Poset Topology Meets Combinatorial Representation Theory

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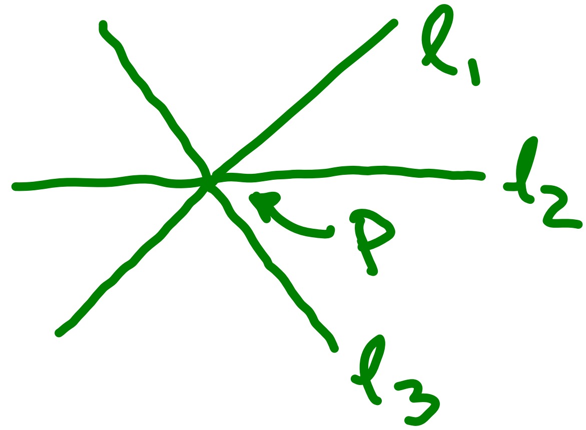
Counting Topologically

e.g. "counting" points in the $\mathbb{R}^2$
complement of $\rightsquigarrow$
yields:



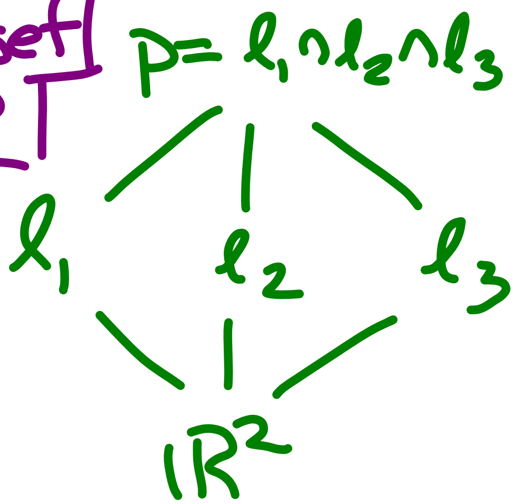
$$\mathbb{R}^2 - \ell_1 - \ell_2 - \ell_3 + 2p$$

- Coefficients $1, -1, -1, -1, 2$ in such inclusion-exclusion counting formula given by "Möbius function" μ

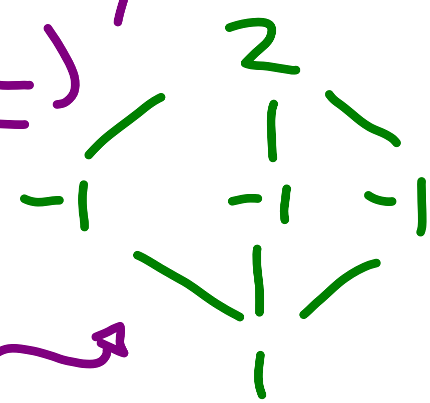
Defn: Möbius function $\mu_P(x,y)$ of partially ordered set (poset) P is defined recursively: $\mu_P(x,x)=1$

and $\mu_P(x,y) = -\sum_{x \le z < y} \mu_P(x,z)$ (for $x \neq y$) (so $\sum_{x \le z \le y} \mu_P(x,z) = 0$)

Poset P



Its Möbius fn $\mu_P(\mathbb{R}^2, -)$



coefficients in $\mathbb{R}^2 - \ell_1 - \ell_2 - \ell_3 + 2P$

Working over $\mathbb{F}_q$: #pts $= q^2 - q - q - q + 2$

$$\sum_{u \in L_A} M(\hat{0}, u)\, q^{\dim V - rk(u)} =: $$ characteristic poly. of the arrangement

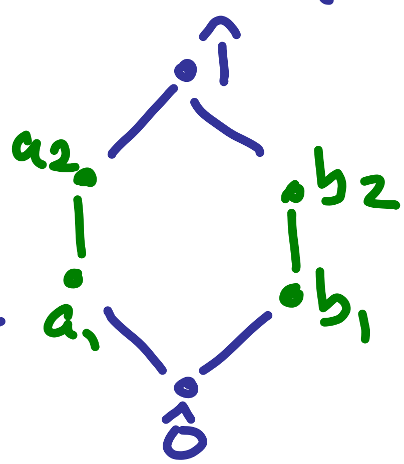
e.g. chromatic poly given by hyperplanes $x_i = x_j \subseteq \mathbb{R}^n$ for each edge e_{ij} in graph G

