δ	Δ^M	Description of Step
		add critical cell labeled 1
		add non-critical pair 2,3 & eliminate via elementary collapse from Δ^M
		add non-critical pair 4,5
		add critical cell labeled 6

Connection to Simple Homotopy Theory

- An elementary collapse is the elimination of pair of cells σ & τ where σ is a "free face" of τ, i.e. $\sigma \leq \bar{\tau}$ but $\sigma \not\leq \bar{\rho}$ for all $\rho \neq \sigma, \tau$.

- The inverse operation is called an anti-collapse.

- K is simple homotopy equivalent to K′ if K′ may be obtained from K by series of elementary collapses & anti-collapses.

Known Implications.

simple homotopy equivalence $\Rightarrow$ discrete Morse theoretic equivalence

discrete Morse theoretic equivalence $\Downarrow$ homotopy equivalence

homotopy equivalence $\not\Rightarrow$ simple homotopy equivalence

Whitehead group captures discrepancy

Qn: Where exactly is discrete Morse theoretic equivalence situated in middle?

Some Further Topics

(as time permits)...

Algebraic Version of Discrete Morse Theory

(see Jöllenbeck-Welker, Memoirs AMS)

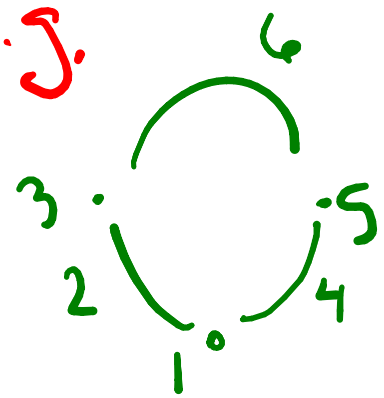
- yields homological results e.g. w/ coefficients in finite char.

Rough Idea: Given an algebraic chain complex & generators of chain groups, build poset on generators with $u \lessdot v \iff \partial v|_u = \text{unit}$. "Acyclic matching" implies chain homotopy to chain complex gen'd by "critical cells" (unmatched elements)

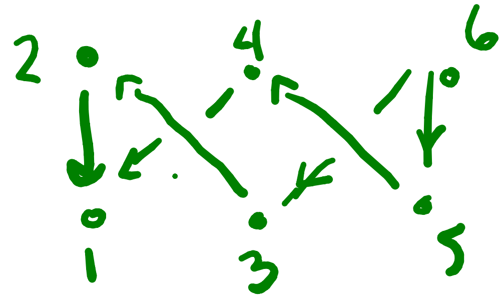
Why Acyclic Matching $\Rightarrow$ Existence of Discrete Morse Function

1. Discrete Morse function on Δ induces acyclic matching on $F(\Delta)$: arrows in direction function (weakly) decreases

e.g.



$\rightsquigarrow$



2. Acyclic matching $\rightsquigarrow$ partial order on faces via $F \leq G \iff F \leftarrow^* G$ where $\leftarrow^*$ is transitive closure of $\leftarrow$. Any linear extension of

this partial order $\leftarrow^*$ is a discrete Morse function giving rise to acyclic matching

More Geometrically:

- Discrete Morse fns on Δ with n faces $\leftrightsquigarrow$ points in $\mathbb{R}^n$.
- Arrows in acyclic matching $\leftrightsquigarrow$ half spaces valid points must lie in.
- Acyclicity $\Rightarrow$ $\cap$ half spaces is nonempty

Exercise: Prove for any shelling order $F_1, \ldots, F_k$ on the facets of a simplicial complex Δ that there is an acyclic matching on each poset $F(\overline{F_j} \setminus \cup_{i<j} \overline{F_i})$ whose union is an acyclic matching on $F(\Delta)$ with critical cells the "homology facets" of the shelling, i.e. those F_j closing off spheres.

Hint: Prove for any filtration $\Delta_1 \subseteq \Delta_2 \subseteq \ldots \subseteq \Delta_k$ of simplicial complexes that a union of acyclic matchings on posets $F(\Delta_j \setminus \Delta_{j-1})$ is acyclic.

References:

- Robin Forman, "A User's Guide to Discrete Morse Theory"
- Jakob Jonsson, "Simplicial Complexes of Graphs"
- John Milnor, "Morse Theory"
- P. Hersh, "On Optimizing Discrete Morse Functions"
- References therein...