Shellability of Uncrossing Posets &
the CW Poset Property

Patricia Hersh
North Carolina State University

(partly joint work with Rick Kenyon)
Outline for Talk

1. Background on CW Posets
   * (Lexicographic) Shellability

2. Two maps of face posets of their images

3. Close Look at Dyer's Reflection Order Labeling for Bruhat order

4. Shellability of uncrossing posets (joint work w/Kenyon)
Motivations & Context

• Want to study images *f*ibers of interesting maps $f : K \to L$ of stratified spaces which induce poset maps
  $\tilde{f} : F(K) \to F(L)$ on face posets

• It helps to prove $F(L)$ is "CW poset" which follows from "thinness" ≠ "shellability"

• Today: Discuss such maps and poset shellability for
  • $F(L) =$ Bruhat order
  • $\tilde{F}(L) =$ uncrossing order 
    (w/ Rick Kenyon)
**CW Complexes & their Face Posets**

*Example.*

\[ F(K) = \{ e_1, e_2, e_3, e_4 \} \]

*Closure poset* or *face poset*

\[(u \leq v \iff u \subseteq \overline{v})\]

**Recall:** A CW complex is comprised of open cells each homeomorphic to an open ball. A regular CW complex further has cell closures homeomorphic to closed balls. e.g. simplicial complexes.
**Defn**: The order complex (or nerve) of a poset $P$ is the abstract simplicial complex $\Delta(P)$ whose $i$-dimensional faces are the $(i+1)$-"chains" $v_0 < \cdots < v_i$ in $P$.

**Examples**: $P = a_1 \prec b_1 \prec a_2 \prec b_2$, $\Delta(P) = \ldots$

**Key Property** (Hall; popularized by Rota):

$$M_P(x,y) = \tilde{\chi}(\Delta_P(x,y)) = -1 + \# \text{ vertices} - \# \text{ edges} + \# \text{ 2-faces} - \cdots$$

- $(u,v) = \Sigma z \in P \mid u < z < v \}$
- $u < v$ means $u < v \land \exists z \text{ s.t. } u < z < v$
- Saturated chains up to $v := u < \cdots < v$
Background on Face Posets & CW Posets

• A graded poset with \( \hat{0} \neq \hat{1} \) is Eulerian if \( M(u,v) = (-1)^{rk(v) - rk(u)} \) for all \( u \leq v \).

• A graded poset \( P \) is a CW poset if
  (1) \( \hat{0} \in P \)
  (2) \( P \) has at least one other element
  (3) \( \Delta(\hat{0}, u) \cong S^{rk(u) - 2} \) for \( u \neq \hat{0} \)

Thm (Björner): \( P \) is CW poset \( \iff \) there exists regular CW complex with \( P \) as poset of closure relins

Cor: CW Poset \( \Rightarrow \) Eulerian
Some CW Posets

- all graded, thin, shellable posets (Danaraj-Klee)

  "thin" $\iff$ \begin{tikzpicture}
  \node (v) at (0,0) {$v$};
  \node (w) at (1,0) {$w$};
  \node (u) at (0.5,0.866) {$u$};
  \draw (v) -- (w);
  \draw (v) -- (u);
  \draw (w) -- (u);
  \end{tikzpicture}

  $rk(v) - rk(w) = 2$
  $|E_{uw}| = 4$

- Bruhat order (Björner-Wachs; Dyer)

- Face posets of stratified spaces of electrical networks

  - conjectured by Thomas Lam
  - proved by H.-Kenyon (a main topic for today's talk)
Shellability

- Simplicial complex is pure of dim. d if all maximal faces ("facets") are d-dimensional.
- Simplicial complex is shellable if there is total order $F_1, F_2, \ldots, F_k$, a shelling, on facets s.t. $F_j \cap (\cup_{i<j} F_i)$ is pure, codimension one subcomplex of $F_j$ for each $j \geq 1$ (hence is $\partial F_j$ or has a cone point).