Recent applications: number theory (Church-Ellenberg-Farb; Matchett-Wood-Vakil)

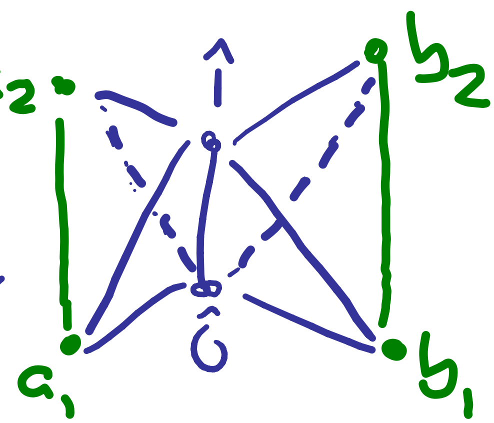
Defn: The **order complex** (or **nerve**) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-dim'l faces are the $(i+1)$-chains $v_0 < v_1 < \dots < v_i$ in P.

e.g.

$P =$



$\Delta(P) =$



Thm (Hall; Popularized by Rota):

$$\mu_P(u,v) = \tilde{\chi}(\Delta(\underbrace{(u,v)}_{\substack{\| \\ \text{subposet } \{z \in P \mid u < z < v\}}}))$$

Some Techniques in Poset Topology

- Quillen fiber lemma

  Use $f: P \to Q$ to show $\Delta(P) \simeq \Delta(Q)$

- (Lexicographic) shellability

  (Björner & Wachs)

  $\Rightarrow \Delta(P) \simeq$ wedge of spheres

- Lexic. discrete Morse fn's

  (Babson–H. ~2001)

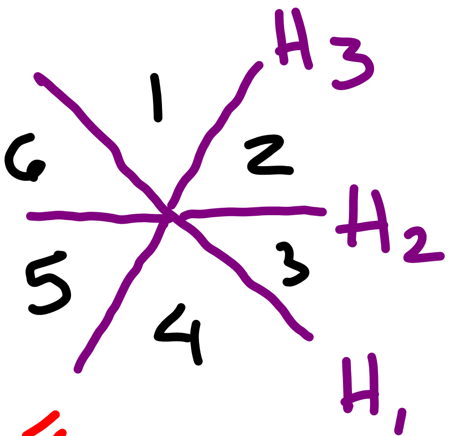
  - Betti # bds, etc. more generally

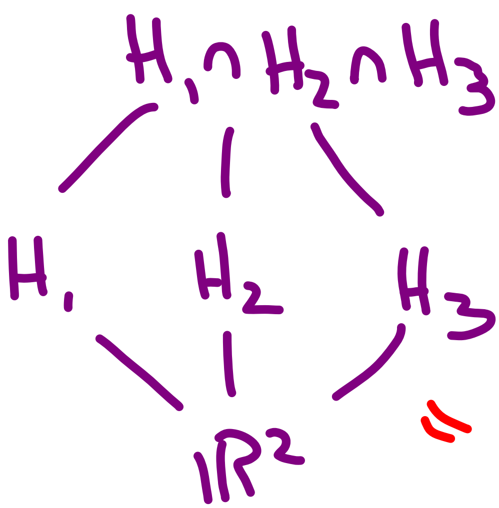
Theorem (Zaslavsky):

$$\#\text{regions} = \sum_{u \in L_A} |\mu(\vec{0}, u)|$$

$$\#\text{bdd regions} = \Big|\sum_{u \in L_A} \mu(\vec{0}, u)\Big|$$

$A =$





$= L_A =$ "intersection poset"

e.g. $\#\text{regions} = 1+3+2$

$\#\text{bdd regions} = 1-3+2$

$\mu(\mathbb{R}^2, \mathbb{R}^2) = 1$

$\mu(\mathbb{R}^2, H_i) = -1$ for $i = 1, 2, 3$

$\mu(\mathbb{R}^2, H_1 \cap H_2 \cap H_3) = 2$

Goresky-MacPherson formula

$$\tilde{H}^i(M_A) \cong \bigoplus_{x \in L_A^{>0}} \tilde{H}_{\mathrm{codim}(x)-2-i}(\hat{0}, x)$$

subspace arr't complement

as groups

intersection semi-lattice

Pf: Stratified Morse theory

Thm (Björner): Intersection posets of central hyperplane arrangements are "shellable", giving formula for M in terms of matroid theory

μ as Topological Shadow

- A graded poset with $\hat{0} \neq \hat{1}$ is **Eulerian** if $\mu(u,v) = (-1)^{rk(v)-rk(u)}$ for all $u \leq v$.

