Fibers of Maps to Totally Nonnegative Spaces

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- joint work with Jim Davis & Ezra Miller

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- slides at: https://pages.uoregon.edu/plhersh/MIT-Dec5-2025.pdf

Def'n: A real matrix is **totally nonnegative (TNN)** if all its minors are nonnegative.

Thm (Anne Whitney in type A, generalized vastly by Lusztig)

The unipotent TNN matrices

$$\left\{ \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & 1 & \\ & & \ddots \end{pmatrix} \right\}$$

are products of exponentiated Chevalley generators

how these look in type A

$$x_i(t) = \underbrace{\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}}_{\text{type A}} \quad \text{for } t \in \mathbb{R}_{\geq 0}$$

← row i

column $i+1$

Today: When are products of $x_i(t)$'s equal?
(in a certain sense)

Map whose fibers we study:

$$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{\geq 0} \to TNN(\overset{\text{unipotent radical}}{\widetilde{U_n}})$$

$$(t_1,\dots,t_d) \mapsto x_{i_1}(t_1)\,x_{i_2}(t_2)\cdots x_{i_d}(t_d)$$

e.g. $f_{\underbrace{(1,2,1,2,1)}_{\text{nonreduced word in } S_3}}(t_1,\dots,t_5) = \begin{pmatrix} 1 & t_1+t_3+t_5 & t_1t_2+t_1t_4+t_3t_4 \\ & 1 & t_2+t_4 \\ & & 1 \end{pmatrix}$

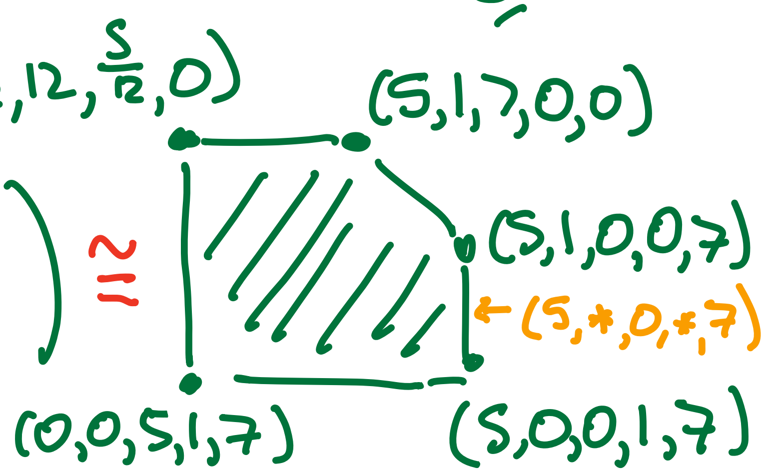
$$\| $$

$$\underbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_1)} \underbrace{\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}}_{x_2(t_2)} \underbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_3)} \underbrace{\begin{pmatrix} 1 & & \\ & 1 & t_4 \\ & & 1 \end{pmatrix}}_{x_2(t_4)} \underbrace{\begin{pmatrix} 1 & t_5 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_5)}$$

Example of a Fiber

$\subseteq \mathbb{R}^5_{\geq 0}$

$$f^{-1}_{(1,2,1,2,1)}\begin{pmatrix} 1 & 12 & 5 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \cong$$



Some Related Past Work

- Lusztig (94): Studied totally nonneg. part in reductive groups (as image of map $f_{(i_1 \ldots i_d)}$) & connected this to canonical bases
- Fomin-Shapiro (00): Results on Bruhat stratification of $\mathrm{im}(f_{(i_1 \ldots i_d)})$ & conjectured it is regular CW ball.
- H. (14): Proof of Fomin-Shapiro Conj.
- Galashin-Karp-Lam (22): Totally Nonneg part of any flag variety is regular CW ball & F-S Conj via Poincaré Conjecture.
- Loosely related: Positroid varieties, cluster algebras, braid varieties ...

Some Motivations:

1. fibers of $f_{(i_1 \ldots i_d)}$ encode nonneg. real relations among exponentiated Chevalley generators in Lie theory
2. "braid relations" among x_i's:

$$x_i(a)x_{i+1}(b)x_i(c) = x_{i+1}\left(\frac{bc}{a+c}\right)x_i(a+c)x_{i+1}\left(\frac{ab}{a+c}\right)$$

tropicalize to change of coords

$$(a,b,c) \mapsto \left(\begin{matrix} b+c \\ -\min(a,c) \end{matrix}, \min(a,c), \begin{matrix} a+b \\ -\min(a,c) \end{matrix}\right)$$

for Lusztig's dual canonical bases.

Fiber Stratification: for each $p \in TNN(U_n)$, the stratification on $\mathbb{R}^d_{\geq 0}$ based on which coordinates are positive and which are 0 induces stratification for $f^{-1}_{(i_1 \ldots i_d)}(p) \cap \mathbb{R}^d_{\geq 0}$

Baby Example of Fiber (& how we Think About It)