OR

- Each facet attachment preserves homotopy type or closes off a new sphere.
**Lexicographic Shellability**

(Björner & Björner–Wachs)

A poset $P$ is **EL-shellable** if it admits labeling $\lambda$ (called an EL-labeling) of its cover relations $x \prec y$ w/ integers s.t. $u < v$ implies:

1) there is unique saturated chain $u \prec u_1 \prec \ldots \prec u_k \prec v$ s.t. $
\lambda(u, u_1) \leq \lambda(u_1, u_2) \leq \ldots \leq \lambda(u_k, v)$

and

2) $(\lambda(u, u_1), \lambda(u_1, u_2), \ldots, \lambda(u_k, v))$

is lexicographically smaller than the label sequences on all other saturated chains from $u$ to $v$. 
**Thm (Björner):** EL-labeling $\Rightarrow$ Shelling

**Idea:** Lexicographic order on maximal chains (breaking ties arbitrarily) induces shelling order on corresponding facets of $\Delta(P)$.

- "descents in labeling" $\iff$ overlap of facets
- "descending" $\iff$ facets attaching along entire boundary $\iff$ spheres
- $M_P(u,v) = \pm \# \text{descending chains } u \text{ to } v$ (for $P$ graded)
Bruhat Order of a (Finite) Reflection Group / Coxeter Group \& Dyer's EL-Labeling

e.g.

\[ S_i \cdot (i, i+1) \]

\[ 321 = s_i s_2 s_1 = s_2 s_1 s_2 \]

\[ S_1 \]

\[ S_2 \]

\[ S_2 s_1 = 312 \]

\[ 213 = s_1 \]

\[ (s_1)^{-1} (s_1 s_2) = s_2 \]

\[ 213 = s_2 \]

\[ S_1 \]

\[ S_2 \]

\[ S_2 = 132 \]

\[ t_{13}, t_{23}, t_{12}, t_{12}^\perp, \ldots \]

EL-labeling: \( u \preceq v = u t \)

\[ \lambda(u, v) := u^t v = t \]
Dyer's EL-labeling (cont)

- Use any "reflection order" to totally order edge labels
- Dyer proved these exist to induce EL-labelings

**Defn**: A total order on positive roots (and associated reflections) is a reflection order if \( \alpha < c_1 \alpha + c_2 \beta < \beta \) or \( \beta < c_1 \alpha + c_2 \beta < \alpha \) for each such triple of positive roots with \( c_1, c_2 > 0 \)

E.g. \((1,2) < (1,3) < (2,3)\) or \((2,3) < (1,3) < (1,2)\) in type A
A Useful Characterization of Bruhat Order Cover Relations (Label Sequences)

Thm (Dyer, preprint 2011; rediscovered H. 2017)

Given \( u \in W \) & reflection \( t_g \in W \),
\[ u < t_g \Leftrightarrow (1) \, g \in R(u) \\
\quad (2) \, \exists \, d, \beta \in R(u) \]
\[ \text{s.t. } g = c_1 d + c_2 \beta \]
\[ \text{for } c_1, c_2 > 0 \]

Recall: \( g \in R(u) \Leftrightarrow l(u \cdot t_g) < l(u) \)

\[ \text{e.g. } e \notin (1,3) \text{ but } (1,2) \prec (1,2) \cdot (1,3) \]
\[ (1,3) \sim \sim e_1 - e_3 \]
\[ l(e_1 - e_2) + (e_2 - e_3) \]
Bruhat Order as Face Poset of Map Image as CW Poset

- \( \chi_i(t) = \text{Int} + tE_{i,i+1} = \exp(t(e_i)) \) (type A)
- (general finite type, exp'd Chevalley generator)
- \( f_{(i_1, \ldots, i_d)} : \mathbb{R}_{\geq 0}^d \rightarrow M_{n \times n} \subseteq \mathbb{R}^{n^2} \)

\( (t_1, \ldots, t_d) \mapsto \chi_{i_1}(t_1) \ldots \chi_{i_d}(t_d) \)

\( \text{e.g. } f_{(1, 2, 1)}(t_1, t_2, t_3) = \chi_1(t_1) \chi_2(t_2) \chi_1(t_3) \)

\( Rk: (i_1, \ldots, i_d) \text{ word for } w_0 = \text{im}(f_{(i_1, \ldots, i_d)}) \) totally nonneg. put unipotent radical of Borel

\[
\begin{pmatrix}
1 & t_1 & 0 & t_2 \\
0 & t_1 & 0 & t_2 \\
0 & 0 & t_1 & t_2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
"Picture" of Map

\[ f_{(1,2,1)}(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 & 1 \\ 1 & t_2 & 1 \\ 1 & t_3 & 1 \end{pmatrix} \]

\[ \mathbb{R}_+^3 \cap (\sum t_i = 1 \text{ hyperplane}) \]

\[ f_{(1,2,1)}(t_1, 0, t_3) = \begin{pmatrix} 1 & t_1 & 1 \\ 1 & 0 & 1 \\ 1 & t_3 & 1 \end{pmatrix} = \chi_i(t_1 + t_3) \]