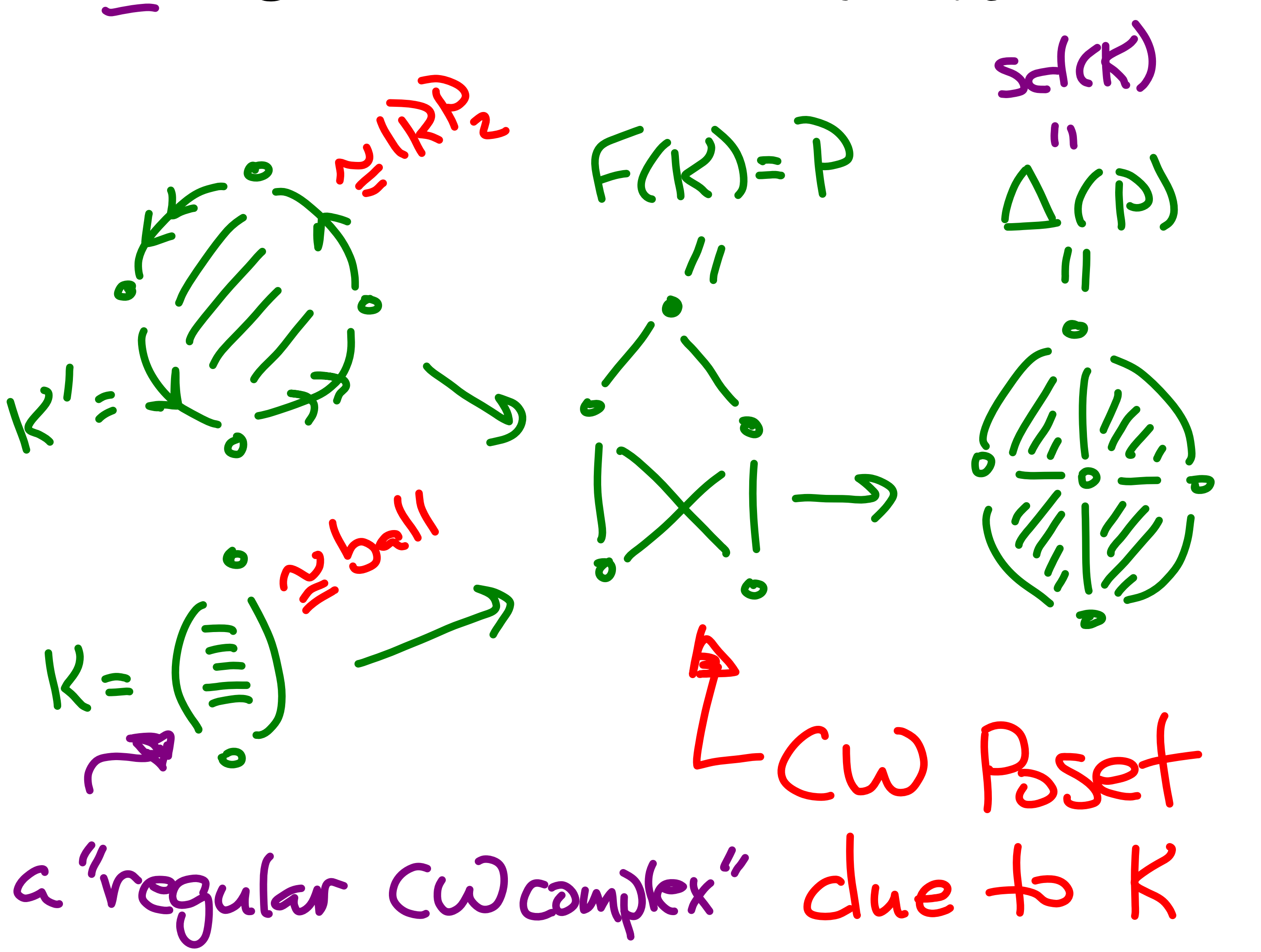
- A graded poset P is a **CW poset** if
  (1) $\hat{0} \in P$
  (2) P has at least one other element
  (3) $\Delta(\hat{0},u) \cong S^{rk(u)-2}$ for $u \neq \hat{0}$

