SHELLABILITY OF FACE POSETS OF ELECTRICAL NETWORKS AND THE CW POSET PROPERTY

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Abstract. We prove a conjecture of Thomas Lam that the face posets of stratified spaces of planar resistor networks are shellable. These posets are called uncrossing partial orders. This shellability result combines with Lam's previous result that these same posets are Eulerian to imply that they are CW posets, namely that they are face posets of regular CW complexes. Certain subposets of uncrossing partial orders are shown to be isomorphic to type A Bruhat order intervals; our shelling is shown to coincide on these intervals with a Bruhat order shelling which was constructed by Matthew Dyer using a reflection order.

Our shelling for uncrossing posets also yields an explicit shelling for each interval in the face posets of the edge product spaces of phylogenetic trees, namely in the Tuffley posets, by virtue of each interval in a Tuffley poset being isomorphic to an interval in an uncrossing poset. This yields a more explicit proof of the result of Gill, Linusson, Moulton and Steel that the CW decomposition of Moulton and Steel for the edge product space of phylogenetic trees is a regular CW decomposition.

1. Introduction

We prove a conjecture of Thomas Lam from [La14a] that partially ordered sets known as uncrossing posets have dual posets that are lexicographically shellable. This implies that the uncrossing posets themselves are also shellable. This conjecture of Lam is proven in Theorem 3.18. Specifically, we prove that these uncrossing posets are dual EC-shellable (see Definition 2.6). Combining this with Lam's result in [La14a] that these posets are Eulerian (see Definition 2.1), we conclude that these are CW posets (see Definition 2.11), namely that they are face posets of regular CW complexes. Moreover, general properties of lexicographic shellings allow us also to conclude that each closed interval in an uncrossing poset is also a CW poset.

These uncrossing posets, denoted P_n for $n \geq 2$, naturally arise as face posets of stratified spaces of planar electrical networks given by planar graphs (as discussed for instance in [Ke] and [La14b]) for planar graphs that are "well-connected" (a notion defined for instance in [CIM]) with n boundary nodes. Our result that these posets are shellable is suggestive that these stratified spaces may be well behaved topologically, and in particular may be regular CW complexes with each cell closure homeomorphic to a closed ball. A proof that the closure of the big cell is homeomorphic to a closed ball was recently announced and outlined in [GKL]. Our shellability result may be seen as a combinatorial first step towards the still-open question of understanding the

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homeomorphism type of all cell closures for all electrical networks whether or not the networks are well-connected.

Another consequence of our shelling for the uncrossing posets is a shelling for each interval in the face poset for the edge product space of phylogenetic trees, namely the Tuffley poset (see Definition 5.3). We give this shelling for each interval of the Tuffley poset in Corollary 5.5. The main result in [GLMS] is a proof of the existence of a shelling for each interval in the Tuffley poset, but that paper left open the question of constructing such a shelling. The shelling existence result in [GLMS] is used within [GLMS] to prove that the CW decomposition for the edge product space of phylogenetic trees given in [MS] is a regular CW decomposition. Our shelling for the uncrossing poset yields an explicit construction of a shelling for each interval in the Tuffley poset, hence also a more explicit proof that the CW decomposition of [MS] is a regular CW decomposition. The point is that each interval in the Tuffley poset is an interval in an uncrossing poset and that any shelling of an entire poset that is induced by a dual EC-labeling (resp. dual EL-labeling) by definition also induces an explicit shelling on each interval by restricting the labeling to the interval. Thus, our work could also shed some new light on the edge product space of phylogenetic trees (in other words for an important compactification of the tree space studied e.g. in [BHV]).

1.1. **Description of the uncrossing posets.** Denote the uncrossing poset on n wires by P_n. Figure 1 shows the uncrossing poset P_3.

Let us first describe the elements of P_n. We place $2n$ nodes around the boundary of a disk, then connect these nodes in pairs using n wires to do so. In addition to P_n including all such wire diagrams with n wires, we adjoin an element $\hat{0}$. The elements of $P_n \setminus \{\hat{0}\}$ are in natural bijection with those permutations of $2n$ letters which are fixed point free involutions. To see this, begin by labeling the wire endpoints (proceeding clockwise around the boundary of the disk from a chosen basepoint) with the integers $1, 2, \dots, 2n$ assigned in ascending order; the fixed point free involution associated to a wire diagram $D \in P_n$ consists of exactly the product of 2-cycles (i, j) where i and j are the endpoints of a wire in D.

Now let us define the order relation. There is a unique maximal element $\hat{1}$ in P_n given by a wire diagram D in which all n strands cross each other. See the leftmost diagram in Figure 2 for the case with $n = 3$. This naturally corresponds to the fixed point free involution which exchanges i with $n + i$ for each $i \in [1, n]$. We may proceed from an element v to an element u with $u < v$ by taking a pair of wires that cross and locally uncrossing the pair of wires in either of the two possible ways. This gives a cover relation $u \prec v$ if and only if this downward step decreases by exactly one the total number of pairs of wires that cross each other in the drawing for u that minimizes this crossing number. In other words, we have $u \prec v$ if and only if the uncrossing of a pair of wires in v to obtain u does not introduce any double crossings of pairs of wires in u. See the rightmost diagram in Figure 2 for an element obtained by uncrossing a pair of wires in the fully crossed diagram $\hat{1}$, indeed yielding a cover relation for $n = 3$. Notice that uncrossing this same pair of wires in the other direction would introduce a double crossing.

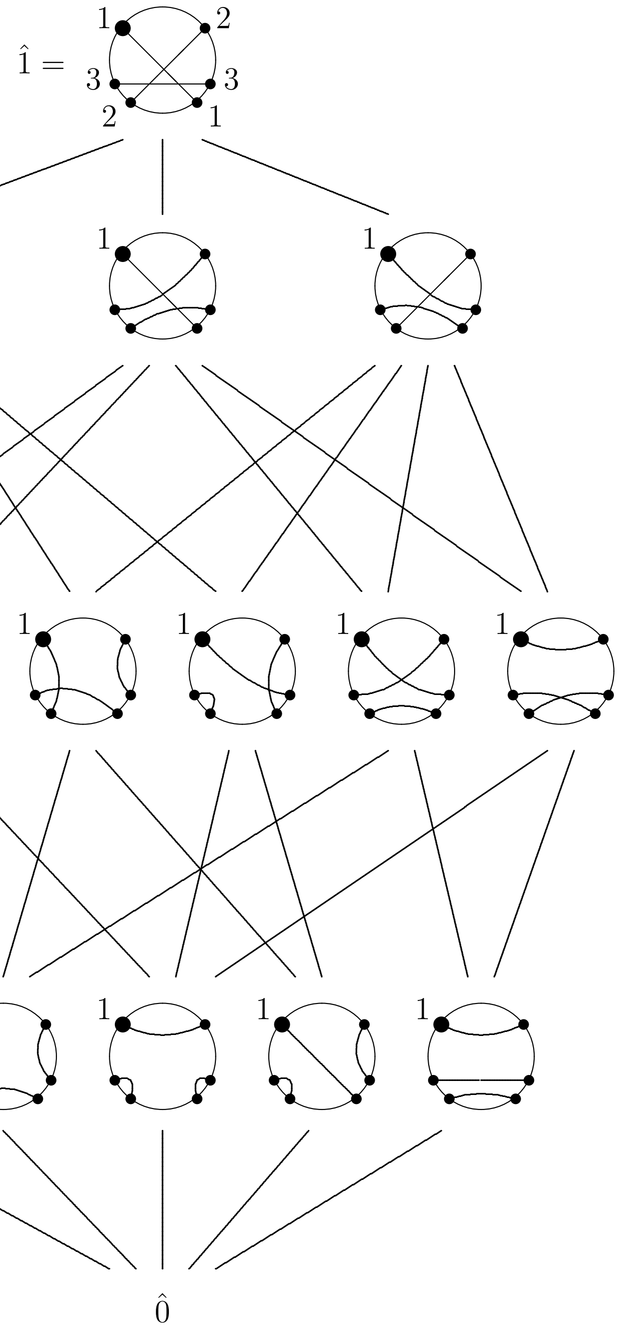


FIGURE 1. The Hasse diagram for P_3

This process of proceeding down cover relations naturally terminates at many different minimal elements given by the various wire diagrams with no crossings. A unique minimal element $\hat{0}$ is artificially adjoined to the poset, rendering the wire diagrams without any crossings as the atoms of the resulting poset P_n. See Lemma 3.16 for a precise combinatorial description for the cover relations in P_n.

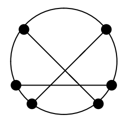
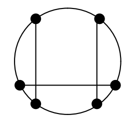
$\hat{1} =$  which covers wire diagram 

FIGURE 2. Two wire diagrams with $n = 3$ wires

Remark 1.1. *The number of atoms in P_n is the n-th Catalan number, namely is* $\frac{1}{2n+1}\binom{2n+1}{n}$.

We observe in Corollary 3.6 that the number of elements in P_n is

$$1 + \frac{(2n)!}{n!2^n}.$$

Remark 1.2. The uncrossing poset P_n is graded by letting the rank of any $D \neq \hat{0}$ be one more than the number of pairs of wires that cross each other in D, with the rank of $\hat{0}$ being 0.

Our main result, conjectured in [La14a] and proven as Theorem 3.18 and Corollary 3.19, is as follows.

Theorem 1.3. *The uncrossing poset P_n is EC-shellable for each $n \geq 2$. Moreover, it is a CW poset.*

The proof exploits a close relationship between P_n and Bruhat order. In particular, the proof uses Dyer's notion of reflection order from [Dy93] to guide the choice of edge labeling. A large class of intervals, namely those preserving what we call the start set of a wire diagram, are proven to be dual isomorphic to type A Bruhat intervals. Our labeling coincides on these intervals with a known Bruhat order reflection order EL-labeling. It might be tempting to think every interval should be isomorphic to a type A Bruhat order interval, but whether this is true seems to be a rather subtle question that remains open.

Another conceptual aspect of the proof is the establishment of an analogue to the notion of the inversion pairs of a permutation (see Definition 3.11 and Lemma 3.16), what we call the noncrossing pairs of a wire diagram. This allows us to construct and justify the validity of a shelling based on a variant of a type A reflection order.

2. Background

We review background on partially ordered sets, shellability, CW complexes and CW posets, reflection orders, Dyer's EL-shelling for Bruhat order, and finally the affine symmetric group. This is done in preparation for the proof of our main result in the following section.

2.1. **Partially ordered sets.** Denote by $u \prec v$ a **cover relation** in a partially ordered set (poset) P, namely an order relation $u < v$ for a pair of elements of $u, v \in P$ such that there does not exist $z \in P$ such that $u < z < v$. We then say v **covers** u. The **Hasse diagram** of a poset P is the graph whose vertices are the elements of P and whose edges are the cover relations $u \prec v$, typically drawn in the plane with each such edge proceeding upward from u to v.

If a poset has a unique minimal element, denote this as $\hat{0}$. Likewise, if a poset has a unique maximal element, denote it as $\hat{1}$. An **atom** is an element that covers $\hat{0}$ while a **coatom** is an element covered by $\hat{1}$. A **chain** is a series $u_1 < u_2 < \cdots < u_k$ of comparable poset elements. A **saturated chain from** u **to** v is a chain $u \prec u_1 \prec \cdots \prec u_k \prec v$ comprised of cover relations.

A **closed interval** $[u, v]$ is the subposet $\{z \in P \mid u \leq z \leq v\}$. An **open interval** (u, v) is the subposet $\{z \in P \mid u < z < v\}$. Any poset P has a **dual poset**, denoted P^*, with the same elements as P and with $u \leq v$ in P^* if and only if $v \leq u$ in P.

A poset is **graded** if $u < v$ implies all maximal chains from u to v have the same number of cover relations, called the **rank** of $[u, v]$. If a graded poset has $\hat{0}$, then the **rank** of each element v is defined to be one more than the rank of each element u covered by v, letting $\hat{0}$ have rank 0.

The **order complex** of a poset P is the abstract simplicial complex, denoted $\Delta(P)$, whose i-dimensional faces are the chains $u_0 < u_1 < \cdots < u_i$ of $i + 1$ comparable poset elements. Denote by $\Delta_P(u, v)$ the order complex of the open interval (u, v) in P. Notice that the saturated chains from u to v will be in natural bijection with the **facets** (namely the maximal faces) of $\Delta_P(u, v)$, a fact that will be important to upcoming "lexicographic shellings".

