Representation Stability & $S_n$-module Structure in the Partition Lattice

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(joint work with Vic Reiner)
**Abstract**

*Representation Theoretic Stability*

**Definition (Church, Farb):** A series of $S_n$-modules $M_1, M_2, \ldots$ stabilizes at $B > 0$ if for each $n > B$, we have $M_n = \sum c_\lambda V(\lambda)$ where $V(\lambda) \cong S^{(\lambda \vdash n, \lambda)}$ with $\lambda^t + m \leq B$ and where $c_\lambda$ does not depend on $n$.

**Example:**

- $M_n \rightarrow \text{Diagram 1} + \text{Diagram 2}$
- $M_{n+1} \rightarrow \text{Diagram 3} + \text{Diagram 4}$
- $M_{n+2} = \text{Diagram 5}$

**Our Focus:** $S_n$-reps from partition lattice
Partition Lattice $\Pi_n$ & its $S_n$-representations

$\Pi_4$

1234

12|3|4 12|3|4 1|234 12|3|4 124|3 23|1|4 2|134

12|3|4 13|2|4 14|2|3 23|1|4 24|1|3 34|1|2

$S_n$ acts by permuting values

E.g. $(13)(12|3|45) = 321|4|5$
Our Starting Point:

Thm (Church-Farb): \( H^i(M_n) \) stabilizes for \( n \geq 4 \), where \( M_n \) is configuration space of \( n \) distinct points in plane if \( i \) is held fixed.

- \( M_n = \) complement of type A (complex) braid arrangement

\[ A_n = \{ x_i = x_j \mid 1 \leq i < j \leq n \} \]

\[ x_2 = x_3 \]

\[ x_1 = x_2 \]

\[ x_1 = x_3 \]

\( \Pi_n = \) intersection lattice of \( A_n \)
• $H^i(M_n) = i^{th}$ graded piece in Orlik-Solomon algebra of type A braid arr't

• $S_n$-module structure for $H^i(M_n)$ translates (via Goresky-MacPherson formula) to "Whitney homology" in $\Pi_n$.

**Note:** Church, Ellenberg & Farb theory of FI-modules for rep’n stability

• applic’s of rep’n stability in number theory (Matchett-Wood, Vatilk,...)
**Defn:** The order complex (or nerve) of a finite poset $P$ is the simplicial complex $\Delta(P)$ whose $i$-dimensional faces are the $(i+1)$-chains $v_0 < \ldots < v_i$ in $P$.

**E.g.**

$$
\begin{align*}
\mathcal{P} &= \{a_1, a_2, b_1, b_2\} \\
\Delta(P) &= \begin{array}{c}
\hat{a_2} \\
\hat{b_1} \\
\hat{b_2} \\
\hat{a_1} \\
\hat{c} \\
\end{array}
\end{align*}
$$

**Notations:** $u < v$, "$u$ is covered by $v$" in a poset $P$ $\iff$ $u < v$ and $\{u, z, v_3\} = \emptyset$.

- $u_0 < u_1 < \ldots < u_i$ is a saturated chain of $P$ if it is a maximal chain $\iff$ gives rise to maximal face (facet) of $\Delta(P - \{e_0, e_1, \ldots, e_3\})$. 

$S_n$-Representations on Chains (i.e. on Faces) \& on Homology

- $S_n$ action on set partitions is order-preserving \& rank-preserving

- Thus, it induces $S_n$-action $\sigma_\zeta$ on chains $u_1 < u_2 < \ldots < u_j$ with $u_\zeta$ of rank $i_\zeta$ for $1 \leq \zeta \leq j$ and $S=\{i_1, \ldots, i_j\}$, in other words on faces of $\Delta(\Pi_n)$ with vertices colored $S$, where vertices in $\Delta(\Pi_n)$ colored by poset rank.
\( S_n \)-action on chains commutes with simplicial boundary map

\[
d (u_0<\ldots<u_r) = \sum_{i=0}^{r} (u_0<\ldots<\hat{u}_i<\ldots<u_r)
\]

Thus, \( S_n \)-action on i-faces induces \( S_n \)-rep\'in on i-th homology.