homeomorphic

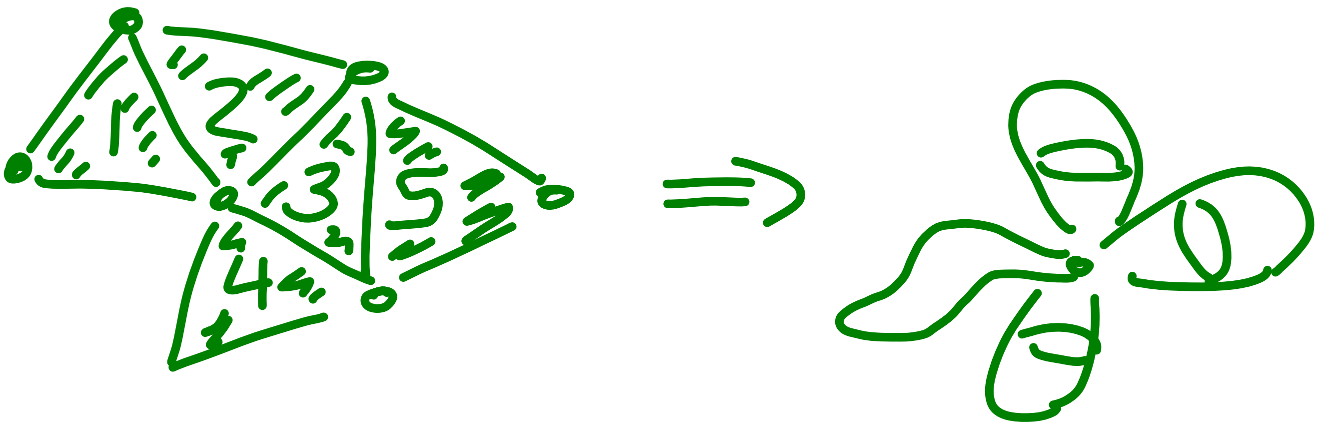
e.g.

Thm (Björner): P is CW poset $\Leftrightarrow$ there exists "regular" CW complex K with P as poset of closure rel'ns, which implies $\Delta(P) = sd(K) \cong K$.

Cor: CW Poset $\Rightarrow$ Eulerian



Thm (Danaraj-Klee): P graded, thin & shellable $\Rightarrow$ P is CW poset
(Victor)



e.g. Bruhat order is CL-shellable CW-poset (proof by Björner-Wachs)

Today: posets with $\mu = 0, \pm 1$

For methods to study homeom. type of K via poset topology of $F(K)$ + topol. data about codim. one incidences, see:

P. H., "Regular CW complexes in total positivity", Invent. Math., 197 (2014), no. 1, 57–114.

"Explains" Bruhat order as CW poset

(Strong) Bruhat Order

Closure poset $F(K)$ for Schubert cell decomposition K of flag variety $\mathcal{F}\ell_n = GL_n/B$; "Schubert varieties" (over $\mathbb{C}$), namely for cell closures