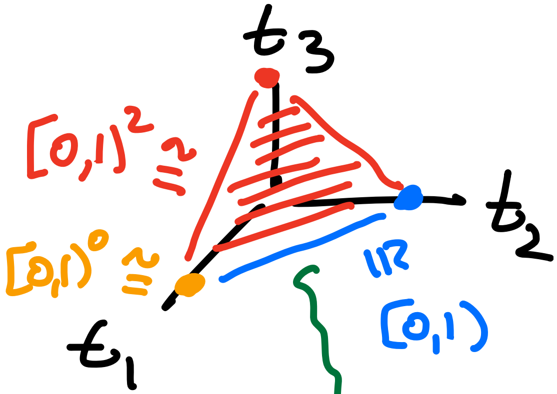
$$\overbrace{x_1(t_1)}^{\begin{pmatrix}1 & t_1 & \\ & 1 & \\ & & 1\end{pmatrix}}\overbrace{x_1(t_2)}^{\begin{pmatrix}1 & t_2 & \\ & 1 & \\ & & 1\end{pmatrix}}\overbrace{x_1(t_3)}^{\begin{pmatrix}1 & t_3 & \\ & 1 & \\ & & 1\end{pmatrix}} = \overbrace{x_1(5)}^{\begin{pmatrix}1 & 5 & \\ & 1 & \\ & & 1\end{pmatrix}}$$

$$\shortparallel$$

$$x_1(t_1+t_2+t_3) = \begin{pmatrix}1 & t_1+t_2+t_3 & 0 \\ & 1 & \\ & & 1\end{pmatrix}$$

$$\shortparallel$$

$$f_{(1,1,1)}(t_1,t_2,t_3)$$



Natural Description of Fiber:

$$f^{-1}_{(1,1,1)}(x_1(5)) = \left\{(t_1,t_2,t_3) \in \mathbb{R}^3_{\geq 0} \;\middle|\; \sum_{i=1}^{3} t_i = 5\right\}$$

More Useful Description (for giving a cell decomposition)

$$f^{-1}_{(1,1,1)}\Big(\overbrace{x_1(s)}^{\begin{pmatrix}1 & s & \\ & 1 & \\ & & 1\end{pmatrix}}\Big)$$

$$\rotatebox{90}{=}$$

$$\left\{(t_1,t_2,t_3)\in\mathbb{R}^3_{\geq 0}\;\middle|\;\begin{array}{l} 0\leq t_1\leq 5 \\ 0\leq t_2\leq \boxed{5-t_1} \\ t_3=\boxed{5-t_1-t_2}\end{array}\right\}$$

Sample Stratum:

$$\underbrace{0<t_1<5}_{\text{range }(0,k)\text{ of values for }k>0} \qquad t_3=\underbrace{5-t_1-t_2}_{\text{cont. fn of } t_1,t_2}$$

$$\underbrace{0<t_2<5-t_1}_{\text{range }(0,t_2^{max})}\text{ for } \underbrace{t_2^{max}=5-t_1}_{\text{cont. fn of } t_1}$$

Note: t_3 is in rightmost reduced word for s_1 in $(1,1,1)$ so determined by parameters to its left

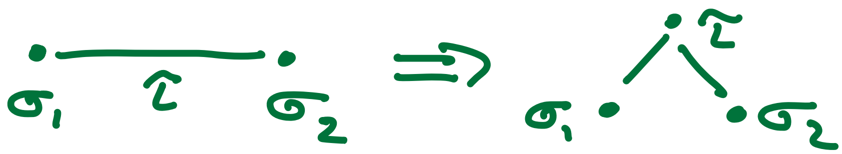
- A cell decomposition of topol. space X is decomp. into disjoint union of cells, namely pieces homeom. to $(0,1)^s \cong \mathbb{R}^s$ for various $s \geq 0$

e.g.

- A cell stratification is cell decomp. with $\sigma \cap \overline{\tau} \neq \emptyset \Rightarrow \sigma \subseteq \overline{\tau}$

- The face poset of a cell stratific. is partial order on cells with $\sigma \leq \tau \iff \sigma \subseteq \overline{\tau}$

e.g.



Main Results for fibers

Topological

- each stratum is homeomorphic to $(0,1)^s$ for some $s \geq 0$.
- Parametrizations for points in unions of strata showing these are each homeomorphic to $[0,1)^s$
- cells form cell stratification

Combinatorial:

- Same face poset as interior dual block complex of subword complex
- these interior dual block complexes are contractible regular CW complexes

Conjectural

- $f^{-1}_{(i_1,\ldots,i_d)}(p)$ is regular CW complex.

Necessary Condition for points to be in Same Fiber: Supports have same Demazure Product

$$x_i(t) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = I_n + tE_{i,i+1}$$

(type A)

$$f_{(i_1,\dots,i_d)}(t_1,\dots,t_d) = x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

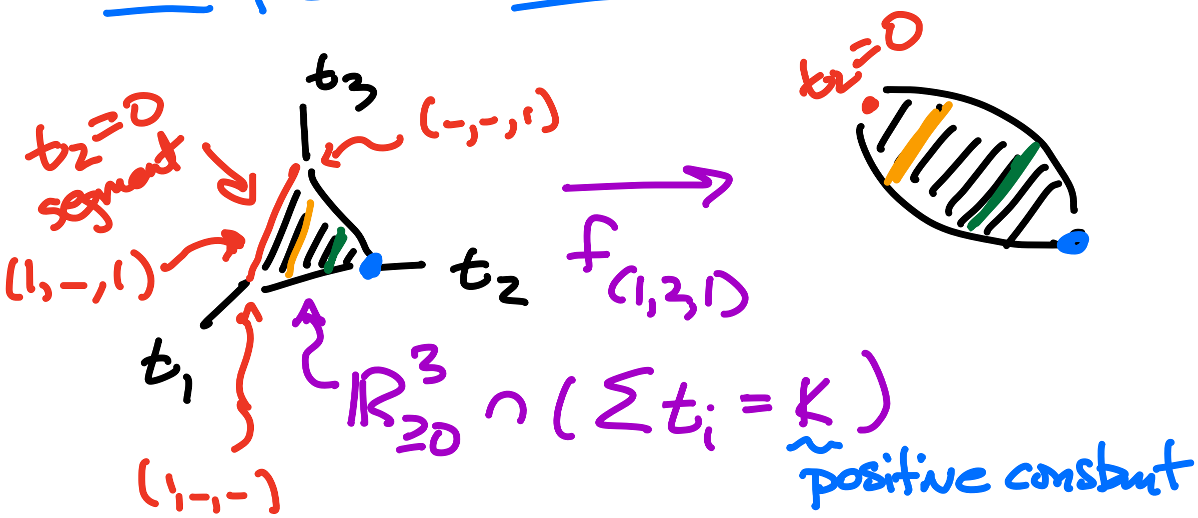
↑ reduced or nonreduced word

e.g.

