Representation Stability in Configuration Spaces via Whitney Homology of the Partition Lattice

Patricia Hersh
North Carolina State University & ICERM

joint work with Vic Reiner

(based on paper to appear in IMRN)
A "Point" in a Configuration Space with $S_n$-rep's on Cohomology

- Manifold = 3-holed torus
- $\eta = 6 = \# \text{distinct labeled points}$
- $S_n$ acts freely on configuration space by permuting pt. labels, inducing rep'n on each cohomology group
**Representation Theoretic Stability**

Defn (Church, Farb): A series of $S_n$-modules $M_1, M_2, \ldots$ for $n=1, 2, \ldots$ stabilizes at $B > 0$ if for each $n > B$, we have

$M_n = \sum c_\lambda V(\lambda)$ where $V(\lambda) = S^{(n-m, \lambda)}$ with $m \leq B$

and where $c_\lambda$ does not depend on $n$.

E.g.

$n-m \rightarrow \begin{array}{c} \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \end{array} \rightarrow \begin{array}{c} \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \end{array}$

$M_n \rightarrow M_{n+1}$

$M_{n+2} = \begin{array}{c} \vdots \\ \vdots \end{array}$

**Our Focus**: $S_n$-rep's from partition lattice
Our Starting Point:

**Thm (Church-Farb):** $H^i(M^n_\infty, \Omega)$ stabilizes for $n \geq 4i$ where $M^n_\infty$ is configuration space of $n$ distinct points in plane & $i$ is held fixed.

**Thm (Church-Farb):** More generally, letting $M^d_n$ be the configuration space of $n$ distinct labeled points on connected orientable $d$-manifold, $H^i(M^d_n, \Omega)$ stabilizes for

$$\begin{cases} 
  n \geq 4i & \text{if } d = 2 \\
  n \geq 2i & \text{if } d \geq 2
\end{cases}$$

Our First Objective: Sharpen these bounds for $M^d = \mathbb{R}^d$
How Representation Stability Typically Arises

- Finite number of irreducible reps $S^\lambda$, $S^\lambda$ 1st appearing in $M_{1\lambda}$
- Each $M_n$ with $n \geq 1\lambda_1$ likewise includes $S^\lambda \otimes \text{triv} \uparrow^\text{Sn}_{n-1\lambda_1} S_{n-1\lambda_1} \times S_{\lambda_1}$
- Church-Ellenberg-Farb prove stability bounds of $n = 2\max 1\lambda_1$
- H-Reiner prove sharp stability bds for $PConf(\mathbb{R}^d)$ at $n = \max (1\lambda_1 + 1, )$

**Pieri Rule:**

\[ S^\lambda \otimes \text{triv} \uparrow^\text{Sn}_{\lambda_1} = \bigoplus \]

![Diagram of Pieri Rule]
Church-Farb Method for Orientable Manifolds

- Use Totaro's $E_2$-page of Leray spectral sequence (showing cohom. of config. space of $n$ distinct pts on manifold $M$ is determined by cohom. of $M + H^c(M; R^d))$ to deduce stability of each page from previous page:

$$E_2^{p,d-1} = \bigoplus H^d(c_{S}(R^d)) \otimes H^p(M^S)$$

S with $|S| = n - 3$

product of subspace arrangement complements for set partition $S$ with 1 partition

e.g. for $S = \{1, 3\}, \{2, 4, 5\}$

- $C_S(R^d) := \{x \in (R^d)^5 | x_1 \neq x_3; x_2 \neq x_4\}$

  $= C_{\{1, 3\}}(R^d)^2 \times C_{\{2, 4, 5\}}(R^d)^3$

- $M^S := \{x \in M^5 | x_1 = x_3; x_2 = x_4 = x_5\}$

  $\Rightarrow E_2^{p,5} = 0$ for $d-1 \neq 3$
Partition Lattice \( \Pi_n \) & its \( S_n \)-representations

\[
\begin{array}{cccc}
1234 \\
123|4 & 12|34 & 1|234 & 1|324 & 124|3 & 23|14 & 2|134 \\
12|3|4 & 13|2|4 & 14|2|3 & 23|1|4 & 24|1|3 & 34|1|2 \\
\end{array}
\]

\( \Pi_4 = 1|2|3|4 \)

- \( S_n \) acts by permuting values
- e.g. \((13))[12|3|45] = 32|1|45\)
Reinterpreting via Subspace Arrangement Complements