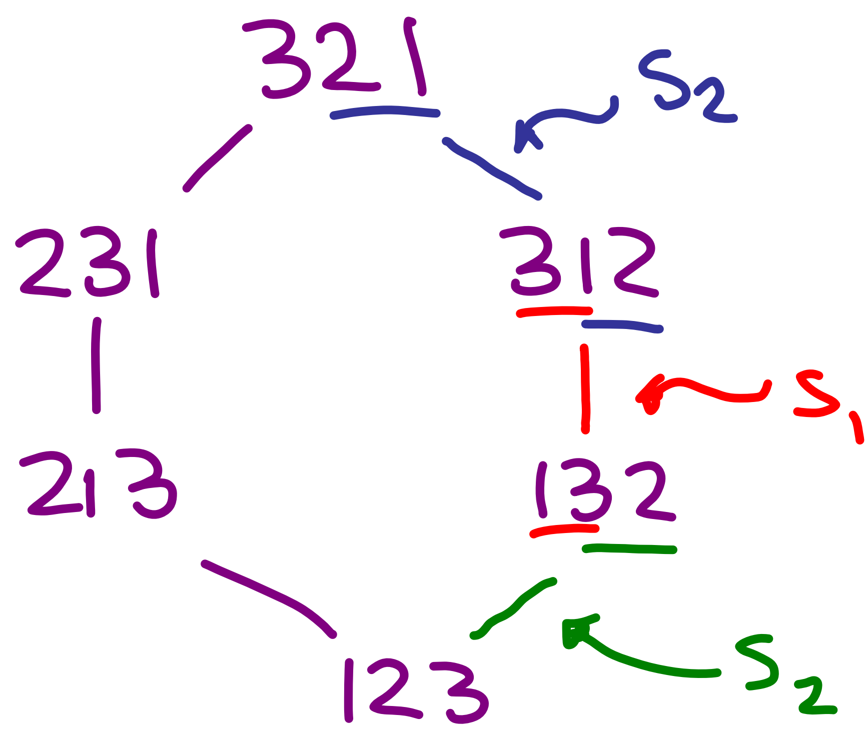
Likewise for G/B in other types.

Remark: Studied e.g. by Billey; not a regular CW complex.

Weak Bruhat Order: Another Partial Order on Permutations

$u \lessdot v$ iff u obtained from v by adjacent transposition $s_i = (i, i+1)$ sorting pair of letters in positions i & $i+1$

e.g.



General Defn: weak order on Coxeter group W is partial order with $u \lessdot v \iff v = s_i u$ for $u, v \in W$ s.t. $\text{length}(v) > \text{length}(u)$ where $\text{length}(u) := \min\{r \mid u = s_{i_1} \cdots s_{i_r}\}$

e.g. $W = S_n$ with relations:

$$s_i^2 = e \;\;\&\;\; \underbrace{s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \;\;\&\;\; s_i s_j = s_j s_i \;\; (\text{for } |j-i|>1)}_{\text{"braid reln's"}}$$

(Left) Weak Bruhat Order for S_3

$$\begin{tikzcd}
& 321 = s_1 s_2 s_1 = s_2 s_1 s_2 & \\
231 = s_2 s_1 \arrow[ur, dash, "s_1 \cdot -"] & & s_1 s_2 = 312 \arrow[ul, dash, "s_2 \cdot -"'] \\
213 = s_1 \arrow[u, dash, "s_2 \cdot -"] & & s_2 = 132 \arrow[u, dash, "s_1 \cdot -"'] \\
& 123 = e \arrow[ul, dash, "s_1 \cdot -"] \arrow[ur, dash, "s_2 \cdot -"'] &
\end{tikzcd}$$

(also is Cayley graph)

"saturated chains" from e to ω $\longleftrightarrow$ "reduced expressions" for ω

Connectedness under Braid Moves

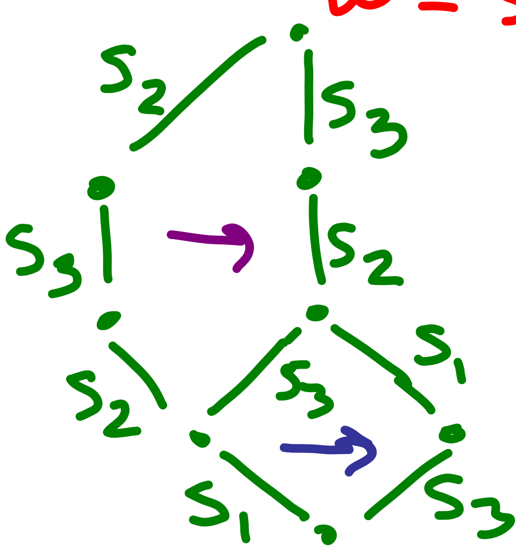
Thm (see e.g. Björner–Brenti book): Let (W,S) be Coxeter system & let $w \in W$. Then every two reduced expressions for w are connected via braid moves.