But homology of \( \tilde{\Pi}_n \) is concentrated in top degree due to shellability.

Likewise, homology of \( \tilde{\Pi}_n^S = \{ u \in \tilde{\Pi}_n | \text{rk}(u)^{\mathbb{S}^3} \} \) also concentrated in top degree.
The virtual rep'$ \beta_S := \frac{\sum_{T \subseteq S} (-1)^{|S-T|}}{15}$ is actual $S_n$-rep on top homology of $\pi_n^S := \{ u \in \pi_n | vh(u) \in S^n \}$ (since lower homology vanishes in $\pi_n^S$)

Note: $\pi_n$ is "shellable", implying:

- $\Delta(\pi_n \setminus \hat{\emptyset}, \hat{1^3})$ is homotopy equivalent to a wedge of $(n-1)!$ top-diml spheres

- $\pi_n^S$ is also shellable
Rank-Selected Homology

Whitney Homology

\[ \beta_S := \text{"rank-selected homology" for rank set } S \]

\[ WH_i(P) := \text{"i-th Whitney homology of } P\]
\[ = \bigoplus_{\text{rank}(u) = i} H_i(\mathcal{O}, u) = \bigoplus_{\text{has } n-i \text{ blocks}} WH_n(P) \]

\[ WH_i(P) := \bigoplus_{u \in P} H_i(\mathcal{O}, u) \quad \text{type}(u) = \lambda \]

**Thm (Sundaram):** \( WH_i(P) = \beta_{i-\lambda}(P) + \beta_{i+\lambda}(P) \)

**Observation:** This implies \( WH_i \) stabilizes at same bound as \( \beta_{i-\lambda} \)

- \( \beta_S \) is subrepresentation of \( \alpha_S \) and will stabilize at least as fast as \( \alpha_S \)
Goresky-MacPherson Formula
(for cohomology of subspace arr't)
\[ \tilde{H}^i(M_A) \cong \bigoplus_{x \in L_A^{20}} \tilde{H}^{\text{codim}(x)-2-i}(\partial, x) \]
subspace arr't complement \( \leftarrow \) as groups \( \leftrightarrow \) intersection lattice
(OS-Algebra = presentation of cohomology ring for complex hyperplane arr't complement)

\( S_n \)-Equivariant Enrichment
(Sundaram & Welker) as it specializes to complex hyperplane arrangements:
\[ \tilde{H}^i(M_A) \cong \bigoplus_{x \in (L_A^{>0})} \text{Ind}^G_{\text{Stab}(x)} \tilde{H}^{\text{codim}(x)-2}(\partial, x) \]
\[ = \omega H; (L_{A_n}) = \omega H; (\Pi_n) \]
Upshot for Stability:

- $\beta_{i,j} (\Pi_n)$ stabilizes at $B>0$
  $\iff$ $WH_i (\Pi_n)$ stabilizes at $B>0$
  $\iff$ $H^i (M_n)$ stabilizes at $B>0$
Past Results on $\Pi_n$:

Thm (Hanlon-Stanley): $\Pi_n \cong \text{sgn}(\delta, i) \cdot \binom{i}{n}$

Method: Calculate $\mu_{\Pi_n}(\delta, i) = \chi_{\Pi_n}^S$.

Thm (Joyal): $\text{lie}_n \cong \mathfrak{S}_n^i \mathfrak{c}_n$

Thm (Barcelo): Explicit $S_n$-equivariant bijection yielding $\Pi_n \cong \text{sgn} \circ \text{lie}_n$

Thm (Krasikiewicz-Weyman):

$$\text{lie}_n \cong \bigoplus S^{\lambda(t)}$$

$T S^{t \gamma}$

with $t \gamma(t) \equiv 1 \pmod{n}$

* Key Fact for Stability: $u \in \Pi_n$ of rank $i$ has at most $2i$ letters in nontrivial blocks
Thm (Sundaram): \( S_j \)-repn on top homology of \( \Pi_j \)

\[
\text{ch}(WH^2) = \Pi \sum_{\substack{\text{odd} \, m_j \, \text{part} \, \text{even} \, \Pi_j}} \Pi \sum_{\text{even} \, m_j} \Pi_j
\]

\[
= (h_{m_1}) (\Pi \sum_{\text{odd} \, m_j} \Pi_j) (\Pi \sum_{\text{even} \, m_j} \Pi_j)
\]