Non-injectivity: results from "nil-moves"

\[ \chi_i(u) \chi_i(v) = \chi_i(u + v) \neq \chi_i(u + v) \quad \text{and} \quad \text{long braid moves} \]
Thm (Fomin-Shapiro): Face poset for image of $f_{(i_0, \ldots, i_d)}$ is Bruhat interval $[0, \omega]$ for $(i_0, \ldots, i_d)$ reduced word for $\omega$.

Thm (H., 2014): Image of $f_{(i_0, \ldots, i_d)}$ is regular CW complex homeomorphic to closed ball. (“Fomin-Shapiro Conj.”)

Lusztig: Connections to “dual canonical bases”

Galashin-Kaup-Lam (July 2017): clever, shorter proof of homeomorphism for closure of “big cell” for $\omega = \omega_0$ in type $A$; other spaces including $Gr_2(n, k)$ & closed big cell for electrical networks in “well-connected graphs.”
**Induced Map of Face Posets**

\[ f : \text{poset of subsets} \rightarrow \text{Bruhat order} \]

\[ F(K) \quad \text{and} \quad F(L) \]

**Obs (Armstrong-H.):** Each subset \( f^{-1}_u \) of \( F(K) \) is dual to face poset of "subword complex" \( \Delta(Q, u) \) of Knutson-Miller, reduced word for \( w \).
Maps Arising from Electrical Networks (see R. Kenyon, "The Laplacian on Planar Graphs & Graphs on Surfaces")

\[
\Delta \begin{pmatrix} v_N \\ v_I \end{pmatrix} = \begin{pmatrix} c_N \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}
\]

\text{vector of currents of voltages}

I = \text{internal nodes} \quad N = \text{boundary nodes}

\[
(A - BC^{-1}B^T) v_N = c_N
\]

"response matrix of network" (entries are rat'ld fns of conductances)
A **Goal**: Given a graph $G$, study the space of response matrices as image of

$$f: \{ \text{conductance vectors} \} \rightarrow \{ \text{response matrices} \}$$

$$(1R_{\mathbb{Z}_2} \cup \{e0^3\})^E$$

**Note:**
contracting $\leftrightarrow$ sending conductance to $\infty$ (i.e. resistance to 0)
deleting $\leftrightarrow$ sending conductance to 0 (resistance to $\infty$)

**Secondary Goal**: Study fibers of $f$
Correspondence: From Graphs to Uncrossing Wire Diagrams "Medial Graphs"

\[ \Gamma' = \]

\[ G(\Gamma') = \] = wire diagram

Uncrossing:

deletion

contraction
Face Poset for Electrical Networks
A Conjecture of Thomas Lam

Thm (Lam): The uncrossing poset is Eulerian.

Conjecture (Lam): The uncrossing poset is lexicographically shellable.

Thm (H.-Kenyon): Uncrossing posets are dual EC-shellable.

Cor: They are CW posets.
The Uncrossing Poset (Face Poset for Electrical Networks)

1. \[ \hat{1} := \text{wire diagram w/ all } \binom{n}{2} \text{ crossings of } n \text{ wires} \]

2. \( u \prec v \) if \( u \) obtained from \( v \) uncrossing pair of wires without introducing double crossing

3. \( \hat{0} \) adjoined below Catalan many atoms
Edge Labeling

Step 1: Define word of wire diagram $D$, denoted $w(D)$, as sequence of $2n$ wire ends $t$ encountered clockwise starting with “root” 1.

e.g. $D = \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} 2 \downarrow \quad 1221 \ 3 \ 4 \ 3 \ 4 \ 1 \ w(D)$

Step 2: Label $D \prec D'$ for $i \leq j$ as

- $(i, j)$ if $ijijj$ in $w(D)$ becomes $ijji$ in $w(D')$
- $(ji)$ if $ijiji$ becomes $ijjj$
Face Poset for Electrical Networks
Label Ordering

- \( \mathcal{E}(i,j) \mid 1 \leq i < j \leq n^3 \) or \( \mathcal{E} L^3 \)
  - \( \mathcal{E}(j,i) \mid 1 \leq i < j \leq n^3 \)