$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ & 1 & t_2 \\ & & 1 \end{pmatrix}$$

Example Continued



$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow{\scriptstyle t_2=0}$$

$$f_{(1,-,1)}(t_1,0,t_3) = \underbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_1)}\underbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}_{x_1(t_3)}$$

$$= \left.\begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}\right\} x_1(t_1+t_3)$$

$$\{x_1(t) \mid t>0\} = \{x_1(t_1)x_1(t_3) \mid t_1,t_3>0\}$$

$$\Downarrow$$

$$x_i = x_i^2$$ (unsigned 0-Hecke algebra or "Demazure" product)

Demazure Product δ (a.k.a. Unsigned 0-Hecke Algebra Product) Governs Which Minors are Positive

- $x_i(t_1)x_i(t_2) = x_i(t_1+t_2)$

  $x_i x_i \to x_i$ "modified nil move"

  $[\delta(s_i, s_i) = s_i]$

- $x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}(t_1')x_i(t_2')x_{i+1}(t_3')$

  for $t_1' = \dfrac{t_2 t_3}{t_1+t_3}$ $\quad t_2' = t_1+t_3$ $\quad t_3' = \dfrac{t_1 t_2}{t_1+t_3}$

  $x_i x_{i+1} x_i \to x_{i+1} x_i x_{i+1}$ "braid move"

  $[\delta(s_i, s_{i+1}, s_i) = \delta(s_{i+1}, s_i, s_{i+1})]$

- $x_i(t_1)x_j(u) = x_j(u)x_i(t)$ for $|j-i|>1$

  $x_i x_j = x_j x_i$

  $[\delta(s_i, s_j) = \delta(s_j, s_i)]$

The Demazure product for Coxeter group W satisfies

$$\delta(s_{i_1}, s_{i_2}, \ldots, s_{i_d}) = \begin{cases} \delta(s_{i_2}, \ldots, s_{i_d}) & \text{if } \ell(u) < \ell(s_{i_1} u) \\ s_{i_1}\delta(s_{i_2}, \ldots, s_{i_d}) & \text{otherwise} \end{cases}$$

where

$u := \delta(s_{i_2}, \ldots, s_{i_d})$

e.g. $\delta(1,2,1,2,1) = s_1 s_2 s_1$

Facts:

subword of positive parameters for this stratum

1. $f_{(i_1 \ldots i_d)}(\mathbb{R}^{Q}_{>0}) = f_{(i_1 \ldots i_d)}(\mathbb{R}^{Q'}_{>0})$

$$\iff \delta(Q) = \delta(Q')$$

2. δ is equivalent to unsigned 0-Hecke algebra product

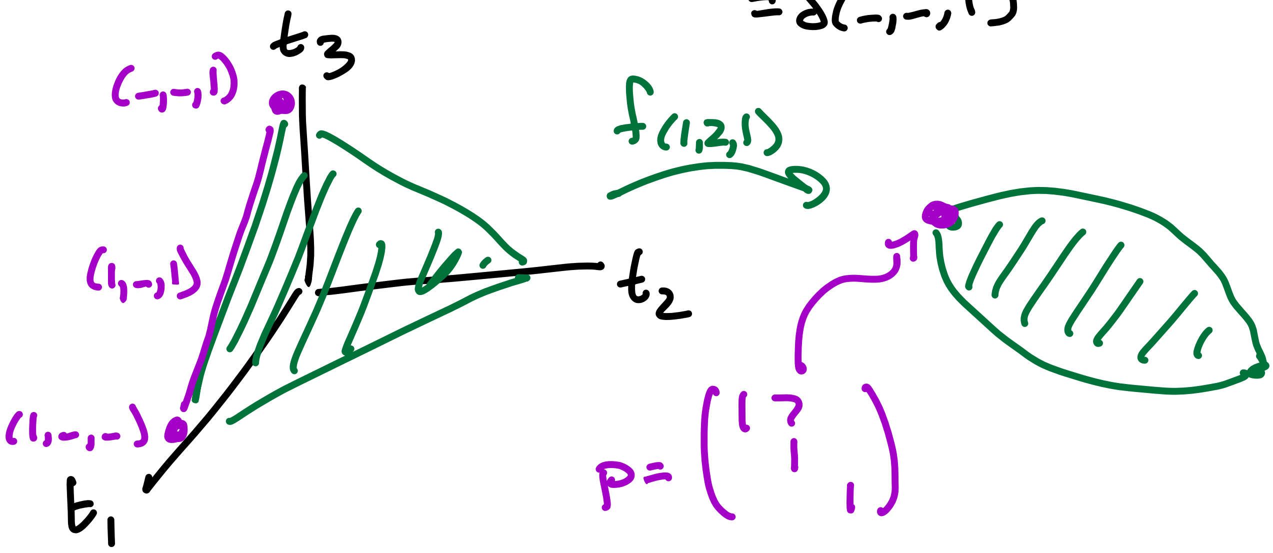
Notation: $U(w) := f_{(i_1,\dots,i_\ell)}(\mathbb{R}^Q_{>0})$

for $w = \delta(Q)$.