"\( \hat{\text{WH}}^2 \)" has degree \( \leq 2i \) by *

where \( \text{ch} = "\text{Frobenius characteristic}" \)

isom from ring of \( S_j \)-repn's to ring of symmetric functions

\( h_n := \text{complete symmetric fn} \)

\[
= \sum x_1 x_2 \cdots x_n = \text{ch}(\text{triv})
\]

\( 1 \leq i_1 \leq i_2 \leq \cdots \)

\( e_n := \text{elementary symmetric fn} \)

\[
= \sum x_{i_1} x_{i_2} x_{i_3} \cdots x_i = \text{ch}(\text{sgn})
\]

\( 1 < i_1 < i_2 < \cdots \)
Facts about Symmetric Functions

- ring of symmetric fn's ≡ ring of $S_n$-repn's with $\rho_1 \otimes \rho_2 \uparrow_{S_m \times S_n}^{S_{m+n}}$ as multiplication

- $S^\lambda \mapsto$ schur fn $s_\lambda = \sum_{T \text{ SSYT}} x^T$
  \text{shape} $\lambda$

for $SSYT =$ semi-standard Young tableaux

\[
T = \begin{array}{c|c|c|c}
1 & 1 & 2 & 2 \\
\hline
3 & 4 & & \\
\end{array}
\]

$\mapsto x_1^2 x_2^2 x_3 x_4$

- wreath $\mapsto$ plethysm of product of symmetric fn's of repn's
**Thm (H-Reiner):** Holding i fixed let $n$ grow, $\beta_{i-i}(\Pi_n), WH_i(\Pi_n)$ stabilize as $S_n$-rep's at $n=3i+1$.

**Idea:** $\hat{WH}_i = \bigotimes_{j=n} WH_j(\Pi_n)$ stabilizes at $n=2i$ since at most $2i$ letters in nontrivial blocks. Obtain upper bound of $i+1$ on length of 1st row in $S^i$ in $\hat{WH}_i$. Pieri Rule lower bd on $j$ s.t. multiplying $\hat{WH}_i = \bigotimes_{j} S^j$ by $h_j$ stabilizes.

**Pieri Rule:**

$$h_n S_\lambda = \sum_{\lambda \vdash \lambda} S_{\lambda}$$

$$= \lambda$$
Thm (H-Reiner): $\beta_3(Tn_n)$ for any fixed $S$ stabilizes for $n \geq 4\max S$.

Idea: Show $\chi_h(x_5(Tn_n))$ has upper bound of $2\max S$ on length of 1st row in "h-free" part of symmetric fn $f$ (analogue of $\hat{WH}_i(n)$).

- This gives stability bound $n \geq 4\max S$ for $x_5(n)$.

- Deduce same bound for $\beta_5(n)$ using that $\beta_5(n)$ is subrep'n of $x_5(n)$.
Stability for the Multiplicity "b₅(n)" of Trivial Repn in β₅(n)

Thm (H-Reiner): b₅(n) = <1, β₅(Πₖₙ)) stabilizes at n ≥ 2max S - (151-1)/2

Partitioning: Δ is partititnable if face poset F(Δ) decomposes into disjoint union of boolean algebras w/ facets as top elements

e.g.,
• $\Delta$ is **pure** of dimension $d$ if all maximal faces are $d$-dim'l.

• Such $\Delta$ is **balanced** if vertices can be colored w/d+1 colors s.t. no two vertices in a face have same color.

• If $\Delta$ is balanced & partitionable, then $h_S(\Delta) = \#$ facets with restriction face (in min'l face in Boolean algebra) colored $S$.