- $M_n = \text{complement of type A (complex) braid arrt } \{x_i = x_j | 1 \leq i < j \leq n\}$

Warning: figure is IIR-picture, need C-picture

$(\text{Config space pt } p_i \leftrightarrow x_i \in \mathbb{C})$

- $\Pi_n = \text{intersection poset } \mathcal{A}(A_{n-1})$

- $S_n$-module structure for $H^i(M_n)$ will translate to "Whitney homology" in $\Pi_n$, $WH_i(\Pi_n)$
$\mathcal{L}(A_2) =$

"poset of intersections of subspaces"

$\Pi_3 =$

"lattice of set partitions"

$x_i = x_j \iff i, j$ in same block
Def'n: The order complex of a finite poset $P$ is the simplicial complex $\Delta(P)$ whose $i$-dimensional faces are the $(i+1)$-chains in $P$.

e.g. $P = \{a_1, a_2, b_1, b_2\}$, $\Delta(P) = \{a_1, a_2, b_1, b_2\}$

Let $\overline{P} = P \setminus \{0, \hat{1}\}$ e.g. for $\Pi_n$

Convention: When we speak of topological properties (homology, etc.) of poset $P$, we mean $\Delta(P)$ or $\Delta(\overline{P})$.

Poset rank: = # steps from bottom
**Goresky-MacPherson Formula**

(for cohomology of subspace arr't)

\[ \tilde{H}^i(M_A) \cong \bigoplus_{x \in \Lambda_R \cap \text{codim}(x) - 2 - i} \tilde{H}^{\text{codim}(x) - 2 - i}(\partial, x) \]

subspace arr't complement \( \Lambda_R \) intersection lattice

**Plan:** Apply to braid arrangement using upcoming \( S_n \)-equivariant version due to Sundaram-Welker, yielding Whitney homology. (See also Blagojević, Lück, Ziegler for more general versions)
\textit{\textbf{S}}_n-\textbf{Representations on Chains (i.e. on Faces)} \& on \textbf{Homology}

- \textit{\textbf{S}}_n \textbf{action on set partitions is order-preserving \& rank-preserving}

(Recall \( P \) is graded if for each \( u < v \), all saturated chains \( u \rightarrow v \) have same length)

- Hence, induces \( S_n \)-action on \( \{ \text{chains } u_1 < u_2 < \ldots < u_j \} \)

\( \{ \text{faces of } \Delta (\uparrow_n) \} \)
\[ d(u_0 < ... < u_r) = \sum_{i=0}^{r} (u_0 < ... < \hat{u}_i < ... < u_r) \]

- Thus, $S_n$-action on $i$-faces (i-th chain gp) induces rep'n on i-th homology

- But homology of $\Pi_n$ is concentrated in top degree due to EL-shellability of $\Pi_n$

(since shellable $\Rightarrow$ homotopy equivalent to wedge of spheres)
G-Equivariant Enrichment of Goresky-MacPherson Formula

Thm (Sundaram-Welker): Let $A$ be a $G$-arrangement of $C$-linear subspaces in $C^n$ for $G$ a finite subgroup of $GL_n(C)$. Then

$$\widetilde{H}^i(M_A) \cong \bigoplus_G \text{Ind}^G_{\text{Stab}(x) \cap \text{codim}(x) \cdot \mathbb{Z}} \widetilde{H}^i(\mathcal{O}, \mathcal{O}_x)$$

(in our case) = "WH\_i(L_{A_n})"

Note: there are numerous variations, e.g. allowing us also to handle config. spaces in $\mathbb{R}^{2d+1}$.
**Whitney Homology (for Graded Posets)**

\[ WH_i(P) := \text{"i-th Whitney homology of } P \]
\[ = \bigoplus_{\text{rk}(u) = i} H_{i-2}^\infty(\mathcal{G}, u) \]
\[ \text{has } n-i \text{ blocks} \]

\[ WH_\lambda(P) := \bigoplus_{\tau \in \mathcal{P}} H_{\tau}^{\top}(\mathcal{G}, u) \]
\[ \text{type}(u) = \lambda \]

\[ \lambda = (3, 1, 1) = \text{1st of block sizes} \]

123 | 41 | 5  \[\rightarrow\] 423 | 115  \[\leftarrow i = 2\]

123 | 41 | 5  \[\rightarrow\] 132 | 41 | 5  \[\rightarrow\] 231 | 41 | 5  \[\rightarrow\] 123 | 41 | 5

**Aside:** \[ WH_i(P) \cong s_n \beta_{i+1}^{\infty}(P) \oplus s_{i-1}^{\infty}(P) \]
**Thm (H-Reiner):** Let $M_n^d = \text{config. space of } n \text{ distinct pts in } \mathbb{R}^d$. Then $H^i(M_n^2)$ stabilizes sharply at $3i+1$.