$= \left[B^- w B^- \cap \text{unipotent subgp of } B\right]_{\geq 0}$

e.g. $f^{-1}_{(1,2,1)}\begin{pmatrix}1&?&0\\0&1&0\\0&0&1\end{pmatrix}$ has nonempty strata given by subwords $(1,-,-)$, $(1,-,1)$, & $(-,-,1)$ of $(1,2,1)$ since

$\begin{pmatrix}1&?&\\&1&\\&&1\end{pmatrix} \in U(s_1)$ for $s_1 = \delta(1,-,1)$
$= \delta(1,-,-)$
$= \delta(-,-,1)$



Thm (Lusztig):

(a) $(i_1,\dots,i_d)$ reduced & $w = \delta(i_1,\dots,i_d) \Rightarrow$

$f_{(i_1,\dots,i_d)} : \mathbb{R}^d_{>0} \to U(w)$ is homeomorphism.

(b) $U(w) \cap U(w') = \emptyset$ for $w \neq w'$.

A Key Step in Cell Stratif. for Fibers:

Substantially generalize (a) above,

e.g. show that map

$$\begin{array}{c} (t_1,t_2,t_3,t_4) \mapsto x_4(1)x_2(5)\underline{x_4}(t_1)x_1(3)\underline{x_2}(t_2)\underline{x_1}(t_3)\underline{x_2}(t_4) \\ \text{\rotatebox{90}{$\in$}} \\ \mathbb{R}^4_{>0} \end{array}$$

rightmost reduced word for

$\delta(4,2,4,1,2,1,2) = s_4 s_2 s_1 s_2$ in $(4,2,\underline{4},1,\underline{2},\underline{1},\underline{2})$

is homeomorphism from $\mathbb{R}^4_{>0}$ to its image

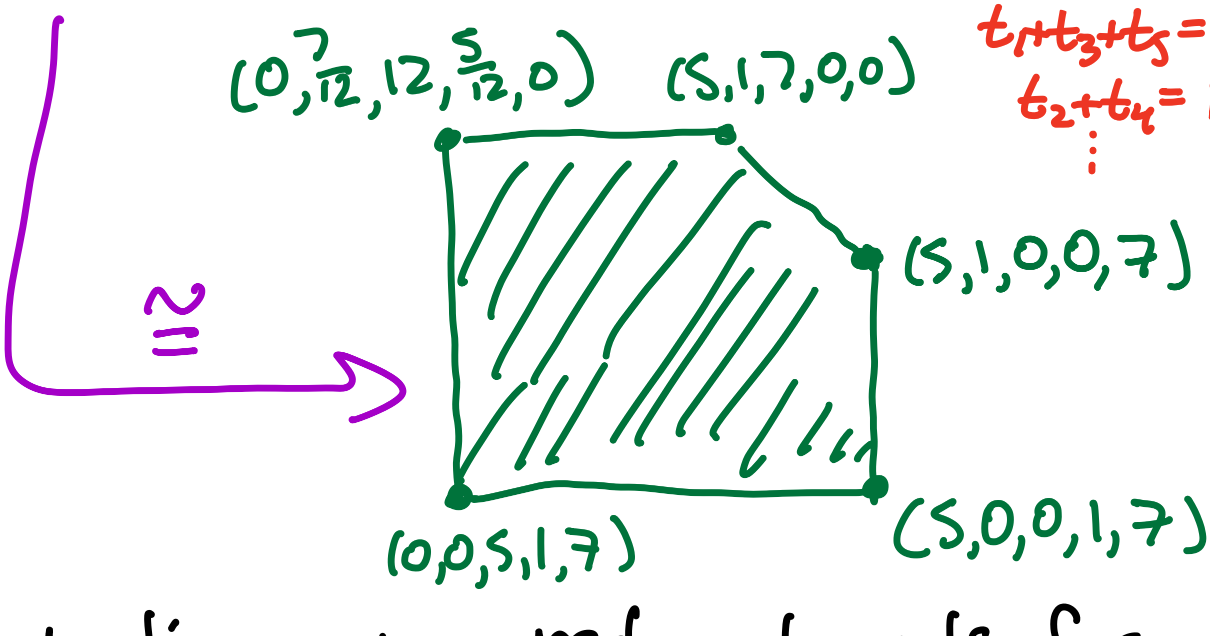
Example of Fiber (Revisited):

$$f_{(1,2,1,2,1)}(t_1,t_2,\dots,t_5) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix} \cdots$$

$$= \begin{pmatrix} 1 & t_1+t_3+t_5 & t_1t_2+t_1t_4+t_3t_4 \\ & 1 & t_2+t_4 \\ & & 1 \end{pmatrix}$$

$f^{-1}_{(1,2,1,2,1)}(M)$ for $M = \begin{pmatrix} 1 & 5 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 & \\ & 1 & \\ & & 1 \end{pmatrix}$

$t_1+t_3+t_5=12$
$t_2+t_4=1$
$\vdots$



vertices $\leftrightsquigarrow$ reduced words for $s_1s_2s_1$ within $(1,2,1,2,1)$

Subword Complexes & their Interior Dual Block Complexes

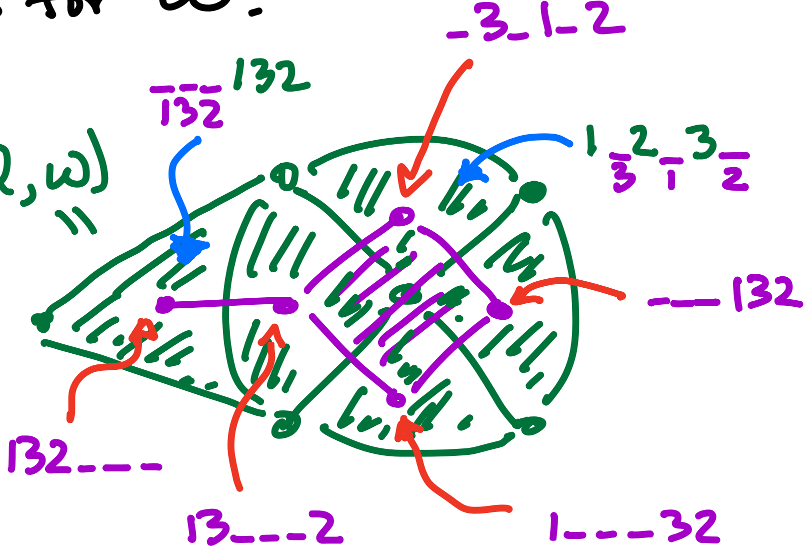
Defn (Knutson-Miller): The subword complex $\Delta(Q,w)$ has letters in Q as vertices & has as its facets the complements of subwords of Q that are reduced words for w.

e.g.