The **Möbius function** μ_P of a poset P is defined recursively by $\mu_P(u, u) = 1$ for each $u \in P$ and

$$\mu_P(u, v) = -\sum_{u \leq z < v} \mu_P(u, z)$$

for $u \neq v$. The Möbius function $\mu_P(u, v)$ is well-known to equal the reduced Euler characteristic $\tilde{\chi}(\Delta_P(u, v))$ (see [Ro]). In particular, if $\Delta_P(u, v)$ is homeomorphic to a d-sphere, this implies $\mu_P(u, v) = (-1)^d$.

Definition 2.1. *A graded poset is* **Eulerian** *if each $u < v$ has $\mu_P(u, v) = (-1)^{r(u,v)}$ where $r(u, v)$ is the rank of $[u, v]$. A graded poset is* **thin** *if each closed interval $[u, v]$ of rank 2 has exactly 4 elements.*

Remark 2.2. *If a graded poset is Eulerian, in particular it is thin.*

2.2. **Shellability.** Call the maximal faces of a simplicial complex the **facets** of it. Define the link of a face F in a simplicial complex Δ, denoted $lk_\Delta F$ to be the subcomplex $\mathrm{lk}_\Delta F = \{G \in \Delta | G \cap F = \emptyset \text{ and } F \cup G \in \Delta\}$. Define the **combinatorial closure** of a face F in an abstract simplicial complex Δ, denoted $\overline{F}$, to be the set of faces $G \in \Delta$ such that $G \subseteq F$.

A simplicial complex is **pure of dimension** d if each facet is d-dimensional. A simplicial complex is **shellable** if there is a total order $F_1, \ldots, F_k$ on its facets, called a **shelling**, such that for each $j \geq 2$ the subcomplex $\overline{F_j} \cap (\cup_{i<j} \overline{F_i})$ is pure of dimension one less than the dimension of F_j.

Shellability of Δ is well known to imply homotopy equivalence to a wedge of spheres in a manner that is convenient for counting the spheres of each dimension, hence calculating reduced Euler characteristic. A shelling for Δ also induces a shelling for the link of each face F in Δ. Since each open interval (u, v) of a finite poset arises as the link of a face in its order complex, shellability is a useful tool for determining poset Möbius function via its interpretation as reduced Euler characteristic.

Now we turn to poset edge labelings λ, namely labelings of the cover relations $u \prec v$ with labels $\lambda(u, v)$ from an ordered label set.

Definition 2.3. *Refer to $u \prec v \prec w$ with $\lambda(u, v) \leq \lambda(v, w)$ a* **weak ascent***. Call $u \prec v' \prec w$ with $\lambda(u, v') > \lambda(v', w)$ a* **descent***. These terms are used both for the saturated chains from u to w themselves and for the associated ordered pairs of labels.*

Definition 2.4. *A saturated chain $x \prec x_1 \prec x_2 \prec \cdots \prec x_k \prec y$ with $\lambda(x, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_k, y)$ is called a* **weakly ascending chain from** *x* **to** *y. If instead we have $\lambda(x, x_1) > \lambda(x_1, x_2) > \cdots > \lambda(x_k, y)$, then this is called a* **descending chain from** *x* **to** *y.*

First we review the notion of EL-labeling, needed for our usage of Dyer's shelling of Bruhat order as an input to our shellability proof for uncrossing orders. Then we turn to the EC-labelings that will be our main tool for uncrossing orders.

Definition 2.5. *A labeling λ on the cover relations of a poset P with a totally ordered set Λ is an* **EL-labeling** *if for each $u < v$ the following conditions are both met.*

(1) *There is a unique saturated chain $u \prec u_1 \prec u_2 \prec \cdots \prec u_k \prec v$ with weakly ascending label sequence, namely with $\lambda(u, u_1) \leq \lambda(u_1, u_2) \leq \cdots \leq \lambda(u_k, v)$. That is, there is a unique weakly ascending chain from u to v for each $u < v$.*
(2) *This label sequence is lexicographically smaller than the label sequence for every other saturated chain from u to v.*

For the uncrossing orders, we will use a relaxation called EC-shelling of the more well known notion of EL-shelling. This idea of EC-labeling and EC-shellability was developed in [Ko] (see also [He03] for the convenient phrasing with topological ascents/descents we will use). The key will be first to relax the notions of ascent and descent in a way that still captures the same topological properties as the ascents and descents of an EL-labeling while allowing a much wider array of possible labelings.

Definition 2.6. *Given an edge labeling λ of the cover relations in a graded poset, we say $u \prec v \prec w$ is a* **topological ascent** *if the ordered pair $(\lambda(u, v), \lambda(v, w))$ of labels is lexicographically smaller than all of the other label sequences for other saturated chains $u \prec v' \prec w$ from u to w. As a word of caution, notice that it might not be the case that $\lambda(u, v) \leq \lambda(v, w)$. We say that $u \prec v \prec w$ is a* **topological descent** *otherwise.*

*An edge labeling is an EC-***labeling** *if each $u < w$ has a unique saturated chain from u to w comprised entirely of topological ascents. Note that this chain is in particular the lexicographically smallest saturated chain from u to v. A poset with such a labeling is said to be EC-***shellable***.*

The saturated chains may be ordered lexicographically, and for the same reasons that EL-labelings induce shellings, the facet orderings induced by EC-labelings will be shelling orders. The topological descents will function in the shelling analogously to how descents function in an EL-shelling, and the topological ascents will function in the shelling just as ascents do in an EL-shelling: the topological descents $u \prec v \prec w$ in a saturated chain will index the vertices v which may be omitted from the facet corresponding to the saturated chain to obtain the codimension one faces in the closure of the facet that are shared with (closures of) earlier facets.

Remark 2.7. *Since $\Delta(P) = \Delta(P^*)$, it suffices to construct an EL-labeling (or EC-labeling) for P^* to deduce shellability for $\Delta(P)$.*

2.3. **Face posets of regular CW complexes.** An **open m-cell** is a topological space homeomorphic to the interior of an m-dimensional ball B^m. Denote the closure of a cell α by $\overline{\alpha}$.

Definition 2.8. A **CW complex** is a space X and a collection of disjoint open cells e_α whose union is X such that:

(1) X is Hausdorff.
(2) For each open m-cell e_α of the collection, there exists a continuous map $f_\alpha : B^m \to X$, called a **characteristic map**, that maps the interior of B^m homeomorphically onto e_α and carries the boundary of B^m into a finite union of open cells, each of dimension less than m.
(3) A set A is closed in X if $A \cap \overline{e}_\alpha$ is closed in $\overline{e}_\alpha$ for each α.

Definition 2.9. A CW complex K is a **regular CW complex** if there exist characteristic maps $\{f_\alpha\}$ (as defined within Definition 2.11) for each of its m-cells e_α for each m such that f_α restricts to a homeomorphism from the boundary of B^m onto a finite union of lower dimensional open cells.

For K a regular CW complex, let $sd(K)$ denote the first barycentric subdivision of K, using the fact that each cell closure in a regular CW complex is homeomorphic to a round ball to make sense of the notion of barycenter in this level of generality and thereby to define $sd(K)$. Notice for K regular that $\Delta(F(K) \setminus \{\hat{0}\}) = sd(K) \cong K$.

Definition 2.10. *The* **face poset** *or* **closure poset** *of a CW complex K is the partial order $\leq$ on the cells of K with $u \leq v$ if and only if u is contained in the closure of v. This poset is denoted $F(K)$.*

See [Bj84] for the introduction of the next notion and the next theorem.

Definition 2.11. *A finite, graded poset P is called a* **CW poset** *if*

(1) $\hat{0} \in P$
(2) $P \setminus \hat{0} \neq \emptyset$
(3) $\Delta_P(\hat{0}, u)$ *is homeomorphic to a sphere of dimension $rk(u) - 2$ for each $u \neq \hat{0}$*

Theorem 2.12 (Björner, [Bj84])**.** *A finite poset P is a CW poset if and only if there exists a regular CW complex having P as its face poset with $\hat{0} \in P$ representing the empty cell.*

This combines with results from [DK] to yield the following result, which is explained in Proposition 2.2 in [Bj84]. We have slightly rephrased this result below by using the fact that a shelling for a poset induces a shelling for each interval $[x, y]$ in it.

Theorem 2.13. *Any finite graded poset P that is thin and shellable and has unique minimal element $\hat{0}$ as well as at least one additional element will be a CW poset.*

2.4. **Reflection order EL-labeling for Bruhat order.** This section reviews an EL-labeling of Dyer for Bruhat order, though we will only need a special case of it as an ingredient to our upcoming EC-shelling for uncrossing orders. We point out below the special case to be used later, one having the symmetric group as our Coxeter group W. For further background on Coxeter groups and root systems, see [BB] and [Hu].

Definition 2.14. *The* **Bruhat order** *is a partial order on the elements of a Coxeter group W with cover relations $u \prec v$ when v is obtained from u by left multiplication by a reflection that increases "length" exactly by one.*

In the case of the symmetric group, the **reflections** *are the transpositions (i, j) and the* **length** *of any $\pi \in S_n$ is the number of inversions, that is, the cardinality of $\{1 \le i < j \le n | \pi(i) > \pi(j)\}$.*

Given a Coxeter system (W, S) with simple reflections S, let T be the set of all its reflections wsw^{-1} for $w \in W$ and $s \in S$. The reflections of (W, S) are in natural bijection with the positive real roots. By way of this bijection, any total order on positive roots will also induce a total order on reflections.

Recall from Definition 2.1 in [Dy93] and remarks shortly thereafter:

Definition 2.15. A total order $<$ on the positive roots of a root system is called a **reflection order** if each triple of roots $\alpha, \beta, c\alpha + d\beta$ for c, d positive real numbers satisfies $\alpha < c\alpha + d\beta < \beta$ or $\beta < c\alpha + d\beta < \alpha$.

Dyer observes in [Dy93] that the following procedure will always yield reflection orders.

Definition 2.16. For W a (not necessarily finite) reflection group, any total order on its simple reflections gives rise to a **lexicographic reflection order** $<_R$ on all positive roots as follows. Each positive root may be written in a unique way as a positive sum of simple roots, hence as a vector in the coordinates given by the simple roots. We use the given order on simple roots to order the coordinates in these vectors. Scale each resulting vector so that its coordinates sum to 1. To obtain $<_R$, order these scaled vectors lexicographically.

Theorem 2.17 (Dyer, Proposition 4.3 in [Dy93])**.** *Any reflection order induces an EL-labeling on Bruhat order by labeling each cover relation $u \prec v$ with the reflection vu^{-1}.*

Now to the case we will use later:

Corollary 2.18. *For the symmetric group S_n, the edge labeling $\lambda(u, v) = vu^{-1}$ induces an EL-labeling with respect to the following ordering on the set of labels, namely on the transpositions (i, j) for $i < j$ in S_n:*

$$(1, 2) < (1, 3) < \cdots < (1, n) < (2, 3) < \cdots < (2, n) < \cdots < (n - 1, n).$$

That is, for $i < j$ and $i' < j'$ we have $(i, j) < (i', j')$ if and only if we have either $i < i'$ or we have $i = i'$ with $j < j'$.

We will also need the following characterization of cover relations in Bruhat order for S_n:

Theorem 2.19. *There is a cover relation $\pi \prec (i,k) \cdot \pi$ for $i < k$ and for $\pi \in S_n$ in Bruhat order for S_n if and only if the following conditions are both met:*

(1) $\pi(i) < \pi(k)$
(2) *For each j satisfying $i < j < k$ either $\pi(j) < \pi(i)$ or $\pi(k) < \pi(j)$.*

Proof. This is a special case of Proposition 4.6 in [DH], but we also include an elementary proof an effort to keep our work self-contained. This will require showing that $(i,k) \cdot \pi$ has exactly one more inversion pair than π does. Notice that (i,k) will be an inversion pair for $(i,k) \cdot \pi$ but not for π, since $\pi(i) < \pi(k)$ whereas applying (i,k) to π to obtain $\tau = (i,k) \cdot \pi$ directly ensures we have $\tau(i) > \tau(k)$. Also observe for each j satisfying $i < j < k$ that j forms an inversion pair with exactly one of the two letters i, k in π, and it also forms an inversion pair with exactly one of the two letters i, k in τ; specifically, the effect of applying (i,k) to π is to exchange for each such j whether it will be in an inversion pair with i or with k. For every other pair (i', k') with $i' < k'$, namely for each pair not equalling (i,k) and not having i' or k' strictly intermediate in value to i and k, notice that (i', k') is an inversion pair for π if and only if (i', k') is an inversion pair for τ. ☐

3. Proof of Lam's Shellability Conjecture

Let us begin by establishing notational conventions that will help us later to assign names to wires in a wire diagram D in an intrinsic way. This will be useful for giving the poset Hasse diagram an edge labeling based on the names of the wires being uncrossed, a labeling that we will eventually prove is an EC-labeling.