e.g. $\underbrace{s_2 s_3 s_2} s_1 \to \underbrace{s_3 s_2 s_3} \overbrace{s_3 s_1}$...

$$s_2 s_3 s_2 s_1 \to s_3 s_2 s_3 s_1 \to s_3 s_2 s_1 s_3$$

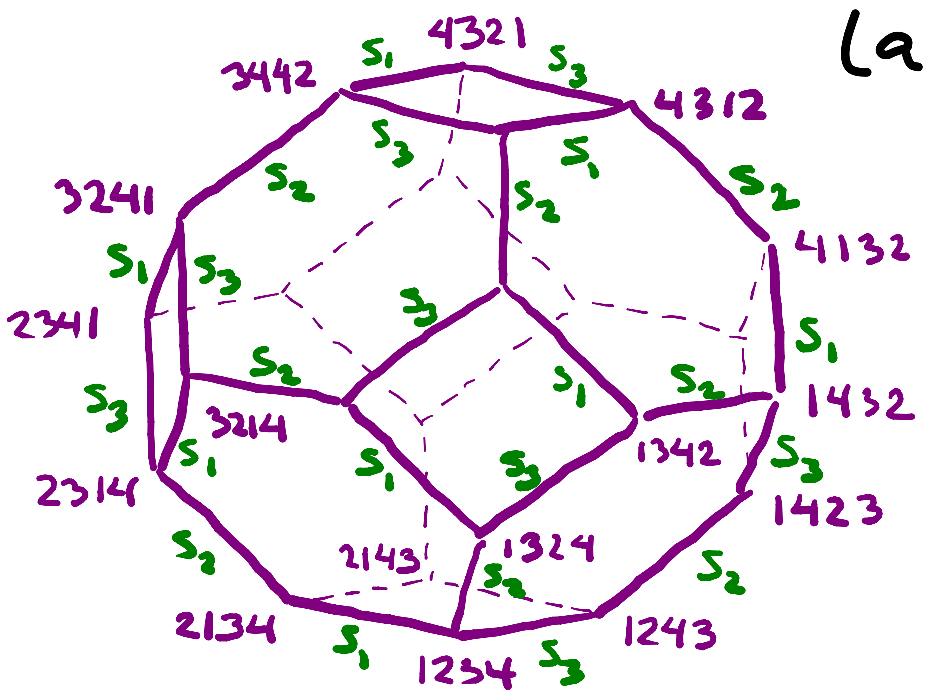
$$w = s_2 s_3 s_2 s_1 = s_3 s_2 s_3 s_1 = s_3 s_2 s_1 s_3$$

left weak order:



Thm (Edelman & Björner): Weak Bruhat order has $\Delta(u,v) \simeq$ ball or sphere, hence $\mu(u,v) = 0, \pm 1$ for all $u \le v$.

Idea: Use Quillen fiber lemma (a.k.a. Quillen Theorem A)



A topological-comb'l tool:

Quillen Fiber Lemma: Given a poset map $f: P \to Q$ s.t. $q \in Q \Rightarrow \Delta(\{p \in P \mid f(p) \leq q\})$ is contractible, then $\Delta(P) \simeq \Delta(Q)$.

Remark: Used extensively e.g. in finite group theory & in combinatorics

Crystal Graphs (as Posets)