$\Delta(Q,w) =$

$Q = (1,3,2,1,3,2)$

$w = s_1 s_3 s_2$

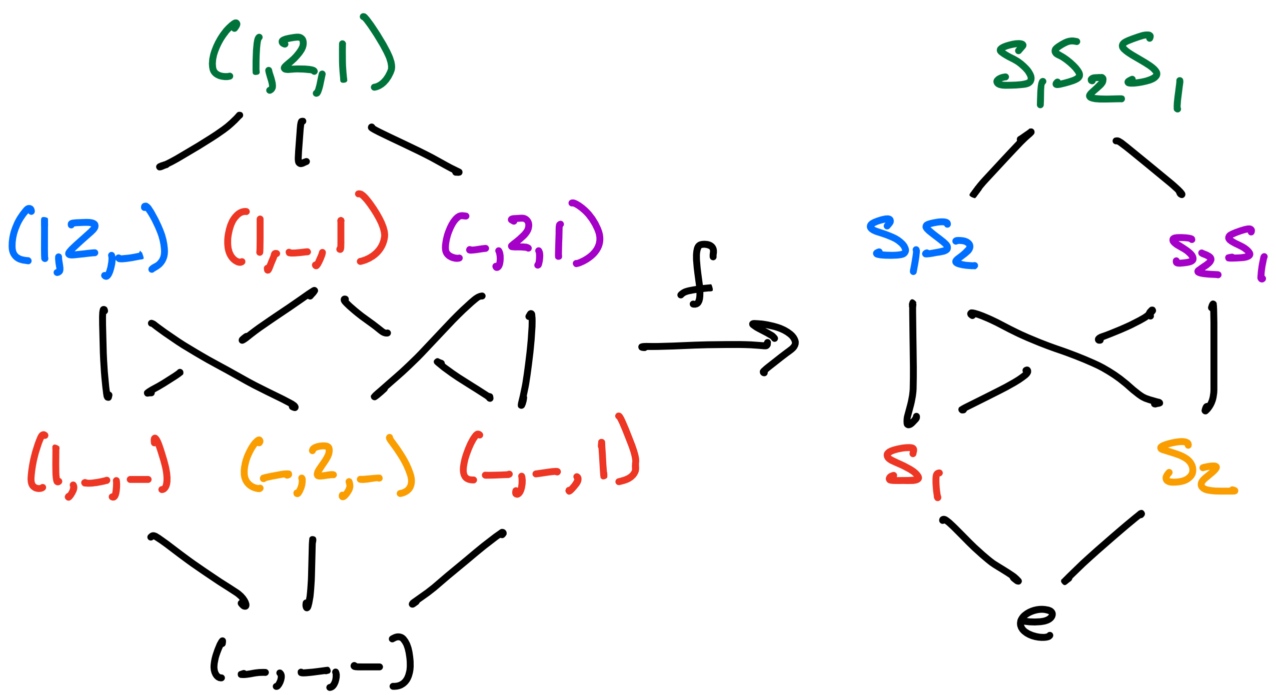


- interior dual block complex in purple

Thm (DHM): For $p \in U(w)$, $f^{-1}_{(i_1 \ldots i_d)}(p)$ has same face poset as the interior dual block complex of $\Delta((i_1, \ldots i_d), w)$.

Thm (DHM): The interior dual block complex (IDBC) of every subword complex is regular CW complex & is contractible.

Idea: Map $f_{(i_1,\dots,i_d)}$ of Topol. Spaces induces Map of Face Posets



Boolean lattice
=
face poset of simplex

Bruhat order
=
face poset of TNN(U_n)

Demazure product

$$f(i_{j_1},\dots,i_{j_\ell}) = \delta(s_{i_{j_1}},\dots,s_{i_{j_\ell}})$$

e.g. $f(1,-,1) = \delta(s_1,s_1) = s_1$

Some Properties of Subword Complexes

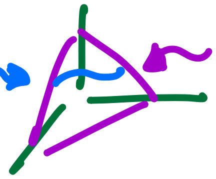
Knutson-Miller: Each subword complex is vertex decomposable, hence shellable & pure.

$\Delta((i_1,\dots,i_\ell),w) \cong$ sphere if $\delta(i_1,\dots,i_\ell)=w$

$\Delta((i_1,\dots,i_\ell),w) \cong$ ball if $\delta(i_1,\dots,i_\ell)\neq w$

Corollary: They are gallery connected, i.e. max faces connected thru codimension one faces.

Application:



$$f^{-1}_{(i_1,\dots,i_d)}(p) \cap \mathbb{R}^d_{\geq 0} = f^{-1}_{(i_1,\dots,i_d)}(p) \cap \Delta^{d-1}_K$$

for some $K>0$

coord. sum K part of $\mathbb{R}^d_{\geq 0}$

Proof:

- show change of coord's $(t_1,\dots,t_d) \longmapsto (t_1',\dots,t_d')$ for braid & modified nil moves preserve sum of parameters

  e.g. $x_1(t_1)x_1(t_2) \mapsto x_1(t_1+t_2)$ $K=t_1+t_2$

- use gallery-connectedness of subword complexes + Matsumoto's Theorem to show states connected by braid & modified nil-moves.