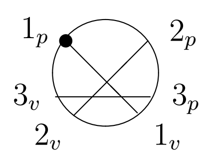


Figure 3. Wire endpoint labels for wire diagram $\hat{1} \in P_3$

Let us choose a fixed wire endpoint in the fully crossed diagram with n wires to serve as a basepoint which we label as 1_p and hold fixed as we proceed up a saturated chain progressively uncrossing more and more pairs of wires. This fixed basepoint 1_p is depicted with a large dot to signify it is the basepoint. This same fixed choice of basepoint 1_p is used in all the wire diagrams arising as poset elements. Read clockwise around D from starting position 1_p. Each time we encounter a new wire, label its endpoint we first encounter as j_p where $j-1$ is the number of distinct wires previously encountered, as depicted in Figure 3. When we reach the second endpoint of a wire whose first endpoint is j_p, label this second endpoint as j_v.

Definition 3.1. Refer to the wire with endpoints labeled j_p and j_v as wire j or as the j-th wire.

Definition 3.2. Define the **word** of a wire diagram D with n strands, denoted $w(D)$, to be the word in the alphabet $1, 2, \ldots, n$ obtained by starting at 1_p and reading clockwise the series of wires encountered. Sometimes it is convenient to suppress the subscripts p and v on the letters, since these subscripts are redundant.

Example 3.3. *The fully crossed diagram D with 3 wires has $w(D) = 123123$. The 5 fully uncrossed diagrams on 3 wires have words*

$$112233; 122331; 123321; 122133; 112332.$$

Remark 3.4. Recall from the definition of the uncrossing order P_n that a downward cover relation $D \to D'$ will uncross a pair of wires i, j such that doing so does not introduce any double-crossings. There are two different potential ways to do this. Notice that for one of them, $w(D')$ is obtained from $w(D)$ by swapping the label i_v with the label j_v for $i < j$. Uncrossing the wires in the other way will first swap the label i_v with the label j_p and then (if necessary) permute the names of the wires so that the labels with subscript p are encountered in increasing order as we proceed from left to right through $w(D')$. In the latter case, observe that in the event that wire names need to be permuted, $w(D')$ will still have a subsequence i, i, j, j, although now j denotes a different wire. Note that such a step preserves the part of the associated word to the left of the letter i_v.

Proposition 3.5. *The word map w gives a bijection from the valid wire diagrams D with n wires to the permutations of alphabet*

$$\{1_p, 2_p, \ldots, n_p, 1_v, 2_v, \ldots, n_v\}$$

with the requirements that i_p appears to the left of j_p for each $i < j$ and that i_p appears to the left of i_v for each i.

Proof. The proof follows directly from observing that exactly these words arise and that the word map is invertible. □

Corollary 3.6. *The number of elements in the uncrossing poset P_n is*

$$|P_n| = 1 + \frac{(2n)!}{n!2^n} = 1 + (2n-1) \cdot (2n-3) \cdot \cdots \cdot 3 \cdot 1.$$

Definition 3.7. *Let us define the* **start set** *of D, denoted $S(D)$, of a wire diagram D with n wires as the n-subset of $\{1, 2, \ldots, 2n\}$ recording the positions in the word $w(D)$ where letters with subscript p appear.*

For example, $w(D) = 12331244$ gives rise to $S(D) = \{1, 2, 3, 7\}$ while $w(D') = 11223344$ yields $S(D') = \{1, 3, 5, 7\}$.

Let us define the **permutation** of D, denoted $\pi(D)$, as the permutation in S_n obtained by taking the restriction of $w(D)$ to the alphabet $1_v, 2_v, \ldots, n_v$ and suppressing subscripts to obtain $\pi(D)$ in one-line notation.

Theorem 3.8. *Each interval $[D_1, D_2]$ in P_n^* satisfying $S(D_1) = S(D_2)$ is isomorphic to the type A Bruhat order interval $[\pi(D_1), \pi(D_2)]$.*

Proof. Notice that the wire crossings in a wire diagram D exactly correspond to the non-inversions in the restriction of $w(D)$ to the alphabet $\{1_v, 2_v, \dots, n_v\}$, so that the desired isomorphism is given by sending D to this restriction of $w(D)$, namely to a permutation in one line notation. To see that this is indeed a poset isomorphism, observe that uncrossing a pair of wires by swapping some i_v with j_v corresponds to applying the reflection (i,j) on the left to the permutation $\pi(D)$ in one line notation given by the restriction of $w(D)$ to the letters $1_v, 2_v, \dots, n_v$. One may use Lemma 3.16 to see that such an uncrossing step creates a double crossing if and only if the number of non-inversions decreases by more than one. Thus, our cover relations in P_n^* are exactly the cover relations of the type A Bruhat order, as is immediate from the characterization of type A Bruhat order cover relations given in Theorem 2.19. $\square$

Define the order $\leq_{lex}$ on subsets of size n of $\{1, \dots, 2n\}$ by $\{i_1, \dots, i_n\} \leq_{lex} \{j_1, \dots, j_n\}$ for $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$ if and only if either $(i_1, i_2, \dots, i_n)$ is a lexicographically smaller vector than $(j_1, j_2, \dots, j_n)$ or the two vectors are equal.

Proposition 3.9. *If $D_1 < D_2$ in P_n^*, then $S(D_1) \leq_{lex} S(D_2)$ for $\leq_{lex}$ the above order on n-subsets of $\{1, 2, \dots, 2n\}$. In other words, $D_1 < D_2$ implies that we have $i_s < j_s$ for some s with $i_r = j_r$ for all $r < s$.*

Proof. Observe that we have $S(D_1) = S(D_2)$ for each cover relation $D_1 \prec D_2$ in which a subsequence i, j, i, j of $w(D_1)$ is transformed to i, j, j, i in $w(D_2)$, namely for each cover relation swapping some pair i_v and j_v in the associated words. Now we turn to the other type of wire uncrossing discussed in Remark 3.4. Observe that replacing i, j, i, j in $w(D_1)$ by i, i, j, j in $w(D_2)$ and then permuting the names of the wires so that wire names are first encountered in ascending order will cause $S(D_2)$ to be obtained from $S(D_1)$ by increasing the value of a single element of $S(D_1)$, namely replacing the position of the first copy of j in $w(D_1)$ by the larger position of the second copy of i in $w(D_1)$. That is, $S(D_2)$ lacks the position of the first copy of j in $w(D_1)$ but instead has the position of the larger copy of i in $w(D_1)$, the latter of which is necessarily larger in order for the i and j wires to cross each other in D_1. $\square$

The proof of Proposition 3.9 also yields:

Corollary 3.10. *Given $D_1 < D_2$ in P_n^* with $S(D_1) = S(D_2)$, then all saturated chains from D_1 to D_2 consist of uncrossing steps which each replace some i, j, i, j in $w(D_1)$ by i, j, j, i in $w(D_2)$. Each $D_1 < D_2$ in P_n^* with $S(D_1) <_{lex} S(D_2)$ has the property that all saturated chains from D_1 to D_2 must use one or more uncrossings of the type which moves i_v in $w(D_1)$ to the position in $w(D_2)$ that is occupied by j_p in $w(D_1)$; in this case, $w(D_1)$ will have a subsequence i, j, i, j and $w(D_2)$ will have a subsequence i, i, j, j.*

Next we introduce for wire diagrams a more refined analogue of the idea of inversion pairs of a permutation.

Definition 3.11. The **noncrossing pair set of** D, denoted $N(D)$, of a wire diagram D equals $N_1(D) \cup N_2(D)$ for the disjoint sets $N_1(D)$ and $N_2(D)$ of ordered pairs defined as follows. $N_1(D)$ consists of those ordered pairs (i,j) for $i < j$ such that $w(D)$ includes subexpression i, j, j, i. $N_2(D)$ consists of those ordered pairs (j, i) for $i < j$ such that $w(D)$ instead has subsequence i, i, j, j.

Remark 3.12. The i,j,j,i and i,i,j,j subsequence requirements for $w(D)$ above which define $N_1(D)$ and $N_2(D)$, respectively, reflect exactly the two different possible ways a pair of wires i and j may be noncrossing. Likewise having the subsequence i,j,i,j in $w(D)$ encodes combinatorially exactly the condition that a pair of wires i and j cross each other.

3.1. **Dual EC-shelling for the uncrossing poset P_n.** Let us now describe an edge labeling for P_n^* whose well-definedness will follow immediately from Lemma 3.16 and which we will prove is an EC-labeling in Theorem 3.18 (based on a series of technical lemmas comprising Section 4).

Definition 3.13. Label $D \prec D'$ in P_n^* as follows. If $w(D)$ has subsequence k,m,k,m, and we uncross wires k and m for $k<m$, to get D' with $w(D')$ having subsequence k,m,m,k, then let $\lambda(D,D')=(k,m)$. (In this case, we have $(k,m)\in N_2(D')$, and the k wire "turns right" upon approaching the point where the wires previously crossed, to avoid crossing, assuming that this approach of the crossing is from a starting point that is the earlier of the two k endpoints within $w(D)$). If $w(D')$ instead has subsequence k,k,m,m, then let $\lambda(D,D')=(m,k)$. (In this case, we have $(m,k)\in N_2(D')$, and the m wire "turns right" upon approaching the previous wire crossing point, now using as the starting point of the approach the later of the two endpoints labelled m in $w(D)$). Finally, we use a special symbol L to label the remaining cover relations with a label L that is defined to be larger than all labels (i,j) for $i<j$ and smaller than all labels (r,s) for $r>s$; specifically, let $\lambda(D,\hat{1})=L$ for each coatom $D\in P_n^*$.

The labels are ordered as follows, denoting by $<_\lambda$ this label order.

Definition 3.14. The ordered pairs (i,j) with $i<j$ are ordered amongst themselves lexicographically, namely with the order $(1,2)<_\lambda(1,3)<_\lambda(1,4)<_\lambda\cdots<_\lambda(1,n)<_\lambda(2,3)<_\lambda\cdots<_\lambda(2,n)<_\lambda\cdots<_\lambda(n-1,n)$. The ordered pairs (r,s) for $r>s$ are ordered amongst themselves reverse linearly based on the second coordinate, breaking ties with reverse linear order on the first coordinate, so as $(n,n-1)<_\lambda(n,n-2)<_\lambda(n-1,n-2)<_\lambda(n,n-3)<_\lambda(n-1,n-3)<_\lambda(n-2,n-3)<_\lambda\cdots<_\lambda(n,1)<_\lambda(n-1,1)<_\lambda\cdots<_\lambda(2,1)$. Finally, $(i,j)<_\lambda L<_\lambda(r,s)$ for each $i<j$ and each $r>s$.

Remark 3.15. The restriction of $<_\lambda$ to labels (i,j) for $i<j$ coincides with the type A lexicographic reflection order (see Definition 2.16) based on the ordering on simple roots induced by the ordering $s_1<s_2<\cdots<s_{n-1}$ on the corresponding type A simple reflections. The label L is set to be larger than all these labels and smaller than all other labels.

These choices will allow the transfer of some established results from [Dy93] related to shellability of Bruhat order to provide useful ingredients to our proof that λ is an EC-labeling for P_n^*.

Next is an analogue to a property of inversions and Bruhat order, namely a characterization of cover relations that will be useful later.