- \((i,j) < L < (r,s)\) for all \(i < j\)
  - and all \(r > s\)

- \((1,2) < (1,3) < (1,4) < \cdots < (1, n) < (2,3)\)
  - \((2,4) < \cdots < (2, n) < (3,4) < \cdots < (3, n)\)
  - \(\cdots < (n-1, n)\)

- \((n, n-1) < (n, n-2) < (n-1, n-2) < (n, n-3)\)
  - \(\cdots < (n-1, n-3) < (n-2, n-3) < \cdots\)
  - \(\cdots < (2,1)\)

Re: finite type A reflection order \(\mathcal{E}(i,j)\)
then \(L\), then reversal for \(\mathcal{E}(j,i)\)
"Start Sets" and Connection to Type A Bruhat Order

The start set of $D$, denoted $S(D)$, is the subset of $\mathbb{Z}, \mathbb{Z}, \ldots, 2n^3$ of positions in $\omega(D)$ where 1st copies of letters occur.

\[
\begin{align*}
\text{e.g., } & \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
3 \\
2 \\
1
\end{array}
\end{array} \\
\omega(D) &= 121233 \\
S(D) &= 31, 32, 33
\end{align*}
\]

Propn: If $D' < D$ and $SCD(D) = S(D)$, then $[D', D] = [\pi(D'), \pi(D)]$

- subwords of $\omega(D') \# \omega(D)$
- uncrossing order labeling coincides with Dyer's Bruhat order labeling
Prop\(\text{in}: D' \leq D \Rightarrow S(D') \leq_{\text{lex}} S(D)\) (very helpful for proving \([D', D]\) has unique topologically ascending chain)

**Noncrossing Sets**

Given a wire diagram \(D\), its noncrossing set is defined as \(N(D) = N_1(D) \cup N_2(D)\) for

- \(N_1(D) = \{ (i, j) \mid \omega(D) \text{ includes } i < j \}\)
- \(N_2(D) = \{ (j, i) \mid \omega(D) \text{ includes } j > i \}\)
New Description of Cover Relations in Uncrossing Order

Given wire diagram $D$, there is $D' \prec D$ uncrossing wires $R \nleq m$ with $2(C(D', D)) = (k, m) \& N(D)$

$\iff (1) \ (m, k) \in N(D)$

$$\iff (2) \ k < m \implies \text{for } k < l < m \begin{array}{c} |\{(k, l), (l, m): (k, m) \in N(D)\}| = 1 \end{array}$$

$$\iff (3) \ m < k \implies \text{for } l < m \text{ or } l > k \begin{array}{c} |\{(k, l), (l, m): (k, m) \in N(D)\}| = 1 \end{array}$$
Type A Specialization of Bnhat Order Cover Relation Description

\[ \Pi = \Pi(1)\Pi(2)\ldots\Pi(n) \] (one line notation)

has

\[ \Pi < \cdot \Pi \cdot (i, k) \] swap letters i and k

• i appears to left of k
• for each j s.t. i < j < k, either j appears to left of i or j appears to right of k

e.g. \[ \Pi = 3 2 5 1 4 \] inversion 5,4

\[ \Pi \cdot (2, 4) = 3 4 5 1 2 \] inversion 5,2
Dyer's Proof: Reflection labelings are EL-labelings

- Interpreted number of ascending chains from $u$ to $v$, namely $u < u_1 < ... < u_k < v$ s.t.
  $\lambda(u, u_1) \leq \lambda(u, u_2) \leq ... \leq \lambda(u_k, v)$,
  as leading coeff. of $\tilde{R}_{u,v}(g)$

- Observed EL-shellability then followed from result of Kazhdan & Lusztig that $\tilde{R}_{u,v}(g)$ is monic.

Thm (H.): Elementary proof in type $A$ (without properties of KL-polys)
\( \tilde{R} \)-poly's: Unique polys \( \{ \tilde{R}_{u,v}(g) \} \)

s.t.,

\[
\tilde{R}_{u,v}(g) = g \tilde{R}_{u,v}(g^{1/2} - g^{-1/2})
\]

used to define Kazhdan-Lusztig poly's

Thm (see e.g. Björner-Brenti 5.3.4)

\[
\tilde{R}_{u,v}(g) = \sum \tilde{R}_{v}(D)
\]

where \( D \in B(u,v) \)

s.t. \( D(\Delta, <) = \emptyset \)