Parametrizing Points in Unions of Strata

- Consider rightmost subword Q of $(i_1, \ldots, i_d)$ that is reduced word for $\delta(i_1, \ldots, i_d)$

$$\text{e.g. } (3, 1, 2, 1, 2, 1, 2)$$

- Parametrize the points in those strata satisfying $t_1, t_3, t_6, t_7 > 0$ in $f^{-1}_{(3,1,2,1,2,1,2)}(p) \subseteq \mathcal{U}(s_3 s_2 s_1 s_2)$ using the set $[0,1)^3$ as follows:

$t_1, t_5, t_6, t_7 > 0 \iff t_2, t_3, t_4$ satisfy:

$$t_2 \in [0, t_2^{max}) \cong [0,1)$$

$$t_3 \in (0, t_3^{max}(t_2)) \cong [0,1)$$

$$t_4 \in [0, t_4^{max}(t_2, t_3)) \cong [0,1)$$

contin. fn of t_2

continuous function of t_2, t_3

e.g.,

$$x_3(t_1)\, x_1(t_2)\, x_2(t_3)\, x_1(t_4)\, x_2(t_5)\, x_1(t_6)\, x_2(t_7)$$

choice of (nonmax.) values $\cong [0,1)$

values determined by other params

Cell Stratification: Key Lemmas
(& Definitions)

Defn: Consider word $(i_1, \ldots, i_d)$.
The letter i_ℓ is redundant in $(i_1, \ldots, i_d)$ if $\delta(i_1, \ldots, i_d) = \delta(i_1, \ldots, \hat{i}_\ell, \ldots, i_d)$ and nonredundant otherwise.

e.g. $\delta(1,2,1,2) = s_2 s_1 s_2 = \delta(2,1,2) \neq \delta(1,2,2)$

redundant

nonredundant

Note:

(1) i_1 nonredundant $\iff$

$$\delta(i_1, i_2, \ldots, i_d) = s_{i_1}\, \delta(i_2, \ldots, i_d)$$

(2) $f_{(i_1 \ldots i_d)}(k_1 \ldots k_d) = p \iff f_{(i_2 \ldots i_d)}(k_2 \ldots k_d) = x_{i_1}(-k_1) p$

Lemma: If i_1 is nonredundant in $(i_1, \ldots, i_d)$, then there is unique value k_1 for t_1 s.t.

$$f_{(i_1, \ldots, i_d)}(k_1, t_2, \ldots, t_d) = p$$

has a solution.

allows "reduction step"

This has $x_{i_1}(-k_1)p \in \mathcal{U}(\delta(i_2, \ldots, i_d))$

Lemma: If i_1 is redundant in $(i_1, \ldots, i_d)$, then there exists $t_1^{max} > 0$ s.t. $f_{(i_1, \ldots, i_d)}(k_1, t_2, \ldots, t_d) = p$ has solution $\iff k_1 \in [0, t_1^{max}]$.

Moreover, $k_1 \in [0, t_1^{max})$ implies $x_{i_1}(-k_1)p \in \mathcal{U}(\delta(i_2, \ldots, i_d))$.

allows "reduction step"

Useful Characterization of which Parameters are Nonredundant (so Uniquely Determined Value)

Lemma: $S^c = \{j_1, \dots, j_s\} \subseteq \{1, \dots, d\}$ indexes rightmost reduced word for $\delta(i_1, \dots, i_d)$ in $(i_1, \dots, i_d)$ $\Longleftrightarrow$

$$S^c = \{ j \in [d] \mid i_j \text{ nonredundant in } (i_j, \dots, i_d) \}$$

e.g. $(1,2,1,2,4,1,5,2,4,5)$ $\quad d=10$

$$\begin{array}{cccccccccc} 1 & 2 & 3 & \underset{\sim}{4} & 5 & \underset{\sim}{6} & \underset{\sim}{7} & \underset{\sim}{8} & \underset{\sim}{9} & \underset{\sim}{10} \end{array}$$

S^c

$S^c = \{4,6,7,8,9,10\}$ since $(2,1,5,2,4,5)$ is rightmost reduced word for $\delta(1,2,1,\dots,4,5)$

$S = \{1,2,3,5\}$

Notation: $S = \{j_1', \dots, j_{d-s}'\}$

$S^c = \{j_1, \dots, j_s\}$

Domain for Homeom. from $[0,1)^{d-s}$ to Union of Strata

Notation for Domain:

$D^{<\max} :=$

$$\left\{ (t_1,\ldots,t_d) \in f^{-1}_{(i_1,\ldots,i_d)}(p) \;\middle|\; \begin{array}{l} t_j < t_j^{\max}(t_1,\ldots,t_{j-1}) \\ \text{for all } j \in S \end{array} \right\}$$

Thm (DHM): $D^{<\max} = \bigcup \sigma$

$\cong$

$[0,1)^{d-\ell(\omega)}$

strata σ s.t. $v \subset \bar{\sigma}$ for v of support S^c

rightmost red word

Corollary: Each stratum is homeomorphic to $(0,1)^i$ for some $i \geq 0$.