Lemma 3.16. *For D with at least one pair of crossing wires, the cover relations $D'\prec D$ downward from D in P_n are given by exactly those wire uncrossings which* *get labeled via Definition 3.13 by ordered pairs* $(k,m) \notin N(D)$ *such that the following conditions met:*

(1) $(m,k) \notin N(D)$
(2) *If* $k < m$*, then for each* l *satisfying* $k < l < m$ *we have*

$$|\{(k,l),(l,m)\} \cap N(D)| = 1.$$

(3) *If* $k > m$*, then for each* l *satisfying* $l < m$ *or* $k < l$ *we have*

$$|\{(k,l),(l,m)\} \cap N(D)| = 1.$$

Proof. The point is to observe that the above combinatorial condition on $N(D)$ translates exactly to the no-double-crossing condition for the diagram D' obtained by performing the uncrossing of wires k and m in the way that is dictated by the label $\lambda(D',D) = (k,m)$ given by Definition 3.13. That is, we use the label (k,m) to dictate the nature of the uncrossing of wires and will show that the above condition describes when this indeed gives a cover relation.

The equivalence of this reformulation to the no-double-crossing condition can be checked by a straightforward consideration of the various cases given by the various words consisting of the letters k,k,l,l,m,m in those orders which may appear as subsequences of $w(D)$ for $k < m$ and then separately for $k > m$; it is important to utilize our assumption that we have either $k < l < m$ or $l < m < k$ or $m < k < l$ to restrict which subsequences need to be considered. In other words, we must consider the various allowable ways these three wires may cross each other or avoid crossing each other under our hypotheses. It may help the reader to draw a picture and calculate the contribution of wires k,l,m to $N(D)$ for the various allowable subsequences of $w(D)$ comprised of the multiset of letters $\{k,k,l,l,m,m\}$. ☐

Lemma 3.17. *Given* $D_1 < D_2$ *with* $S(D_1) = S(D_2)$*, then the restriction of* λ *to the interval* $[D_1,D_2]$ *in* P_n^* *with label ordering* $<_\lambda$ *is exactly the Dyer reflection order EL-labeling for type A Bruhat order resulting from the lexicographic reflection order given by the ordering*

$$(1,2) < (1,3) < \cdots < (1,n) < (2,3) < \cdots < (2,n) < \cdots < (n-1,n)$$

on the type A positive roots.

Proof. This is immediate from the definition of our labeling together with our earlier isomorphism in Theorem 3.8 which maps an allowable uncrossing $D \prec D'$ of a pair of wires i and j, namely one with $S(D) = S(D')$, to the application of the reflection (i,j) to the corresponding element of Bruhat order. ☐

See Definition 2.6 for the notions of EC-shellability, topological ascent and topological descent, used heavily in the remainder of this section and all of the next section.

Theorem 3.18. P_n^* *is EC-shellable via edge labeling* λ *(see Definition 3.13) for* P_n^* *with respect to the ordering* $<_\lambda$ *(see Definition 3.14) on edge labels. Therefore,* P_n *is shellable.*

Proof. First note that shellability of P_n^* will imply shellability of P_n since these posets have the same chains and hence the same order complex as each other.

To prove EC-shellability of P_n^*, we need to prove for any $u < v$ there is a unique topologically ascending saturated chain from u to v. As a word of caution, when we leave off the adjective "topologically" below, this is deliberate, and we really do mean traditional ascents and descents rather than topological ones in that case.

Lemma 4.2 proves for $u < v < \hat{1}$ in P^* (in other words for $u > v > \hat{0}$ in P) that there is a unique saturated chain from u to v not having any topological descents, which therefore must be the lexicographically first one.

For $v = \hat{1}$ we need a separate argument: Lemmas 4.5 and 4.6 prove that the lexicographically first saturated chain from u to $\hat{1}$ has weakly ascending labels and that every other saturated chain from u to $\hat{1}$ has at least one descent. Lemma 4.1 proves that each descent $\lambda(x,y) > \lambda(y,z)$ for $z \neq \hat{1}$ is a topological descent, implying that each saturated chain with such a descent has a topological descent; moreover, any descent $\lambda(x,y) > \lambda(y,z)$ for $z = \hat{1}$ is a topological descent because the wire diagram x then has a single crossing with $\lambda(x,y) = (j,i) > (i,j) = \lambda(x,y')$ for $i < j$ the two wires comprising the unique wire crossing in x. Thus, the lexicographically first saturated chain from u to $\hat{1}$ is the only topologically ascending chain. □

Corollary 3.19. *The uncrossing order P_n is a CW poset.*

Proof. Our proof of Lam's shellability conjecture given in Theorem 3.18 will imply that uncrossing posets are CW posets, due to the fact that they are by definition graded posets (see Remark 1.2) and were already proven to be Eulerian in [La14a]. Thus, Theorem 2.13 applies. □

This shelling for P_n will also induce a shelling for each interval in the face poset for the edge product space of phylogenetic trees, namely a shelling for each interval in the so-called Tuffley poset. This consequence of our shelling for P_n is explained and justified in Section 5.

4. Key Technical Lemmas

Now we turn to the heart of the proof that our labeling λ for P_n^* is an EC-labeling, namely the lemmas which together provide the technical details of the proof. First we handle intervals $[u,v]$ for $v < \hat{1}$, proving each such interval has a unique topologically ascending chain. We begin by showing how traditional descents always yield topological descents, thereby considerably restricting the possibilities for how topologically ascending chains may arise.

Lemma 4.1. *For each $u < v < \hat{1}$ in P_n^*, any descent in any saturated chain from u to v (with respect to the edge labeling λ) is a topological descent.*

Proof. It suffices to prove the following: given $x \prec y \prec z$ in P_n^* with labels $\lambda(x,y) = (p,q)$ and $\lambda(y,z) = (r,s)$ such that $(p,q) >_\lambda (r,s)$, then there is a saturated chain $x \prec y' \prec z$ with lexicographically smaller label sequence from x to z. We break the proof of this assertion into cases, based on the various ways a descent $\lambda(x,y) >_\lambda \lambda(y,z)$ may arise.

First suppose there are four different wires involved in the two consecutive wire uncrossings $x \prec y \prec z$ in P_n^* comprising a descent $\lambda(x,y) >_\lambda \lambda(y,z)$. Notice that these two uncrossings may be carried out in the other order yielding some $x \prec y' \prec z$ in P_n^*, since reversing the order in which the two uncrossings are carried out will not impact the fact that z has exactly two fewer crossings than x, forcing y' to have exactly one more crossing than z and one fewer crossing than x. Reversing the order in which these two uncrossings are carried out preserves both the wire name at the earlier of the two endpoints for the smallest of the four wires involved in the two uncrossings as well as preserving the property that this endpoint belongs to the smallest of the four wires involved in the two uncrossings. Letting $\lambda(x,y') = (a,b)$ and $\lambda(y',z) = (c,d)$, we claim that we have $p < q$ if and only if we have $c < d$ and likewise we have $r < s$ if and only if we have $a < b$; these observations follow from the fact that deleting other wires not involved in the uncrossings being performed does not impact which of these wires have endpoints that are encountered first in clockwise order proceeding from our basepoint.

These observations together with our label ordering and the fact that the pair of uncrossing steps uses four distinct wires will yield $(a,b) <_\lambda (p,q)$ from $(p,q) >_\lambda (r,s)$, just as needed, as we now check by running through the various possible cases. The case with $p > q$ and $r > s$ must have $q < s$ in order for $x \prec y \prec z$ to have a descent $(p,q) = \lambda(x,y) >_\lambda \lambda(y,z) = (r,s)$ in its labels. Hence, such a descent must have q as the overall smallest wire amongst the four wires involved in the two uncrossing steps. This yields the result in this case whether we have $a < b$ (which implies $(a,b) <_\lambda (p,q)$ due to having $a < b$ and $p > q$) or we have $a > b$ (since in this case we have $b > q$ with $a > b$ and $p > q$, hence $(a,b) <_\lambda (p,q)$). See Figure 4. This same analysis also applies in the case with $p > q$ and $r < s$ in the event that we also have $r > q$. If we instead have $p > q$ and $r < s$ with $r < q$, then this implies $a < b$ with $r = a$, by our observations above, yielding the result. See Figure 5. Finally, for $p < q$, then having a descent in $x \prec y \prec z$ means we also must have $r < s$ with $r < p$, which implies $a < b$ with $r = a$, giving the result in this case. See Figure 6. This completes the proof for all possible cases with four different wires involved in two consecutive wire uncrossings carried out by cover relations $x \prec y \prec z$.

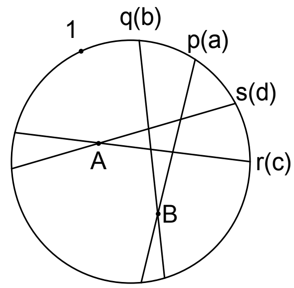


FIGURE 4. $u \prec z$ and $x \prec y'$ uncrossings at A; $x \prec y$ and $y' \prec z$ uncrossings at B

Now to $x \prec y \prec z$ carrying out two uncrossings involving a total of three wires. All of the possible cases in P_n^* correspond naturally (by restriction to these three wires) to

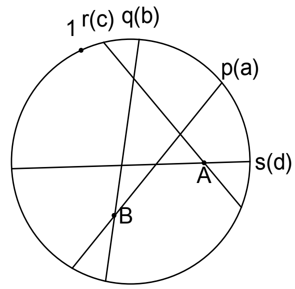


FIGURE 5. $x \prec y'$ and $y \prec z$ uncrossings at A; $x \prec y$ and $y' \prec z$ uncrossings at B

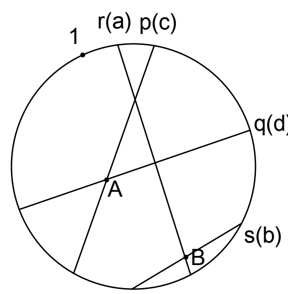


FIGURE 6. $x \prec y$ and $y' \prec z$ uncrossings at A; $x \prec y'$ and $y \prec z$ uncrossings at B

cases that arise in P_3^*. This description of various cases involving three wires according to how they restrict to P_3^* seems to be a good way to organize these cases for P_n^*. We will prove that each such descent in P_n^* restricts to a descent in P_3^*. The authors have checked by hand that all descents in P_3^* are topological descents. See Figure 7 for this edge labeling for P_3^*, from which the interested reader may also check this claim quite easily for P_3^*; it is important to note that one must traverse the cover relations downward rather than upward in Figure 7 so as to consider saturated chains in P_3^* rather than in P_3. We will also prove for the inclusion map from P_3^* to P_n^* that is inverse to the aforementioned restriction map that each topological descent in P_3^* includes into P_n^* as a topological descent in P_n^*. Once these claims are proven, this will yield the desired result.

Consider an edge label (a, b) for an uncrossing in P_n^* arising in the case of a descent $x \prec y \prec z$ involving a total of three wires in the two consecutive uncrossings. Also consider the unique uncrossing $x \prec y'$ for $y' \neq y$ and $y' \prec z$, noting that $x \prec y' \prec z$ also carries out uncrossings involving only these same three wires. Let us show now that passing back and forth between P_3^* to P_n^* by wire inclusion and by restriction to these three wires, respectively, will not impact the relative order of the labels $\lambda(x, y)$ and $\lambda(x, y')$. For convenience in doing this, let us denote by (a', b') the corresponding edge label for P_3^* obtained by restriction to these three wires. This desired result will