More generally, $H^i(M_n^{2d})$ stabilizes sharply for $n \geq 3 \left( \frac{i}{2d-1} + 1 \right)$ and $H^i(M_n^{2d+1})$ stabilizes sharply for $n \geq 3 \left( \frac{i}{2d} \right)$.

**Idea:** Determine stability of $\hat{\mathcal{H}}_i \oplus \hat{\text{Lie}}_i$

**Thm (H-Reiner):** $\langle H^i(M_n^d), S^{(n-1\lambda \lambda, \nu)} \rangle$ vanishes for $1 \nu \leq 2i$ and becomes constant for $n \geq n_0 = \frac{3\nu + 1}{d}$ if $d$ odd and $\nu + 1$ if $d$ even.
Proof Techniques & Results We'll Use

**Thm (Hanlon-Stanley):**  \( \Pi_n \cong \text{sgn} \otimes (\xi_n^* \circ \xi_n) \)

**Thm (Joyal):**  \( \text{lie}_n \cong \xi_n^* \circ \xi_n \)

**Cor:**  \( \Pi_n \cong \text{lie}_n \otimes \text{sgn} \)

**Thm (Krasikiewicz & Weyman):**

\[
\text{lie}_n \cong \bigoplus \mathbb{S}^2(T)
\]

\[
T \text{ symw} \quad \text{with } \text{maj}(T) \equiv 1 \pmod{n}
\]

**Thm (Sundaram):**

\[
\text{ch}(WH_2) = \prod h_{m_j} \left[ \pi_j \right] \prod e_{m_j} \left[ \pi_j \right]
\]

\[
= (h_{m_1}) \left( \prod h_{m_j} \left[ \pi_j \right] \right) \left( \prod e_{m_j} \left[ \pi_j \right] \right)
\]

\[
j \text{ odd, } m_j \geq 1, \quad j > 1
\]

\[
j \text{ even}
\]
**Thm (Sundaram):** \( \text{S}_j \)-rep’n on top homology of \( \Pi_j \)

\[
\text{ch}(W_{H_2}) = \prod_{j \text{ odd}} h_{m_j}^{\Pi_j} \prod_{j \text{ even}} e_{m_j}^{\Pi_j}
\]

\[
= (h_{m_1})(\prod_{j > 1} h_{m_j}^{\Pi_j})(\prod_{j > 1} e_{m_j}^{\Pi_j})
\]

where \( \text{ch} = \text{"Frobenius characteristic" iso m.} \)

\[ \text{ch}(f) = \sum_{\pi} f(\pi) \frac{\mathbb{P}_n}{\mathbb{S}_n} \text{ from } S_n \]

class functions to ring of symmetric fn’s

\[ h_n := \sum x_{i_1} x_{i_2} \cdots x_{i_n} = \text{ch (trivial rep’n)} \]

\[ e_n := \sum x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_n} = \text{ch (sgn rep’n)} \]

**Obs:** \( \Pi_n \) has 1st row upper bd \( n-1 \) for \( n > 2 \)

\[ e_m[\Pi_2] = e_m[\pi_2] \text{ has 1st row upper bd m+1} \]
**Key Fact for Stability:** \( u \in T_n \) of rank \( i \) has at most \( 2i \) letters in nontrivial blocks