First Example, Revisited

$$x_1(t_1)\,x_1(t_2)\,x_1(t_3) = x_1(5)$$

$$\overset{''}{f_{(1,1,1)}(t_1,t_2,t_3)}$$

$D^{<\max}$ part of fiber (i.e. $t_3>0$ part)

- $0 \le t_1 < t_1^{\max} = 5$
- $0 \le t_2 < t_2^{\max}(k_1) = 5-k_1$
- $t_3 = 5-k_1-k_2 =$ "f_3" (k_1,k_2)

$$[0,1)^2 \cong \Big\{ (\underbrace{k_1}_{[0,1)}, \underbrace{k_2}_{[0,1)}, k_3) \;\Big|\; \begin{matrix} 0 \le k_1 < 5 \\ 0 \le k_2 < 5-k_1 \\ k_3 = 5-k_1-k_2 \end{matrix} \Big\}$$

this uses continuity of $t_2^{\max}$ & f_3
& well-definedness of f_3 (discussed next)

Generalization of Lusztig Result

Lemma: Given $\delta(i_1,\ldots,i_d)=\omega$ & $D_{k_{j_1},\ldots,k_{j_s}} :=$

$$\left\{ (t_1,\ldots,t_d)\in\mathbb{R}^d_{\geq 0} \;\middle|\; \begin{array}{l} t_{j'_\ell} > 0 \text{ for } j'_\ell \in S^c \\ t_{j_r} = k_{j_r} \text{ for } j_r \in S \end{array} \right\}$$

for any fixed $k_{j_1},\ldots,k_{j_s} \geq 0$, then

$f_{(i_1,\ldots,i_d)}\big|_{D_{k_{j_1},\ldots,k_{j_s}}}$ is a homeomorphism

to its image within $U(\omega)$.

e.g.

$$\underset{\in D_{3,7}\cong\mathbb{R}^4_{>0}}{(t_2,t_4,t_5,t_6)} \mapsto \underset{\in U(s_4 s_1 s_2 s_1)}{x_1(3)x_4(t_2)x_2(7)x_1(t_4)x_2(t_5)x_1(t_6)}$$

(since $\delta(1,\underline{4},2,\underline{1},\underline{2},\underline{1}) = s_4 s_1 s_2 s_1$)

Consequence: Within $D^{<\max}$ the redundant parameter values determine the nonredundant parameter values.

Continuity Lemmas

Lemma: If i_ℓ non-redundant in $(i_\ell, \dots, i_d)$ and $k_1, \dots, k_{\ell-1} \geq 0$ satisfy

(1) $k_i < t_i^{max}(k_1, \dots, k_{i-1}) \;\; \forall\, i < \ell$ with $i \in S$

(2) $\exists\, (k_1, \dots, k_{\ell-1}, t_\ell, \dots, t_d) \in f^{-1}_{(i_1, \dots, i_d)}(p) \cap \mathbb{R}_{\geq 0}$

then f_ℓ is continuous for $f_\ell(k_1, \dots, k_{\ell-1}) = k_\ell$ with $(k_1, \dots, k_{\ell-1}, k_\ell, \dots, t_d) \in f^{-1}_{(i_1, \dots, i_d)}(p) \cap \mathbb{R}^d_{\geq 0}$.

e.g. $f^{-1}_{(1,2,1,2)}\left(\begin{pmatrix} 1 & 5 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 1 & 7 & \\ & 1 & \end{pmatrix} \begin{pmatrix} 1 & 3 & \\ & 1 & \\ & & 1 \end{pmatrix} \right)$

$\quad = \left\{ (t_1, t_2, t_3, t_4) \mid f_{(1,2,1,2)}(t_1, \dots, t_4) = \begin{pmatrix} 1 & 8 & 35 \\ & 1 & 7 \\ & & 1 \end{pmatrix} \right\}$

- $t_1^{max} = 5$
- $f_3(k_1, k_2) = 8 - k_1$
- $f_2(k_1) = \dfrac{21}{8 - k_1}$
- $f_4(k_1, k_2, k_3) = 7 - k_2$

Lemma: If i_ℓ is redundant in $(i_\ell, \ldots, i_d)$, then t_ℓ^{max} is continuous fn of $k_1, \ldots, k_{\ell-1}$

Proof Idea:

$$\underbrace{t_\ell^{max}(k_1, \ldots, k_{\ell-1})}_{\text{for } f^{-1}_{(i_1 \ldots i_d)}(p)} = \underbrace{f_\ell(k_1, \ldots, k_{\ell-1})}_{\text{for } f^{-1}_{(i_1, \ldots i_\ell, Q)}(p)}$$

where (i_ℓ, Q) is reduced word for $S(i_\ell, \ldots, i_d)$

so f_ℓ continuous $\Rightarrow$ t_ℓ^{max} continuous

Example: $f_{(1,\underline{1},1,2)}(t_1, t_2, t_3, t_4) = \underbrace{x_1(7)x_2(5)}_{P}$

$t_2^{max}(k_1) = 7 - k_1$ (for $f^{-1}_{(1,\underline{1},1,2)}(p)$)

$f_{(1,\underline{1},-,2)}(t_1, t_2, -, t_4) = \underbrace{x_1(7)x_2(5)}_{P}$

$$t_2^{max}(k_1) = f_2(k_1) \quad \left(\text{for } f^{-1}_{(1,\underline{1},-,2)}(p)\right)$$

Remark: We focus on $D^{<\max}$ because $t_\ell = k_\ell^{\max}$ part has seemingly unmanageable structure/combinatorics.

Summary: $[0,1)^S \xrightarrow{\cong} \bigcup_{v \le \sigma \le \tau} \sigma$ which restricts to $(0,1)^S \xrightarrow{\cong} \tau$, showing each stratum is open ball.