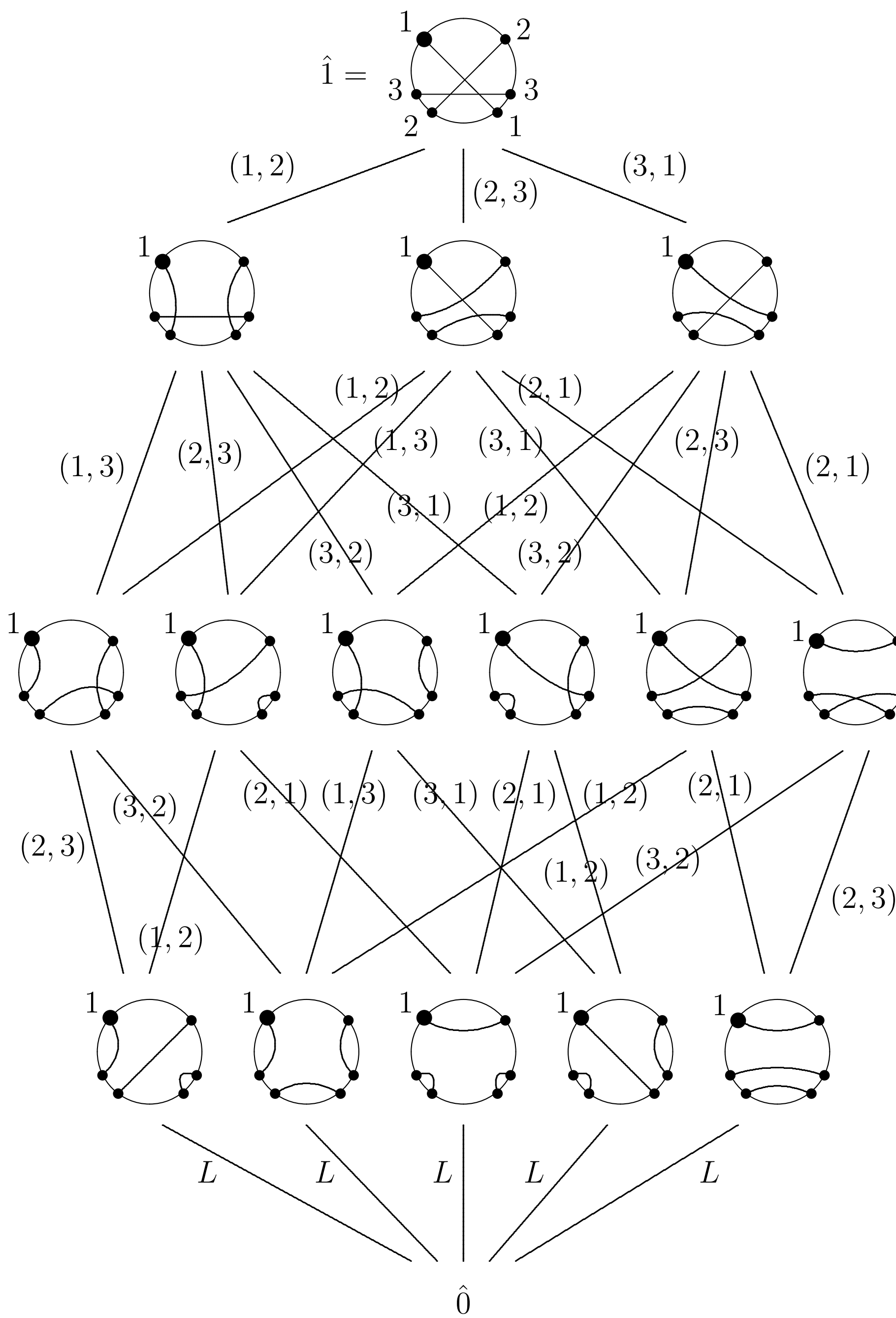


FIGURE 7. Dual EC labeling for P_3

follow directly from the following three facts that are themselves immediate from the definitions of the labels for uncrossing steps and of the label ordering $<_\lambda$:

(1) A label (a, b) has $a < b$ (resp. $a > b$) if and only if the label (a', b') has $a' < b'$ (resp. $a' > b'$).

(2) Two labels (a, b) and (c, d) in P_n^* for uncrossings involving a total of three wires that either occur in consecutive steps $x \prec y \prec z$ or in steps $x \prec y$ and $x \prec y'$ will satisfy $\min\{a, b\} < \min\{c, d\}$ if and only if the labels for the corresponding uncrossings in P_3^* satisfy $\min\{a', b'\} < \min\{c', d'\}$.
(3) For $\lambda(x, y) = (a, b)$ and $\lambda(x, y') = (c, d)$, we have $\min\{a, b\} = \min\{c, d\}$ if and only if $\min\{a', b'\} = \min\{c', d'\}$. In this case of equality, we also have $\max\{a, b\} < \max\{c, d\}$ if and only if $\max\{a', b'\} < \max\{c', d'\}$.

If we can show that each scenario producing a descent $\lambda(x, y) >_\lambda \lambda(y, z)$ in P_n^* with three wires involved in the wire uncrossings corresponds to a situation also giving a descent in P_3^*, we can use the above observations to deduce that each such descent in P_n^* is a topological descent by the following chain of reasoning. Having a descent in P_n^* will restrict to one in P_3^* which will then imply there is a lexicographically earlier label sequence from x to z in P_3^*. By virtue of the preservation of relative order of labels on $x \prec y$ and $x \prec y'$ upon restriction from P_n^* to P_3^* and the inverse operation of inclusion of P_3^* into P_n^*, a topological descent in P_3^* will correspond via wire inclusion to a topological descent in P_n^*. That is, the lexicographically earlier label sequence in P_3^* from x to z (guaranteed to exist in P_3^* by virtue of $x \prec y \prec z$ being a topological descent in P_3^*) will imply the existence of a corresponding lexicographically earlier label sequence from x to z in P_n^* by inclusion of $x \prec y'$ into P_n^* by wire inclusion. This will ensure that $x \prec y \prec z$ will be a topological descent in P_n^*.

Now to the claim about descents in P_n^* restricting to descents in P_3^* for $x \prec y \prec z$ with uncrossings involving a total of three wires. Suppose we have label $\lambda(x, y) = (r, s)$ and then $\lambda(y, z) = (p, q)$ for $(r, s) >_\lambda (p, q)$ in P_n^*. If we have $r < s$, then the uncrossing step with label (r, s) renames only the later endpoints (in clockwise order proceeding from basepoint) of the wires being uncrossed. But we must have $p < q$ in this case in order to have a descent and also must have $p \leq r$; these properties are not impacted in passing from p, q, r, s to p', q', r', s', so the descent stays a descent upon restriction to P_3^*, completing the $r < s$ case. Now suppose $r > s$. If we have a descent comprised of $\lambda(x, y) = (r, s)$ and $\lambda(y, z) = (p, q)$ for $p < q$ in P_n^*, then the corresponding consecutive labels (r', s') and (p', q') in P_3^* also comprise a descent in P_3^*, since $r > s$ implies $r' > s'$ and $p < q$ implies $p' < q'$ in this case. Likewise, for $r > s$ and $p > q$ with $q > s$, restricting from P_n^* to P_3^* will yield $r' > s', p' > q'$ and $q' > s'$ by the three observations listed earlier in this proof. Finally, observe that it is not possible to have consecutive labels $\lambda(x, y) = (j, i)$ and then $\lambda(y, z) = (k, i)$ for $i < j < k$ in any saturated chain in P_n^*, since the uncrossing step $x \prec y$ given by (j, i) will cause the wires i and k no longer to cross each other, rendering $y \prec z$ with label $\lambda(y, z) = (k, i)$ impossible. This completes the $r > s$ case, and hence completes the case in which a total of three wires are involved in the two consecutive uncrossing steps comprising a descent in P_n^*. □

The next lemma will complete the proof that each interval $[u, v]$ for $v < \hat{1}$ has a unique topologically ascending chain, namely its lexiocographically earliest saturated chain.

Lemma 4.2. *Given $u < v < \hat{1}$ in P_n^*, there is a unique saturated chain from u to v with no topological descents (with respect to edge labeling λ).*

Proof. Let C be a saturated chain from u to v which has no topological descents. At least one such saturated chain exists, since the lexicographically first saturated chain from u to v takes this form. Since Lemma 4.1 proved that each descent is a topological descent, we are assured that the labels for C are weakly ascending. For C comprised of cover relations $u = v_0 \prec v_1 \prec v_2 \prec \cdots \prec v_r \prec v_{r+1} = v$, this means we have

$$\lambda(u, v_1) \leq \lambda(v_1, v_2) \leq \cdots \leq \lambda(v_r, v).$$

By definition of our label ordering $<_\lambda$, all of the labels on C of the form (i, j) for $i < j$ must occur lower on the saturated chain than all of its labels which are of the form (j', i') for $j' > i'$. The labels of the form (i, j) with $i < j$ must proceed upward in C from smallest to largest value of i, breaking ties by proceeding from smallest to largest value of j. Otherwise, we would have a descent, and hence by Lemma 4.1 we would have a topological descent. Likewise, the labels of the form (j', i') for $j' > i'$ must proceed upward in C from largest to smallest value of i', breaking ties by proceeding from largest to smallest value of j'.

Our plan is to show that such a saturated chain C from u to v comprised entirely of topological ascents is uniquely determined by the associated words $w(u)$ and $w(v)$. When we have $S(u) = S(v)$, then the result follows immediately from Proposition 3.9, Corollary 3.10, and Lemma 3.17 since these results show in this case that $[u, v]$ is isomorphic to a type A Bruhat interval with our labeling restricting to a Bruhat order reflection order EL-labeling for this interval (using Dyer's results in [Dy93] guaranteeing this is indeed an EL-labeling for Bruhat order); in particular, this guarantees there is a unique saturated chain with weakly ascending labels in this case. Therefore, we may assume henceforth that $S(u) \neq S(v)$. In fact, it will suffice by this same reasoning to prove that the portion $v_m \prec v_{m+1} \prec \cdots \prec v_r \prec v$ of C having (1) $S(v_m) = S(u)$ and (2) $S(v_{m+1}) \neq S(u)$ is uniquely determined. By definition of our labeling, this will be exactly the part of C using labels (j', i') with $j' > i'$. Now we turn to this task, in particular showing that such v_r will be uniquely determined by u and v. Once we do that, the same argument may be applied repeatedly to determine v_{r-1} then v_{r-2} and so on, until eventually reaching some v_m with $S(v_m) = S(u)$. At that point, $u \prec v_1 \prec v_2 \prec \cdots \prec v_m$ is uniquely determined by our above argument handling the $S(u) = S(v)$ case.

Let us begin by making and verifying two observations to be used later. The first observation is that proceeding up any cover relation $v_i \prec v_{i+1}$ anywhere in C either fixes all letters of the form j_p (namely with subscript p) in passing from $w(v_i)$ to $w(v_{i+1})$ or else moves one or more such letters rightward while moving a single letter i_v leftward. The former situation is what happens for each cover relation labeled (k, l) for $k < l$, since such a cover relation exchanges some k_v with some l_v, while the latter describes cover relations labeled (j, i) for $j > i$ which move one or more letters of the form j_p rightward while moving a single letter i_v leftward to the position of the leftmost of these letters moving rightward. In either case, it cannot happen that a letter j_p moves leftward. In particular, this implies that each letter j_p whose position is the same in $w(u)$ and in $w(v)$ must be fixed throughout each saturated chain from u to v. Our second observation is that when a cover relation $v_i \prec v_{i+1}$ labeled (j, l) for $j > l$

moves a letter l_v leftward in passing from $w(v_i)$ to $w(v_{i+1})$ by moving the letter to a position that was occupied by some j_p in $w(v_i)$, then we claim that any letter k_v which appears to the left of l_v in $w(v_i)$ but to the right of l_v in $w(v_{i+1})$ must have $k > l$, as explained next. Otherwise the cover relation $v_i \prec v_{i+1}$ would introduce a double crossing of the wires labeled k and l in $w(v_{i+1})$ by virtue of $w(v_i)$ necessarily having the subsequence $k_p, l_p, j_p, k_v, l_v, j_v$. But this would contradict $v_i \prec v_{i+1}$ being a cover relation, completing the proof of this claim. See Figure 8.

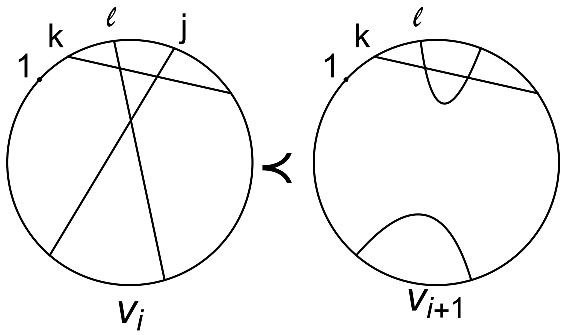


FIGURE 8. $\lambda(v_i, v_{i+1}) = (j, l)$

Let us now show that the cover relation $v_r \prec v$ in C must have label (j, l) for $j > l$ for a uniquely determined value l. Specifically, we will show that l must be as small as possible among letters l_v which either appear at a position in $w(v)$ that is in $S(u)$ or where $w(v)$ has a subsequence l_p, m_p, m_v, l_v with m_v appearing at a position in $w(v)$ that is in $S(u)$; in the latter case, l_v is then the right endpoint of a wire in v having nested below it such a letter m_v. See Figure 9. If l were not as small as possible with this property, then C would necessarily have a label (j', l) for the same value l and some $j' > l$ at some point lower in the saturated chain, since eventually the saturated chain must move the letter j_p either to the location occupied by l_v in $w(v)$ or to the nested m_v position described above where j_p appears in $w(v_m)$ in that case. But this label (j', l) lower on C will guarantee the existence of a larger label (j', l) lower in the label sequence for C than the label $\lambda(v_r, v) = (j', l')$ with $j' > l' > l$ appearing at a higher position in C. In particular, this ensures a descent (and hence a topological descent by Lemma 4.1) somewhere in C, a contradiction, The upshot is that the smaller value l in the label (j, l) is uniquely determined as described just above.