**Significance:** Gives upper bound of \( 2i \) on \( |\lambda| \), where sharp stability bound is \( \max 3|\lambda| + 2, 3 \)

\[
\begin{align*}
12|34|56|78 & \quad \text{max \# letters in nontriv. blocks} \\
 12|34|56|718 & \quad 2\text{-rank} = 2i \\
 12|34|5|6|718 & \\
 12|3|4|5|6|7|8 & \quad \lambda = (3, 1, 1, 1, 1, 1) \\
 1|2|3|4|5|6|7|8
\end{align*}
\]
\textbf{Thm (Sundaram)}: \quad S_j\text{-rep'n on top homology of } \Pi_j

\[ \text{ch}(WH_2) = \prod_{j \text{ odd}} h^{m_j}_{\Pi_j} \prod_{j \text{ even}} e^{m_j}_{\Pi_j} \]

\[ = \left( h^{m_1}_{\Pi_1} \right) \left( \prod_{j \text{ odd}} h^{m_j}_{\Pi_j} \right) \left( \prod_{j \text{ even}} e^{m_j}_{\Pi_j} \right) \]

\[ \text{ch(triv}_{\Pi_1}) \]

\[ \text{"Wh}_2\text{" has degree } \leq 2i \text{ by } \star \]

where \( \text{ch} = \text{"Frobenius characteristic" isom.} \)

\[ \text{ch}(f) = \sum_{\pi} f(\pi) \frac{\pi}{\pi} \]

from \( S_n \)

class functions to ring of symmetric fn's

\[ h_n := \sum_{1 \leq i_1 < i_2 < \ldots} x_{i_1} x_{i_2} \ldots x_{i_n} = \text{ch (trivial rep'n)} \]

\[ e_n := \sum_{1 \leq i_1 < i_2 < \ldots} x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_n} = \text{ch (sgn rep'n)} \]

\textbf{Obs:} \( \Pi_n \) has 1st row upper bd \( n-1 \) for \( n > 2 \) \& \( e_m [\Pi_2] = e_m [h_2] \) has 1st row upper bd \( m+1 \)
Key Properties of Symmetric Functions

- $s^\lambda \overset{ch}{\leftrightarrow}$ Schur fn $s^\lambda = \sum x^T$
  
  "Frobenius charact." TSSYT shape $\lambda$

- $\chi^\lambda$ isom. $\chi^\lambda \lambda' \rightarrow \chi^\lambda \chi^\lambda' \rightarrow \chi^\lambda + \chi^\lambda' \rightarrow \chi(1,1,2,2,3,4) \rightarrow x_1^2 x_2^2 x_3 x_4$

$\Rightarrow$ $s^\lambda$ includes monomial divisible by $x_1^3$ but not $x_1^4$.

- Wreath $\langle \ldots \rangle$ plethysm of product symmetric functions of rep's

$\Rightarrow$ $f$ includes $x_1^a$ & $g$ includes $x_1^b$ then $f \cdot g$ includes $x_1^{a+b}$ while $f[\langle g \rangle]$ cannot include $x_1^{(\deg f) b+1}$
Wiltshire-Gordon Conjectures & Related Results

**Defn (Wiltshire-Gordon):**

\[ V_n^k = \bigoplus_{\lambda \vdash n} WH_{\lambda} (\Pi_n) \]

- \( \lambda \vdash n \)
- \( \ell(\lambda) = n-k \)
- \( \lambda \) has no parts of size 1

**Theorem (H-Reiner):**

\[ \text{Ind} \left( \text{Res} \left( V_n^k \bigoplus V_{n-1}^k \right) \right) \cong \text{Res} \left( V_{n+1}^{k+1} \right) \]

(conjectured by Wiltshire-Gordon)

**Example:**

- \( n+1 = 5 \) \( k+1 = 3 \)
- Dimension formula:
  \[ 4 \cdot \left( \binom{4}{2} + (3-1)! \right) = \binom{5}{3} \cdot (3-1)! \cdot (2-1)! = 20 \]
Key Question: Decompose $V_n^k$ into irreducible reps, since this would exactly give the $S^3$ irrep's yielding $S^3$ trivial rep's comprising $k$-th cohomology for config. space of $n$ distinct, labeled pts in $\mathbb{R}^2$.

Progress (Next Theorem): Answer instead for $\bigoplus_{k} V_n^k$.