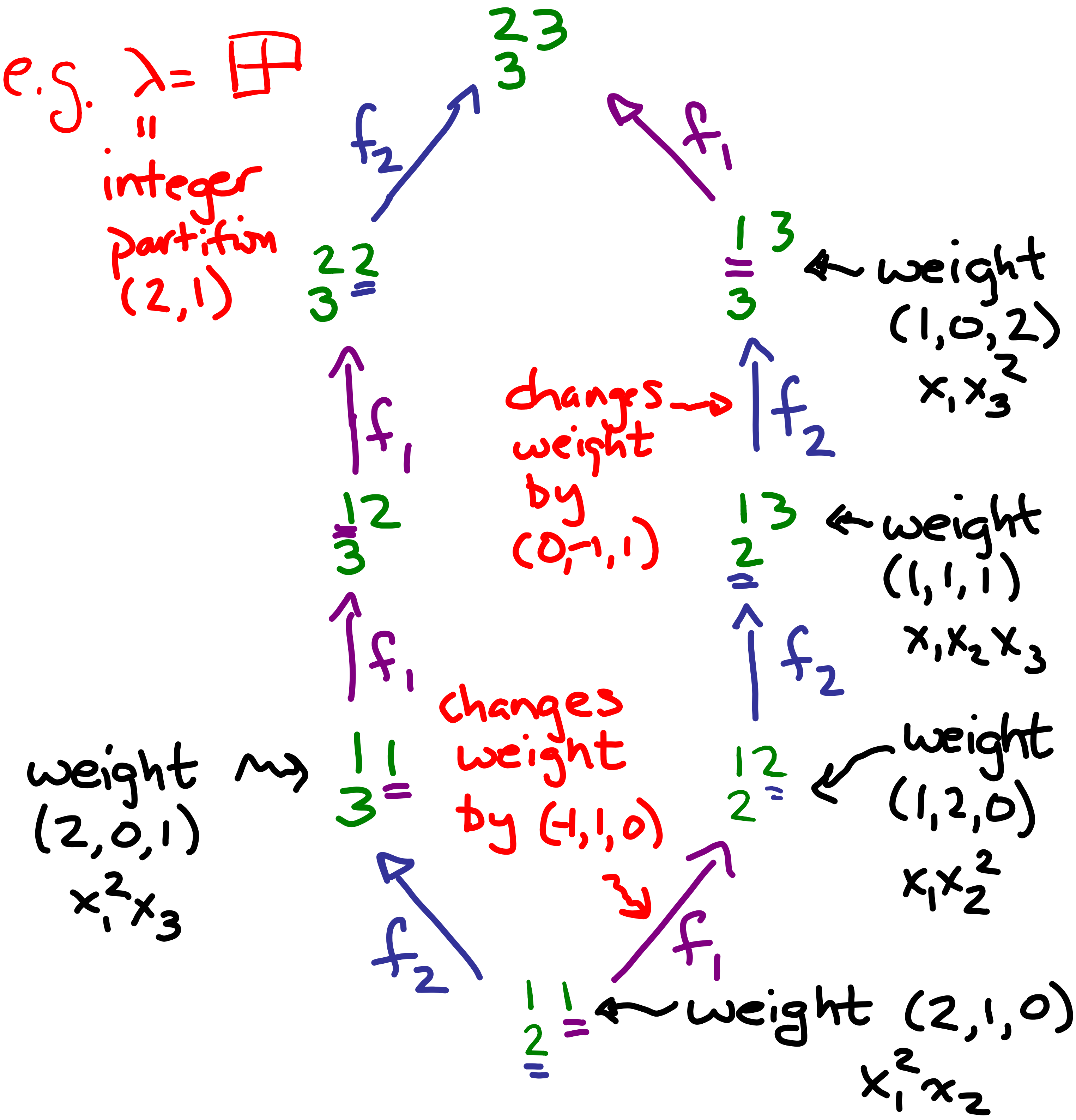
- poset elts $\leftrightsquigarrow$ basis vectors for the various weight spaces

  (guaranteed to exist by properties of Kashiwara's "crystal basis")