\( \tilde{R}_{u,v}(g) = g^3 + g \)
Appendix: Some Further Details

Slides available at:
• http://www4.ncsu.edu/nphersh/

Thank you!
Turning to Fibers via "Electrical Equivalence"

1) \( a \sim b \)

2) \( \emptyset \sim \)

3) \( \frac{a}{b} \sim \frac{ab}{a+b} \)

4) \( a \sim \frac{ab}{a+b} \)

5) "Y-Δ moves"

\( \frac{cc}{a+b+c} \sim \frac{a+b}{a+b+c} \)

\( \frac{bc}{a+b+c} \)
Connection to Bruijnt Oeler in Affine Type A

\[ S_{2n} := \{ f : \mathbb{Z} \to \mathbb{Z} | f \text{ bijective s.t.} \]
\[ f(i+2n) = f(i) + 2n \]
\[ \sum f(i) = 2n + \sum i \]
\[ i=1 \]

\[ \Rightarrow (23)(14)(58)(69)(710) \]

\[ \exists \in S_{2n} \]

fixed pt
free involution

\[ g_2 \in S_{2n} \]

\[ g_2(2) = 3 \quad g_2(1) = 4 \]

\[ g_2(3) = 2 + 5 \quad g_2(4) = 1 + 5 \]

...
Lam's Dual Embedding of Uncrossing Order into $\tilde{S}_{2n}$ Braid Order

- $D_e = \text{fully crossed} \Rightarrow 1D e \tilde{\in} S_{2n}$
  diagram

- $D < \cdot D'$ where $D$ obtained from $D'$ by
  $i,j$ wire endpoint swap

$D' = \begin{array}{c}
\begin{array}{c}
\circ \circ \\
2 & 3 \\
\end{array}
\end{array}$

$D = \begin{array}{c}
\begin{array}{c}
\circ \circ \\
1 & 2 \\
\end{array}
\end{array}$

$g_D = (1,2) g_{D'} (1,2)$

$g_D = (4,5) g_{D'} (4,5)$
Minors of Response Matrix via "Gours"

\( \Pi(G) := \) set partition of boundary graph nodes into connected components

e.g. \( \Pi(\text{graph}) = 12 \mid 3/4 \)

Thm (Special case of Next Result):

\[
L_{ij}(G) = \frac{\sum_{G' \leq G} wt(G')}{\Pi(G') = \text{ij pairs of singletons}}
\]

\( wt(G') = \text{product of edge weights (i.e. conductances)} \)
Thm (Kenyon-Wilson; Curtis-Ingerman-Morrow)

For $|S|=|R|$ with $S \cap R = \emptyset$,

$$\det \left( L_{\frac{S \cup T}{R \cup T}} \right) = (-1)^{|T|} \cdot \sum_{p \in S \cup R} \sgn(p) \cdot K_p$$

For $K_p = \sum wt(G')$

$$\frac{G' \leq G}{\pi(G') = r_1, \pi(r_1), r_2, \pi(r_2), \ldots, r_k, \pi(r_k)} | \text{singletons}$$

$$\sum wt(G')$$

$$G' \leq G$$

$$\pi(G') = \text{all singletons}$$
Tempting Idea Which Doesn't (Quite) Work

- label uncrossing of wires
  - \( i \) and \( j \) for \( 1 \leq i < j \leq n \) as \( (i,j) \) or \( (j,i) \)
  - exchange \( i \) with \( j-n \)
  - exchange \( i \) and \( j \)

\( \mathcal{S}_n = \{ f : \mathbb{Z} \to \mathbb{Z} \mid f(i+n) = f(i) + n \} \)
\[ \sum_{i=1}^{n-1} f(i) = \binom{n+1}{2} \]

\[ (i,j) \quad \overset{\text{exchange}}{\leftrightarrow} \quad (j,i) \]

\[ n=3 \]

\[ (1,3) \leftrightarrow (e_1 - e_2) + (e_2 - e_3) \]
\[ 1 \to 3 \to 1 \quad 2 \to 2 \quad -2 \to 0 \ldots \]

\[ (3,1) \leftrightarrow 8 - (e_1 - e_3) \]
\[ 3 \to 1 \to 0 = 3 - 3 \to 1 \quad 2 \to 2 \ldots \]
**Subword Complexes** (introduced by Knutson & Miller)

\( Q := \text{(not necessarily reduced) expression} \)

\( w := \text{Coxeter group element} \)

Facets of \( \vartriangle(Q, w) \) are the subwords of \( Q \) whose complements are reduced words for \( w \).