Next observe that the value j in the label $\lambda(v_r, v) = (j, l)$ having $j > l$ with $v_r \prec v$ in a topologically ascending chain is uniquely determined by $w(v)$ and l, as follows. The position of j_p in $w(v_r)$ is the position of l_v in $w(v)$, allowing us to determine j from $w(v)$ and l by virtue of $w(v_r)$ necessarily coinciding with $w(v)$ to the left of this position, as explained in Remark 3.4.

Now suppose there are two distinct cover relations $v_r \prec v$ and $v_r' \prec v$ downward from v both having the same label (j, l) for $j > l$; moreover, suppose that v_r and v_r' both belong to topologically ascending chains from u to v. Let us first check that this necessarily implies that $w(v)$ has a subsequence (a) l, l, j, j, t, t or (b) l, l, j, t, j, t for

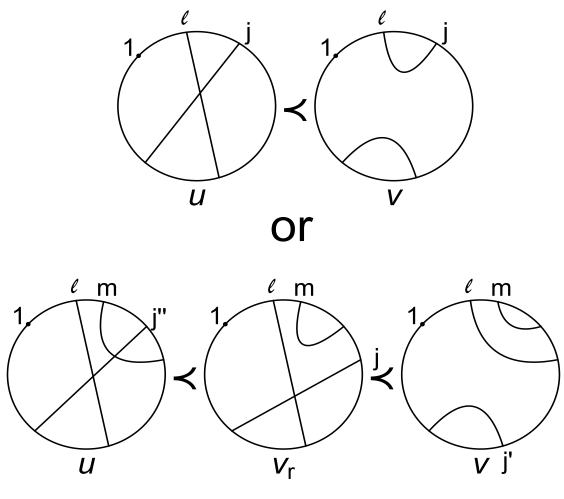


FIGURE 9. $\lambda(v_r, v) = (j, l)$

$l < j < t$; the other possibility, namely having the subsequence l, l, j, t, t, j appearing in $w(v)$, is ruled out by virtue of the fact that a cover relation must eliminate a single crossing. See Figure 10. Specifically, the need for cover relations precludes nesting between the j and t wires in v, since the existence of distinct v_r and v_r' necessarily means that among the downward cover relations $v_r \prec v$ and $v_r' \prec v$, one of these must cross the l and j wires from v while the other must cross the l and t wires from v. One thing that may be confusing here is that both cover relations do receive the same label (l, j) in spite of one of them involving the l and t wires; this is because the names for the wires, for purpose of labeling a cover relation, are determined at the lower element of the cover relation. Regardless of whether we are in case (a) or (b), let us make the convention that v_r is obtained from v by crossing the l and j wires from v, while v_r' is obtained from v by crossing the l and t wires from v.

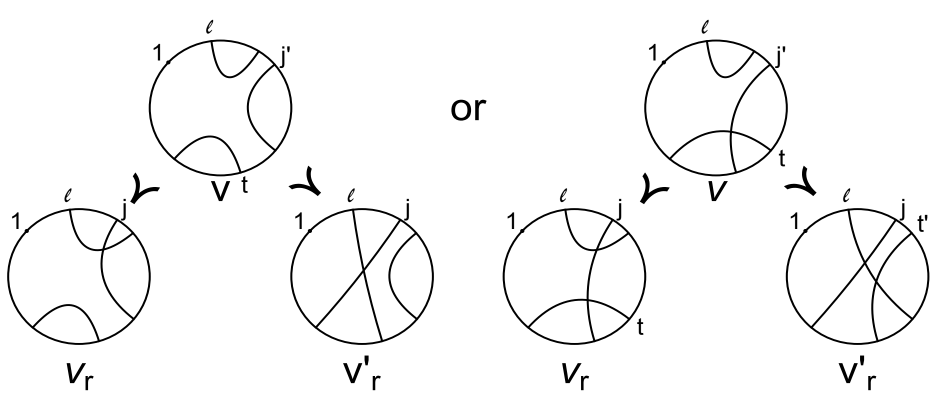


FIGURE 10. $v_r \prec v$ and $v_r' \prec v$

Now we turn to the task of ruling out (a), namely the case where v_r replaces subsequence l,l,j,j,t,t in $w(v)$ having $l<j<t$ with subsequence l,j,l,j,t,t in $w(v_r)$ while v_r' instead replaces l,l,j,j,t,t with subsequence l,j,t,t,l,j in $w(v_r')$. We will use the fact that saturated chains downward from v_r to u and from v_r' to u eventually do reach a common element below both of them, so in particular a single shared start set at this common element; each topologically ascending saturated chain which includes v_r will therefore need a label (t,l) somewhere lower in the saturated chain so as to move the leftmost copy of t leftward to its position in this common start set. To see why we definitely will need such a label (t,l), it helps to notice that the part of the chain which impacts the start set is limited to downward steps which each move a single label of the form i_v to the right, moving a label of the form j_p into the position i_v had occupied; although it is possible that the positions of l_p,j_p,t_p could move even farther to the left prior to reaching a common lower bound for v_r and v_r', that would necessitate a larger label than (j,l) lower in each saturated chain, forcing descents (and hence topological descents by Lemma 4.1) in the aforementioned proposed topologically ascending saturated chains containing v_r and $v_{r'}$, enabling us to rule out that possibility. Thus, we deduce the clam about needing the label (t,l) lower in our saturated chain downward from v_r to a common lower bound.

This lower copy of the label (t,l) in the proposed saturated chain involving v_r will force a topological descent somewhere in the saturated chain, as we explain next. Proposition 4.3 directly handles the possibility of consecutive labels $\lambda(v_{r-1},v_r)=(t,l)$ and $\lambda(v_r,v)=(j,l)$ of the form described above with t chosen as small as possible, by the case analysis in the proof of Proposition 4.3 yielding a topological descent in the case that describes our scenario (which translates to case (c) in the proof of Proposition 4.3). In the "non-consecutive case", namely the case where (t,l) appears lower in the saturated chain rather than directly below the label (j,l) with the further assumption that the intermediate labels are not all of the form (t',l) for $j<t'<t$, we use the fact that there will be one or more other labels at intermediate positions. This would necessarily force a descent (and hence a topological descent by Lemma 4.1) somewhere on the segment of labels beginning and ending with these two labels, by virtue of some label at an intermediate position necessarily either being smaller than both of these labels (t,l) and (j,l) or being larger than both of these labels, due to our very assumption about the labels with second coordinate l and larger first coordinate being non-consecutive. The upshot is that we get a contradiction to having $v_r\prec v$ and $v_r'\prec v$ both labeled (j,l) and both belonging to topologically ascending chains from u to v when we are in case (a) above, namely the case with $w(v)$ including subsequence l,l,j,j,t,t.

Case (b), namely the case with subsequence l,l,j,t,j,t in $w(v)$ again having $l<j<t$, is likewise ruled out by a completely analogous argument which will be largely left to the reader. What makes the argument work again in this case is that $w(v_r)$ now has subsequence l,j,l,t,j,t and $w(v_{r-1})$ has subsequence l,j,t,l,j,t in the event of consecutive labels $\lambda(v_r,v)=(j,l)$ and $\lambda(v_{r-1},v_r)=(t,l)$, which means that when we now apply Proposition 4.3 in this case, we again find ourselves in a scenario giving a descent

(and hence a topological descent by Lemma 4.1), yielding a contradiction; that is, we find ourselves in the scenario labeled as case (b) within the proof of Proposition 4.3. □

Next we handle a situation that required special care within the proof of Lemma 4.2, hence was split off as a separate proposition.

Proposition 4.3. *Given $u \prec v \prec w$ in P_n^* with $\lambda(u,v) = (k,i)$ and $\lambda(v,w) = (j,i)$ for $i < j < k$ carrying out a pair of consecutive uncrossing steps involving a total of three wires, then there exists $u \prec v' \prec w$ either with label sequence $\lambda(u,v') = (j,i)$ and $\lambda(v',w) = (j',k')$ for some $j' < k'$ or with label sequence $\lambda(u,v') = (j,k)$ and $\lambda(v',w) = (j,i)$. In the former case, $u \prec v \prec w$ comprises a topological ascent, and in the latter case $u \prec v \prec w$ comprises a topological descent.*

Proof. The existence of $u \prec v$ with $\lambda(u,v) = (k,i)$ implies that $w(u)$ has subsequence i,k,i,k. The $i < j < k$ requirements implies that the first copy of i is to the left of the first copy of j which is to the left of the first copy of k. These restrictions imply that the only viable possibilities for the subsequence of $w(u)$ with letters i,j,k are (a) i,j,k,i,k,j, (b) i,j,k,i,j,k, (c) i,j,k,j,i,k or (d) i,j,j,k,i,k. See Figures 11, 12, 13 and 14. In each case, we will use the result of Thomas Lam from [La14a] that P_n (and hence P_n^*) is Eulerian; it follows immediately from this and the gradedness of P_n^* that $\mu_{P_n^*}(u,w) = (-1)^2$. This in turn implies for each saturated chain $u \prec v \prec w$ the existence of a unique element v' satisfying $u \prec v' \prec w$, by Remark 2.2.

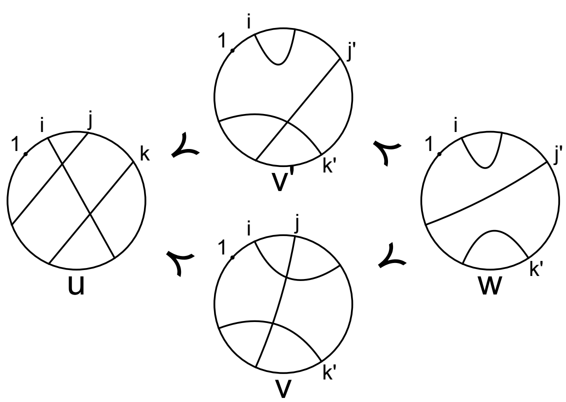


FIGURE 11. $u \prec v \prec w$ topological ascent

In case (a), namely the case with $w(u)$ having the subword i,j,k,i,k,j, the i wire crosses both the j wire and the k wire in u, but there is no crossing of the j and k wires in u. The three wires i,j,k have a total of two crossings, which may be uncrossed in either order to obtain w. Observe that one of the uncrossing sequences yields the labels $\lambda(u,v) = (k,i)$ and $\lambda(v,w) = (j,i)$, while the other uncrossing sequence yields $\lambda(u,v') = (j,i)$ and $\lambda(v',w) = (j',k')$ for some $j' < k'$. These are the only saturated chains from u to v since both these uncrossings must be accomplished. The latter saturated chain from u to v gives a descent and indeed is the lexicographically

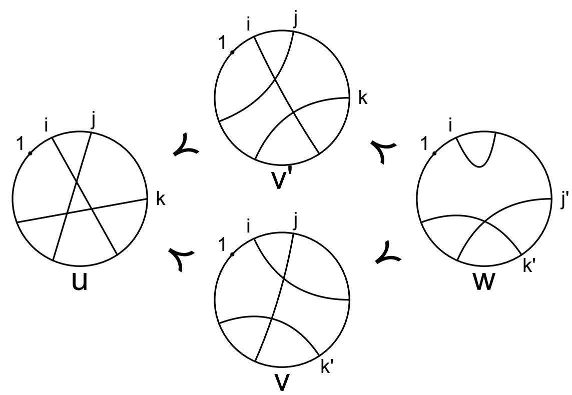


FIGURE 12. $u \prec v \prec w$ topological descent

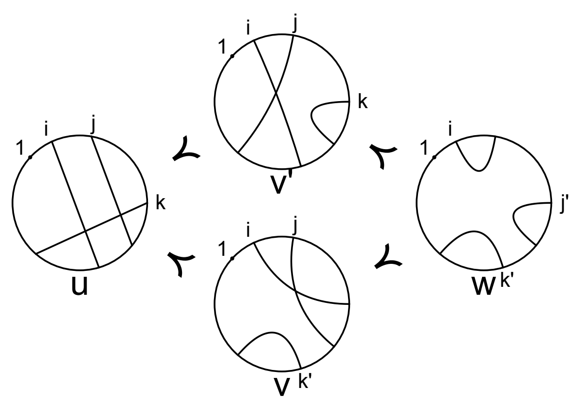


FIGURE 13. $u \prec v \prec w$ topological descent

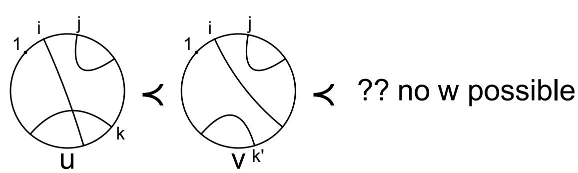


FIGURE 14. Case that cannot arise due to contradiction

later of the two sequences, hence a topological descent. Thus, $\lambda(u,v) = (k.i)$ and $\lambda(v,w) = (j,i)$ together give a topological ascent in this case.