Open Qu: Analogous results for $\mathbb{R}^d$ for $d>2$?
Thm (H-Reiner):

\[ \nu_n = \text{ch} \left( \bigoplus_{k} V_k \right) = \bigoplus_{T} S^{\lambda(T)} \]

where \( T \) is Whitney generating if either

1. \( T = \emptyset \) or \( 12 \) or \( \begin{array}{c} 12 \\ 13 \end{array} \)

or

2. \( T \in \{4,2,3,43\} \) is one of the four shapes:

\[ T_1 = \begin{array}{c} 12 \\ 13 \\ 41 \end{array} \quad T_2 = \begin{array}{c} 124 \\ 3 \end{array} \]
\[ T_3 = \begin{array}{cccc}
1 & 2 \\
3 & 4 
\end{array} \quad T_4 = \begin{array}{cccc}
1 & 2 & 3 \\
4 
\end{array} \]

with the following further restrictions:

(a) If \( T_3 \), then the first ascent \( R \) with \( R \geq 4 \) is odd

(b) If \( T_4 \), then the first ascent \( R \) with \( R \geq 4 \) is even

ascent := i such that \( i+1 \) in weakly higher row

Idea: Both sides satisfy same recurrence: categorified
\[ d_n = nd_{n-1} + (1)^n \]
\[ \hat{\omega} H_n = \hat{\omega} H_{n-1} + \uparrow^{S_n} + (-1)^n \hat{\omega} \]

for
\[ \hat{\omega} \hat{\omega} = \chi(3,1^{n-3}) - \chi(2,2,1^{n-4}) \text{ for } n \geq 4 \]
Motivations from Number Theory for Repin Stability for PConf($\mathbb{R}^d$)

- Church-Ellenberg-Farb
- Matchett-Wood-Vakil
- Others:

$$\langle H^i(\text{PConf}_n(C)), V \rangle_{S_n} = \lim_{q \to \infty} H^i_{\text{et}}(\text{Conf}_{ni}, \mathbb{Q}_p, \text{coeffs twisted by } V)$$

yielding various counting formulas over finite field via "Grothendieck-Lefschetz formula" counting fixed pts of Frobenius map.

E.g., $\lim_{n \to \infty} (\# D\text{-free degree } n \text{ polys}) = 8^n - 8^{n-1}$

Remarks: Applications to number theory focus on $M = \mathbb{R}^2$ case

- We improve error bound in these limits
Translating "Polynomial Characters" into Symmetric Fns (to get Improved Power Saving Bounds)

- Any polynomial $P(x_1, x_2, x_3, \ldots)$ gives a class fn for $S_n$ by letting $x_i = \# i$-cycles in conjugacy class.

- The elements $(\lambda) = (x_1^{m_1}) (x_2^{m_2}) \ldots$ where $\lambda$ has $m_i$ parts of size $i$ form a basis for $\mathbb{C}[x_1, x_2, x_3, \ldots]$.

**Prop in (H-Reiner):** $\text{ch}(x_P) = \sum \frac{P_\lambda}{z_\lambda} h_{n-1\lambda}$ for $n \geq |\lambda|$

for $P = (\lambda) = (x_1^{m_1}) (x_2^{m_2}) \ldots$
Combining with Earlier Results...

- guarantees for all $P \in \mathbf{Q}[x_1, x_2, \ldots]$, 
  $x_P = M \left( \sum_{\lambda} c_{\lambda} x^\lambda \right)$ s.t. $1M1 \leq \deg(P) \forall \lambda$.

- analyze $\langle x_P, H^i(M^n_{2d}) \rangle$ via:

\[
\text{Thm (H-Reiner): } \langle H^i(M^n_{2d}), S^{(n-1\nu_1, \nu)} \rangle \text{ vanishes for } 1\nu_1 \leq 2i \text{ and becomes constant for } n \geq n_0 = \begin{cases} 1\nu_1 + i & \text{for } d \text{ odd} \\ 1\nu_1 + i + 1 & \text{for } d \text{ even} \end{cases}
\]

\[\text{Upshot: } \langle x_P, H^i(D_{\text{Conf}(C)}) \rangle_{S_n} \text{ is constant for } n \geq \max \{ 2 \deg(P), \deg(P) + i + 1 \}.\]
\textbf{Thm}: \( \langle \beta_s(TT_n), \text{triv} \rangle \) is constant for \( n \geq 2\max(s) - \left( \frac{15s - 1}{2} \right) \).

\textbf{Note}: This follows from partitioning of \( \Delta(TT_n)/S_n \) giving combinatorial interpretation for \( \langle \beta_s(TT_n), \text{triv} \rangle \) (i.e. from 2003 result of H.), our point of entry to this topic.

\textbf{Conjecture (H-Reiner)}: For fixed \( S \subseteq \{1, 2, \ldots, n-2\} \) with \( i = \max(S) \), the rank-selected homology \( \beta_s(TT_n) \) stabilizes sharply at \( n = 4i - 15 + 1 \).