\( \vartriangle(Q, w) = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
(12)_c
\end{array}
\begin{array}{c}
(12)_c
\end{array}
\begin{array}{c}
(-12)_c
\end{array}
\end{array}
\end{array} \)

\( Q = (1, 2, 1, 2) \)

\( w = s_1 s_2 \)

**Thm (Knutson-Miller):** \( \vartriangle(Q, w) \) is "vertex decomposable" (hence shellable) ball or sphere.

(Used to study matrix Schubert varieties via "Gröbner degeneration")
Kazhdan–Lusztig Polynomials

**KL-poly's:** Unique \( P_{u,v}(q) \in \mathbb{Z}[q] \) st.
1. \( P_{u,v}(q) = 0 \) for \( u \not\leq v \)
2. \( P_{u,u}(q) = 1 \)
3. \( \deg(P_{u,v}(q)) \leq \frac{1}{2}(l(u,v)-1) \) for \( u < v \)
4. \( \sum_{a \in [u,v]} R_{u \leftarrow a}(q) P_{a,v}(q) = \prod_{a \in [u,v]} P_{a,v}(q) \)

KL-poly \( P_{u,v}(q) \)

"local intersection homology Euler characteristic of \( \widetilde{\Omega}_v \) at generic pt in \( \Omega_u \)"
\( R\)-poly's: Unique \( \exists R_{u,v}(g) \) s.t.

1. \( R_{u,v}(g) = 0 \) for \( u \neq v \)
2. \( R_{u,u}(g) = 1 \)
3. for \( s \in D_R(v) \), then
   \[
   R_{u,v}(g) = \begin{cases} 
   \sum R_{u,s,v}(g) & \text{if } s \in D_R(u) \\
   gR_{u,s,v}(g) + \text{ otherwise} \\
   (g-1)R_{u,v}(g) & \text{if } s \in D_R(u) 
   \end{cases}
   \]
Recall: A real matrix is totally nonnegative if all minors are nonnegative.

e.g. \[
\begin{pmatrix}
1 & t_1 + t_3 & t_1 t_2 \\
0 & 1 & t_2 \\
0 & 0 & 1
\end{pmatrix}
; \quad t_1, t_2, t_3 \geq 0
\]

Since \( t_1 + t_3 \geq 0 \) \( t_2 (t_1 + t_3) - t_1 t_2 \geq 0 \) \( t_2 \geq 0 \) \( t_1, t_2 \geq 0 \)

also:
\[
\begin{pmatrix}
1 & t_2' & t_2' t_3' \\
0 & 1 & t_1' + t_3' \\
0 & 0 & 1
\end{pmatrix}
; \quad t_1', t_2', t_3' \geq 0
\]

Unipotent radical e.g. \[
\begin{pmatrix}
1 & * & * \\
0 & 1 & * \\
0 & 0 & 1
\end{pmatrix}
\]
First: A Motivation for Nonneg. Real Part of Unipotent Radical

e.g. \((t_1, t_2, t_3) \mapsto \left( \frac{t_2 t_3}{t_1 + t_3}, \frac{t_1 t_3}{t_1 + t_3}, \frac{t_1 t_2}{t_1 + t_3} \right)\)

('simply laced' case) \(t_1, t_2, t_3\)

• Tropicalizes to change-of-basis map for Lusztig's "canonical bases":

\((a, b, c) \mapsto (b + c - \min(a, c), \min(a, c), a + b - \min(a, c))\)

(applying braid move to reduced expression for \(w_0 \) w.r.t. which canonical basis is defined)

• Given quantized env. alg. \(U = U \otimes \mathfrak{u} \otimes \mathfrak{h} \otimes \mathfrak{v}_\lambda \otimes \mathfrak{v}_\mu\)

then canonical basis is a basis \(B\) for \(U^-\)

such that highest weight module with highest weight vector \(v_\lambda\) has basis

\(\{v_{a+b} \mid v_{a+b} \neq 0\}^3\) for each \(\lambda\).
Some Other Related Work

- Lauren Williams:  
  - shelling face posets of nonneg flag varieties (2007)

- Galashin-Karp-Lam
  - homeomorphism type for closure of type A big cell  
  for $W_0$ & for big cell for well-connected electrical networks (top element of full uncrossing poset)  
  (preprint, July 2017)