For the cases (b) and (c), namely the cases with $w(u)$ having subsequences i,j,k,i,j,k and i,j,k,j,i,k, respectively, a similar analysis applies, yielding that $u \prec v \prec w$ is a topological descent in these cases. The k wire crosses both the i wire and the j wire in u in each of these cases. One may check both for case (b) and for case (c) that the unique v' satisfying $u \prec v' \prec w$ yields $\lambda(u,v') = (j,k)$ with $j < k$. Our order

$<_\lambda$ implies $(j,k) <_\lambda (k,i)$ since the former has $j < k$ while the latter has $k > i$. One may also observe that in each case we have $\lambda(v', w) = (j,i)$, yielding the desired lexicographically earlier $u \prec v' \prec w$. In particular, in each of the cases (b) and (c), we see that $u \prec v \prec w$ is a topological descent, as desired.

In case (d), the case with $w(u)$ containing the subsequence i,j,j,k,i,k, we deduce from $\lambda(u,v) = (k,i)$ that $w(v) = i,j,j,i,k,k$. This contradicts the existence of a cover relation $v \prec w$ uncrossing the i and j wires, since these wires are nested rather than crossing in v. Thus, (d) is ruled out. □

Remark 4.4. Figure 4 also exhibits the fact that the edge labeling λ is not an EL-labeling in general, because there are rank 2 intervals having two different ascending chains, the lexicographically later of which is a topological descent but not an actual descent. Such examples are what led us instead to prove that λ satisfies the more relaxed requirements to be an EC-labeling, which still yields a lexicographic shelling.

Now we turn to intervals $[D, \hat{1}]$. We begin by showing that the lexicographically first saturated chain has weakly ascending labels, which will be a key tool to proving in Lemma 4.6 that this is the only topologically ascending saturated chain from D to $\hat{1}$.

Lemma 4.5. *For each $D < \hat{1}$ in P_n^*, the lexicographically first saturated chain from D to $\hat{1}$ (with respect to edge labeling λ) has weakly ascending labels.*

Proof. We may assume D has at least one crossing, since otherwise D is covered by $\hat{1}$, a vacuous case. Now let us show how to construct a saturated chain from D to $\hat{1}$ with weakly ascending labels. We start by greedily choosing the smallest possible i for which there exists at least one wire k that crosses the wire i for $k > i$. Among such wires that cross wire i, choose the smallest j such that wires i and j cross. See Figure 15. Lemma 3.16 justifies that we may proceed up a cover relation $D \prec D'$ in P_n^* by uncrossing these wires i and j in such a way that $w(D')$ has the subsequence i,j,j,i. This ensures $\lambda(D, D') = (i,j)$ with $i < j$,

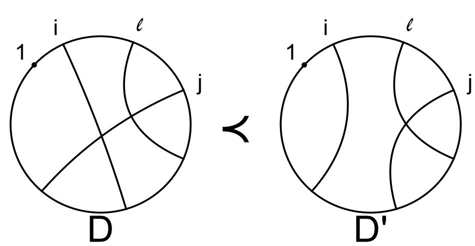


FIGURE 15. $\lambda(D, D') = (i,j)$

Next we show that the smallest available label on any cover relation upward from the diagram D' obtained this way is no smaller than (i,j). To this end, we analyze the impact of the exchange of i_v and j_v together with the corresponding uncrossing of wires i and j; specifically, we need to constrain how this may change the names of any pairs of wires that still cross each other in D' from what their names are in D. If such renaming of wires were to cause a smaller label than (i,j) to be available for a cover relation upward from D', this new label would necessarily result from the renaming of some (i,k) crossing as (j,k) for $j<k$ and the renaming of some (j,l) crossing as (i,l) for $i<l$, by virtue of exchanging portions of wires i and j. Only the new potential label (i,l) could be smaller than (i,j), and this would only happen for $i<l<j$. But the fact that the i and l wires do not cross in D, together with the fact that the pairs (i,j) and (j,l) both do cross in D may all be combined to deduce that $w(D)$ has the subsequence i,l,j,l,i,j. That is, we have l_v before i_v which is before j_v as we proceed clockwise from starting point 1_p. Exchanging i_v and j_v to obtain $w(D')$ from $w(D)$ will yield D' that preserves the fact that l_v comes earlier than i_v in $w(D')$. This contradicts the availability of the label (i,l) for a cover relation upward from D', since we have just shown that the wires i and l do not cross each other in D'.

Applying the above argument repeatedly as we proceed up a saturated chain greedily choosing the lexicographically smallest available label at each step, we may conclude that each pair of consecutive labels $\lambda(x,y)$ and $\lambda(y,z)$ for $z<\hat{1}$ is weakly ascending. By virtue of the construction above, also notice that each label $\lambda(y,z)=(i,j)$ for $z<\hat{1}$ in the lexicographically first saturated chain has $i<j$, implying the label is smaller than L. Thus, we also will get an ascent for the pair of labels $\lambda(x,y)$ and $\lambda(y,\hat{1})$ at the last step in our lexicographically first saturated chain. $\square$

Now we complete the case of intervals $[D,\hat{1}]$.

Lemma 4.6. *For each $D<\hat{1}$ in P_n^*, every saturated chain from D to $\hat{1}$ that is not lexicographically first (with respect to edge labeling λ) has a topological descent.*

Proof. Consider a saturated chain $N = D \prec u_1 \prec \cdots \prec u \prec v \prec \cdots \prec \hat{1}$ from D to $\hat{1}$ such that there is $u \prec v$ in N with $\lambda(u,v') < \lambda(u,v)$ for some $u \prec v' < \hat{1}$. This implies that there is a lexicographically earlier saturated chain from u to $\hat{1}$ involving v' instead of v. By induction on $rk(\hat{1}) - rk(v)$, we may assume that the restriction M of N to the interval $[v,\hat{1}]$ is the lexicographically earliest saturated chain from v to $\hat{1}$. The labeling of M must then consist entirely of labels (i,j) having $i<j$ prior to our final label L, by Lemma 4.7 and Lemma 4.5.

In particular, N has a descent at v unless $\lambda(u,v)$ is of the form (r,s) for some $r<s$. When there is some $z \in P_n^* \setminus \{\hat{1}\}$ that is greater than both v and v', we may deduce the desired result from Lemmas 4.2 and 4.1. We confirm shortly that we will indeed have such an upper bound z unless v and v' are obtained from u by uncrossing the same pair of wires in the two different possible ways; but this uncrossing of the same pair of wires p and q in opposite ways would give labels $\lambda(u,v')=(p,q)$ and $\lambda(u,v)=(q,p)$ for $p<q$, contradicting the fact that $\lambda(u,v)=(r,s)$ for some $r<s$ with $\lambda(u,v')<\lambda(u,v)$.

Now to the outstanding claim. In the event that we do not uncross the same pair of wires in different ways to obtain v and v', we either move from u to v and u to v' by uncrossing disjoint pairs of wires, or via crossings that share one wire in common. In either of these cases, there exists a wire diagram that is an upper bound for v and v' covering v and v' by doing both of these uncrossings and no other uncrossings. $\square$

Finally, we give a simple technical tool that was used in the proof of Lemma 4.6.

Lemma 4.7. *For each $D < \hat{1}$ in P_n^*, every saturated chain from D to $\hat{1}$ that uses any labels $\lambda(u,v)$ of the form $\lambda(u,v) = (j,i)$ for $j > i$ has a descent.*

Proof. Note that any cover relation label $\lambda(x,y) = (j,i)$ for $j > i$ is larger than L, while every saturated chain upward from D to $\hat{1}$ has L as its final label. This already guarantees the presence of a descent on any saturated chain from D to $\hat{1}$ involving a label (j,i) for $j > i$. □

5. Shelling all intervals in Tuffley posets

First we discuss in general terms the edge product space of phylogenetic trees and then we define more precisely its face poset, the *Tuffley poset*. These are both discussed in much more detail in for instance [MS] and [GLMS]; [MS] gives a CW decomposition for this space while [GLMS] proves that this is a regular CW decomposition by a proof that involves first proving the existence of a dual EC-shelling for each interval in the Tuffley poset. We will give a much more explicit shelling for each interval in the Tuffley poset, by showing that each interval is isomorphic to an interval in the poset P_n (which we have already proven to be shellable in the earlier sections of this paper).

This will require the notion of the minors of a graph, as well as notions of edge contraction, edge deletion, and safe edge deletion.

Definition 5.1. *An* **edge contraction** *shortens an edge to length 0, identifying its two endpoints with each other and merging their associated (possibly empty) sets of labels to give the set of labels assigned to the new merged vertex.*

An **edge deletion** *simply eliminates an edge while keeping its two endpoints, disconnecting a tree into two separate trees. An edge delection preserves the vertex label sets. An edge deletion is said to be* **safe** *if it does not result in any vertices with empty label set dropping to degree exactly 2.*

A **minor** *of a graph G is any graph obtained from G by a series of edge contractions followed by a series of safe edge deletions. A minor of one of our upcoming labelled trees will be enriched with a vertex labeling induced by the labeling of the leaves in the tree for which it is a minor.*

Two minors are said to be **combinatorially equivalent** *if there is a graph isomorphism from one to the other that maps the vertex labeling of one minor to the vertex labeling of the other minor.*

Definition 5.2. *The* **edge product space** $\varepsilon(X)$ **of phylogenetic trees** *with leaf label set $X = \{1, 2, \dots, n\}$ is a stratified space comprised of cells. The maximal open cells of $\varepsilon(X)$ are indexed by the combinatorial equivalence classes of trees T with n leaves (i.e. n nodes of degree 1) such that each leaf is assigned a distinct label from X and each non-leaf node in T has degree exactly 3; two such trees are said to be* **combinatorially equivalent** *if they are isomorphic as trees with labeled leaves. One may check that the trees T indexing the maximal cells all have exactly $2n-3$ edges, i.e. have $|E(T)| = 2n-3$.*

The open cell $C(T)$ given by tree equivalence class T with $|E(T)|$ edges in T may be parametrized as the points in $\mathbb{R}_{>0}^{|E(T)|}$, with the $|E(T)|$ coordinates recording the lengths *of the corresponding edges in T. The lower dimensional cells $C(T')$ in the closure of $C(T)$ are given by exactly those combinatorial equivalence classes of labeled forests T' obtained from T as minors of T; we define combinatorial equivalence for forests with label sets on some of the vertices by again requiring a graph isomorphism carrying the vertex labels of one forest to the vertex labels on the other forest.*

Thus, the space $\varepsilon(X)$ also has an open cell homeomorphic to $\mathbb{R}_{>0}^{|E(T')|}$ for each (combinatorial equivalence class of) labeled forest T' obtainable by choosing some T as above indexing a maximal cell and letting T' be one of its minors obtained by letting one or more of the edge lengths in T degenerate either to 0 (via an edge contraction) or to infinity (via a safe edge deletion), in either case reducing the number of edges by one for each such contraction or safe deletion.