**Key Idea:**

$$f_S(\Delta(\Pi_n)/S_n) = \# S_n\text{-orbits of faces with color set } S$$

$$= \langle 1, \alpha_S(\Pi_n) \rangle$$
\[ \therefore h_S(\Delta(\Pi_n)/\hat{S}_n) = \sum_{T \leq S} (-1)^{|S-T|} <1, a_T(\Pi_n)> \]
\[ = \sum_{T \leq S} (-1)^{|S-T|} \beta_S(\Pi_n) \]

**Thm (H., 2003):** $\Delta(\Pi_n)/\hat{S}_n$ is partitionable, giving combinatorial interpretation for
\[ h_S(\Delta(\Pi_n)/\hat{S}_n) = <1, \beta_S(\Pi_n)> \]
as # saturated chain orbits with "topological descent set" $S$. 
Thm (H.-Reiner): \( \langle 1, \beta_5(\Pi_n) \rangle \) stabilizes for \( n \geq 2 \max S - \left( \frac{15(1)}{2} \right) \)

Idea: Use partitioning for \( \Delta(\Pi_n) / S_n \)
and consequent combinatorial interpretation for \( \langle 1, \beta_5(\Pi_n) \rangle \).

- Injection \( \varphi_n : \{ \text{facets} \} \) contrib. to \( b_{5}(n) \) \( \rightarrow \) \( \{ \text{facets} \} \) contrib. to \( b_{5}(n+1) \)
  eventually also a surjection.

Rk: This is sharp for \( S = \{ 5 \} \)
but not for every single choice of \( S \).

\( \text{(Hanlon)}: \langle 1, \beta_{\{1, \ldots, 5\}}(\Pi_n) \rangle = 0 \quad \text{for } n > 2. \)
Wiltshire-Gordon Conjectures & Related Results

Defn (Wiltshire-Gordon):
\[ V_n^k = \bigoplus \text{WH}_\lambda^n(Tn) \]
- \(|\lambda|=n\)
- \(\ell(\lambda)=n-k\)
- \(\lambda\) has no parts of size 1

Thm (H-Reiner):
Ind (Res (V_n^k) \oplus V_{n-1}^k) \cong Res (V_{n+1}^{k+1})

(conjectured by Wiltshire-Gordon)

Idea: Symmetric funts generating funs
Thm (H-Reiner):

\[ v_n = \text{ch} \left( \bigoplus V_n^k \right) \cong \bigoplus S^\lambda(T) \]

where \( T \) is Whitney generating if

- 1 and 2 both appear in 1st row
- if 3 in 1st row, then 1st "ascent" \( \text{rk} \geq 2 \) is even (or there is no ascent if \( n \) is even)
- if 3 \( \not\equiv 4 \) in 2nd row, then 1st ascent odd (or none exists if \( n \) is odd)

**ascent**: \( i \) such that \( i+1 \) in higher row

**Idea**: Both sides satisfy same recurrence.
**Conjecture (H-Reiner):** The multiplicity of $S^{(n-k, k)}$ within $H^2(M_n)$ stabilizes at $n^2 i + k + 1$.

**Intermediate Question:** (work in progress)

Find basis for $a_T b_T V$'s

$\neq a_T b_T V$'s

Young symmetrizer

For $a_T = \sum_{\sigma \in \mathcal{T}} a_T$; $b_T = \sum_{\tau \in \mathcal{T}} \text{sgn}(\tau) b_T$

(Since $\langle S^\lambda, V \rangle = \text{dim}(a_T b_T V)$ for $T$ of shape $\lambda$ )
Some Further Questions

1. (Farb) How fast does the multiplicity of any particular $V(\lambda)$ stabilize within $M_n$?

2. (H-Reiner) How fast does the multiplicity of $V(\lambda)$ stabilize within $\beta_S(\Pi_n)$ as $S$ is held fixed and $n$ grows?

(Note: Qn 1 is special case of Qn 2 with $S = \{1,2,\ldots,i\}^3$)

3. (Farb) What rep\'s do we get after stabilization occurs?