An Open Qu: Given word Q, subword $(i_{j_1},\ldots,i_{j_\ell})$ that is reduced word for $\delta(Q)$ and constants $\{k_j \ge 0 \mid j \notin \{j_1,\ldots,j_\ell\}\}$, is h given by $(t_{j_1},\ldots,t_{j_\ell}) \xmapsto{h} f_Q(u_1,\ldots,u_d)$ for $u_j = \begin{cases} t_j & \text{if } j \in \{j_1,\ldots,j_\ell\} \\ k_j & \text{otherwise} \end{cases}$ a homeomorphism to $\operatorname{im}(h)$?

e.g. $(t_1,t_4,t_5) \mapsto x_1(t_1)x_2(3)x_1(2)x_2(t_4)x_1(t_5)$

Conjecture (Davis-H.-Miller):

For each $w \in W$ and each $p \in \mathcal{U}(w)$, $f^{-1}_{(i_1,\ldots,i_d)}(p) \cap \mathbb{R}^d_{\geq 0}$ is regular CW complex homeomorphic to the interior dual block complex of the subword complex $\Delta((i_1,\ldots,i_d),w)$.

Thanks for Listening!

Lemma: Given i_j nonredundant in $(i_j,\dots,i_d)$ & given $k_1,\dots,k_{j-1} \geq 0$ s.t.

$$x_{i_{j-1}}(-k_{j-1})\cdots x_{i_1}(-k_1)\,p \in U(\delta(i_j,\dots,i_d))$$

then there is a unique value

$k_j = f_j(k_1,\dots,k_{j-1}) \in \mathbb{R}_{\geq 0}$ for t_j s.t.

$$f_{(i_1,\dots,i_d)}(k_1,k_2,\dots,k_{j-1},k_j,t_{j+1},\dots,t_d) = p$$

has solution with $t_{j+1},\dots,t_d \in \mathbb{R}_{\geq 0}$.

e.g. $$f^{-1}_{(1,2,1,2)}\Big(\underbrace{x_1(5)\,x_2(7)\,x_1(3)}_{\begin{pmatrix} 1 & 8 & 35 \\ & 1 & 7 \\ & & 1 \end{pmatrix}}\Big)$$

has $t_1 \in [0,5]$ and

- $f_2(k_1) = \dfrac{21}{8-k_1} = \text{“}k_2\text{”}$
- $f_3(k_1,k_2) = 8-k_1$
- $f_4(k_1,k_2,k_3) = 7-k_2$

Lemma: Given i_j redundant in $(i_j,\dots,i_d)$ & any $k_1,\dots,k_{\ell-1}\geq 0$ s.t.

$$x_{i_{j-1}}(-k_{\ell-1})\cdots x_{i_1}(-k_1)\,p\in U(\delta(i_j,\dots,i_d))$$

then t_j takes exactly the values in $[0,\underbrace{K}_{= \text{"}t_j^{max}(k_1,\dots,k_{j-1})\text{"}}]$ for some $K>0$.

Example: $M=\begin{pmatrix}1 & \pi & e\\ & 1 & 14\\ & & 1\end{pmatrix}$ then

$$f^{-1}_{(1,2,1,2)}(M) = \{(t_1,t_2,t_3,t_4)\mid x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4)=M\}$$

achieves every $t_1\in[0,\underbrace{\tfrac{e}{14}}_{t_1^{max}}]$

Equations w/ Unique Solution (by Lusztig)

$$(t_1,t_4,t_5)\mapsto \underset{s_1}{\begin{pmatrix}1&t_1&\\&1&\\&&1\end{pmatrix}}\underset{s_2}{\begin{pmatrix}1&&\\&1&t_4\\&&1\end{pmatrix}}\underset{s_1}{\begin{pmatrix}1&t_5&\\&1&\\&&1\end{pmatrix}} = \underset{U(s_1s_2s_1)}{M}$$

$$t_1+t_5=\underset{\text{"}s_1\text{" minor}}{M_{12}} \qquad t_4=\underset{\text{"}s_2\text{" minor}}{M_{23}} \qquad t_1t_4=\underset{\text{"}s_1s_2\text{" minor}}{M_{13}}$$

Modified Equations for:

$$(t_1,t_4,t_5)\mapsto \begin{pmatrix}1&t_1&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&&\\&1&3\\&&1\end{pmatrix}\begin{pmatrix}1&2&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&&\\&1&t_4\\&&1\end{pmatrix}\begin{pmatrix}1&t_5&\\&1&\\&&1\end{pmatrix}$$

$$M = \underbrace{\underbrace{\underset{s_1}{x_1(t_1)}\,\underset{s_2}{x_2(3)}}_{s_1s_2}\,\underbrace{\underset{s_1}{x_1(2)}\,\underset{s_2}{x_2(t_4)}}_{s_1s_2}}_{s_1s_2}\,\underset{s_1}{x_1(t_5)}$$

$$t_1+2+t_5=\underset{\text{"}s_1\text{" minor}}{M_{12}} \qquad 3+t_4=\underset{\text{"}s_2\text{" minor}}{M_{23}}$$

$$t_1\cdot 3+t_1t_4+2t_4=\underset{\text{"}s_1s_2\text{" minor}}{M_{13}}$$

Idea for Continuity of f_ℓ:

$$x_{i_1}(t_1)\cdots x_{i_{\ell-1}}(t_{\ell-1})\, x_{i_\ell}(t_\ell)\cdots x_{i_d}(t_d) = p$$

$$\Updownarrow$$

$$x_{i_\ell}(t_\ell)\cdots x_{i_d}(t_d) = x_{i_{\ell-1}}(-t_{\ell-1})\cdots x_{i_1}(-t_1)\, p$$

so $\{(t_\ell,\dots,t_d) \mid (t_1,\dots,t_{\ell-1},t_\ell,\dots,t_d) \in f^{-1}_{(i_1,\dots,i_d)}(p)\}$

$$\|$$

$$f^{-1}_{(i_\ell,\dots,i_d)}\big(\underbrace{x_{\ell-1}(-t_{\ell-1})\cdots x_1(-t_1)\,p}_{\text{continuous function of } t_1,\dots,t_{\ell-1} \text{ for fixed } p}\big)$$