It may be helpful to note that the above notion of combinatorial equivalence classes of trees may alternatively be defined using the set of "splits" of a tree. A **split** of a tree T with n leaves labeled $1, 2, \ldots, n$ is a set partition of $\{1, 2, \ldots, n\}$ into two blocks that is obtained by deleting a single edge from T and letting the blocks of the set partition be the sets of leaf labels for leaves in the same connected component as each other in the resulting forest (comprised of exactly two trees). A pair of trees with labeled leaves are then said to be combinatorially equivalent if they have the same splits.

Next we define the face poset for the edge product space of phylogenetic trees. This is denoted by $S(\{1, 2, \ldots, n\})$ in [GLMS], but we instead denote it by $T(n)$ to avoid confusion with our notation for "start set" earlier in the paper.

Definition 5.3. *The* **Tuffley poset** *$T(n)$ is the face poset for the edge product space of phylogenetic trees with n leaves. It has as its maximal elements the various combinatorial equivalence classes of trees T with n leaves labeled 1 to n with the further requirement that each non-leaf node has degree exactly 3. The non-maximal elements of $T(n)$ other than $\hat{0}$ are those forests F (with collections of labels on some of the leaves) which may be obtained from a maximal element of $T(n)$ by repeatedly proceeding down cover relations as follows. Given any $v \in T(n)$, one first does a (possibly empty) series of edge contractions, each going down a cover relation, then does a (possibly empty) series of safe edge deletions. Elements u obtained in this manner which are combinatorially equivalent give the same poset element.*

This process terminates at graphs which do not have any edges and which have collections of labels assigned to each surviving vertex; these combinatorial equivalence classes of edge-free graphs with nonempty sets of labels on each surviving vertex are in natural bijection with the set partitions of $\{1, 2, \ldots, n\}$. Finally, a unique minimal element in $T(n)$, denoted $\hat{0}$, is adjoined.

Now we are equipped to deduce the shellability of each interval in the Tuffley poset $T(n)$ as a corollary to Theorem 3.18 by verifying the requisite poset isomorphisms.

Example 5.4. *Small examples that could be helpful to keep in mind while reading the proof of Corollary 5.5 are the Tuffley poset $T(2)$ for a tree with two leaves and no non-leaf vertices and the Tuffley poset $T(3)$ for a tree with three leaves and one degree 3 non-leaf vertex. The former tree will be the medial graph in the proof below for a* *wire diagram with two wires that cross each other, leading to each interval in $T(2)$ being isomorphic to an interval in the uncrossing poset P_2. The latter tree will be the medial graph for a wire diagram with three wires all crossing each other, leading to each interval in $T(3)$ being isomorphic to an interval in P_3.*

Corollary 5.5. *Each interval in $T(n)$ is dual EC-shellable by way of a poset isomorphism to an interval in the uncrossing poset P_n. In particular, this shelling may be combined with other known properties to imply that the CW decomposition given by Moulton and Steele in* [MS] *for the edge product space of phylogenetic trees is a regular CW decomposition.*

Proof. Our main task will be to give an isomorphism of each interval of the Tuffley poset $T(n)$ to an interval in an uncrossing poset P_n. Then we may use the fact that our EC-shelling on the dual to the uncrossing poset induces an EC-shelling on each of its intervals. The fact that an EC-shelling of a poset induces an EC-shelling on each of its intervals is immediate from the definition of EC-labeling.

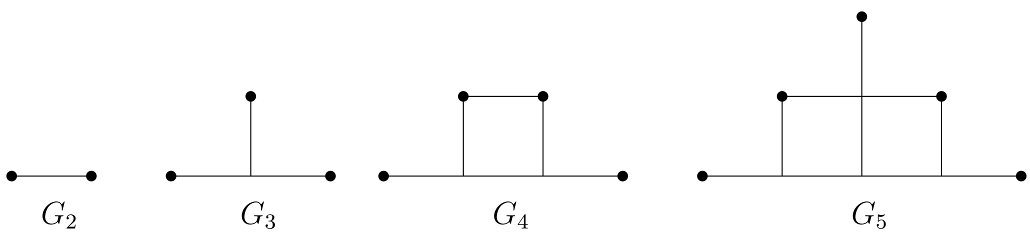


FIGURE 16. The well connected graphs G_2, G_3, G_4 and G_5

In constructing a poset isomorphism, we will rely heavily on the notion of the medial graph associated to a wire diagram (see e.g. Section 4.4.1 in [Ke] and Figure 3 in [Ke] for a specific example of this construction, in which context the term strand diagram is used for what we are calling a wire diagram). Roughly the idea for this association is to have each crossing of two wires in a wire diagram go through the middle of an edge of the associated medial graph, with the edges of the medial graph in bijection with the wire crossings in the associated wire diagram. We will also use the well known fact that the medial graph associated to one choice of embedding of a wire diagram having all n wires cross each other, i.e. a wire diagram associated to the maximal element in the dual uncrossing poset P_n^*, is a so-called well-connected graph G_n. Changing the embedding of the wire diagram by passing a wire over a crossing (a sort of analogue of a braid move) will correspond to a so-called Y-Δ move in the associated graph (see [Ke]) which produces a different well-connected graph.

Results in Section 3 of [KW] imply that each planar graph (and in particular each tree equivalence graph T with unlabelled leaves) arises as a minor of a well connected graph, namely as an element $v \neq \hat{1}$ in some dual uncrossing poset P_m^* for $m \geq n$. It is not difficult to prove this directly in our setting where we restrict to trees with n leaves such that each non-leaf node has degree 3, by embedding any such tree into the well-connected graph G_n (see Figure 16). Next observe that embellishing such a tree T by assigning labels to its leaves does not interfere with the map from [KW] mentioned above identifying each T with an element v in P_m^*. It is also straightforward to check that this leaf labeling will not interfere with the poset interval isomorphism, as we describe and justify next.

Specifically let us now show that the interval $[\hat{0}, v]$ in P_m is naturally isomorphic to the interval $[\hat{0}, T]$ in the Tuffley poset, focusing on the situation without leaf labels and then leaving it to the reader to think through the fact that leaf labels behave just as they should, using our labeled version of edge deletion and contraction and only allowing safe edge deletions. The point is to convert the strand diagram for $v \in P_m$ to a minor G_v of a well-connected graph using the bijection appearing in Section 4.4.1 (sending a wire diagram to its medial graph) in [Ke] to guide us. Observe that the two different ways of uncrossing a pairs of wires exactly correspond to deletion (for one type of uncrossing) and contraction (for the other way of uncrossing the crossing) of the graph edge corresponding to that wire crossing in the associated medial graph. Next we claim (and will soon justify) that such an uncrossing yields a double crossing if and only if the corresponding edge deletion or contraction yields what is called in [Ke] a non-reduced graph. We will show next that because we start with a tree, the only way we can get a non-reduced graph as a minor is to have the edge deletion or construction lead to the existence of an internal vertex (i.e. a vertex not having any leaf labels on it) of degree 1 or 2. This situation corresponds to exactly the edge deletions which are not safe (namely exactly those which are not allowed for cover relations in the Tuffley poset) and cannot happen at all for edge contractions for graphs that are trees or forests. This will suffice since all minors of trees are themselves either trees or forests. To see this claim, notice that a double crossing of wires means that the same two wires cross in the middle of one graph edge and also in the middle of another edge, which can only happen as a result of either a degree 2 vertex (where it happens at the two edges incident to this vertex) or from a series of connected edges (comprising a graph path) that one wire crosses and a different series of graph edges (comprising a different path from the same starting point) that the other wire crosses, where eventually these two paths meet in a vertex leading into the second edge that both wires cross; in particular, this pair of graph paths meeting at both ends implies there is a cycle in the graph (a contradiction to the medial graph being a tree or a forest, as is the case for each minor in our setting). Thus, we have shown that the edge contractions and the safe edge deletions correspond not only to the cover relations in the Tuffley poset but also in the uncrossing poset. Putting this together, we have shown that this map from elements of the interval $[\hat{0}, v]$ in P_m to corresponding graph minors of the medial graph G_v for v is the desired poset interval isomorphism.

It is proven in Section 4 of [GLMS] that the existence of a shelling for each interval of the Tuffley poset implies that the CW decomposition of Moulton and Steel for the edge product space of phylogenetic trees is a regular CW decomposition. Thus, our shelling for each interval of $P(n)^*$ (and hence for each interval in $P(n)$) in Theorem 3.18 yields regularity of the CW decomposition of Moulton and Steel in an explicit way, which is an improvement on the previous non-constructive result in [GLMS] that a shelling for each poset interval exists. $\square$

References

[BHV] L. Billera, S. Holmes and K. Vogtmann, Geometry of the space of phylogenetic trees, *Advances in Applied Math*, **27** (2001), 733–767.

[Bj80] A. Björner, Shellable and Cohen-Macaulay partially orderd sets, *Trans. Amer. Math. Soc.*, **260** (1980), 159–183.

[Bj84] A. Björner, Posets, Regular CW complexes and Bruhat order, *European J. Combin.* **5** (1984), no. 1, 7–16.

[BB] A. Björner and F. Brenti, Combinatorics of Coxeter Groups, Springer, 2005.

[BW] A. Björner and M. Wachs, Bruhat order of Coxeter groups and shellability, *Adv. Math.* **43** (1982), no. 1, 87–100.

[CIM] E. Curtis, D. Ingerman, and J. Morrow, Circular planar graphs and resistor networks, *Linear Algebra Appl.*, **283** (1998), 115–150.

[DK] G. Danaraj and V. Klee, Shellings of spheres and polytopes, *Duke Math. Journal* **41** (1974), 443–451.

[deV] C. de Verdiere, Réseaux électriques planares, I., *Comment. Math. Helv.* **69** (1994), no. 3, 351–374.

[Dy93] M. Dyer, Hecke algebras and shelling of Bruhat intervals, *Compositio Math.* **89** (1993), no. 1, 91–115.

[DH] M. Dyer and C. Hohlweg, Small roots, low elements, and the weak order in Coxeter groups, *Adv. Math.* **301** (2016), 739–784.

[GKL] P. Galashin, S. Karp, and T. Lam, The totally nonnegative Grassmannian is a ball, preprint 2017, arXiv:1707.02010.

[GLMS] J. Gill, S. Linusson, V. Moulton and M. Steel, A regular decomposition of the edge-product space of phylogenetic trees, *Adv. Applied Math.* **41** (2008), 158–176.

[He03] P. Hersh, Lexicographic shellability for balances complexes. *J. Algebraic Combinatorics* **17** (2003), no. 3, 225–254.

[He14] P. Hersh, Regular cell complexes in total positivity. *Inventiones Mathematicae* **197** (2014), no. 1, 57–114.

[Hu] James Humphreys, Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics, **29**, *Cambridge University Press, Cambridge*, 1990, xii+204 pp.

[Ke] R. Kenyon, The Laplacian on planar graphs and graphs on surfaces. *Current developments in mathematics, 2011*, 1-55, *Int. Press, Somerville, MA*, 2012.

[KW] R. Kenyon and D. Wilson, The space of circular planar electrical networks, *SIAM J. Discrete Math.* **31** (2017), no. 1, 1–28.

[KLS] A. Knutson, T. Lam, and D. Speyer, Postroid varieties: juggling and geometry, *Compos. Math*, **149** (2013), 1710–1752.

[Ko] D. Kozlov, General lexicographic shellability and orbit arrangements, *Ann. Comb.* **1** (1997), no. 1, 67–90.

[La14a] T. Lam, The uncrossing partial order on matchings is Eulerian, *Journal of Combinatorial Theory, Ser. A* **135** (2015), 105–111.

[La14b] T. Lam, Electroid varieties and a compactification of the space of electrical networks, preprint at arXiv:1402.6261.

[Ma] I. G. Macdonald, Affine root systems and Dedekind's ν-function, *Inventiones Mathematicae* **15** (1972), 91–143.

[MS] V. Moulton and M. Steel, Peeling phylogenetic 'oranges', *Adv. Applied Math.* **33** (2004), 710–727.

[Po] A. Postnikov, Total positivity, Grassmannians and networks, arXiv:math.CO/0609184

[Ro] G.C. Rota, On the foundations of combinatorial theory I. Theory of Möbius functions, *Z. für Wahrscheinlichkeitsth*, **2**, 340–368.

[Wi] L. Williams, Shelling totally nonnegative flag varieties, *J. Reine Angew. Math.* **2007**, Issue 609 (2007), 1–22.

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