- cover rel'ns $\leftrightsquigarrow$ crystal (lowering) f_i operators

- weights of irreducible $\mathfrak{sl}_n$-crystals are schur fns

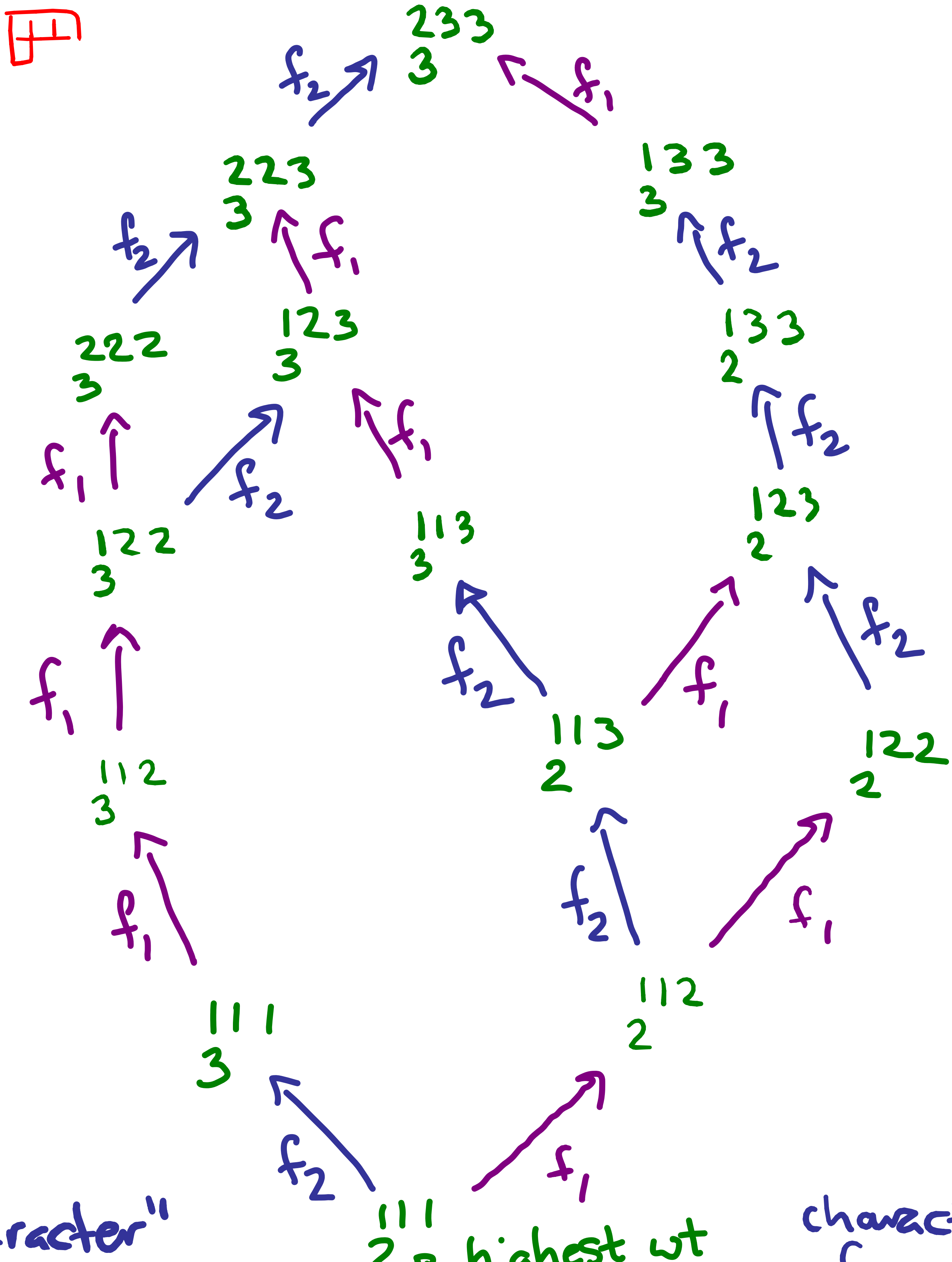
(Type A) Crystals of Highest Weight Rep'ns & their Kashiwara Lowering Operators



Purpose of Crystal Graphs

- Study rep'n theory of Kac-Moody algebras (e.g. affine Lie algebras) A by passing to univ. env. alg. $\mathcal{U}(A)$ & quantized algebra w/ parameter q
- Crystals arising from highest weight rep'ns are posets.

$\lambda=$



highest wt vector $(3,1,0)$

"character" of crystal

character of rep'n

$$= x_1^3x_2 + x_1^2x_2^2 + \dots = \mathrm{weight}\begin{pmatrix}1\,1\,1\\2\end{pmatrix} + \mathrm{weight}\begin{pmatrix}1\,1\,2\\2\end{pmatrix} + \dots$$

Type A crystal for highest weight rep'n of shape λ

1. $\hat{0} = \begin{matrix} 1 1 1 \ldots 1 \\ 2 2 \ldots 2 \\ 3 3 \ldots \\ \vdots \end{matrix}$ of shape λ

"highest weight vector"

2. $u \xrightarrow[i]{} v$ has v obtained from u by incrementing to $i+1$ rightmost i not in "parenthesization pair" with an $i+1$

e.g.

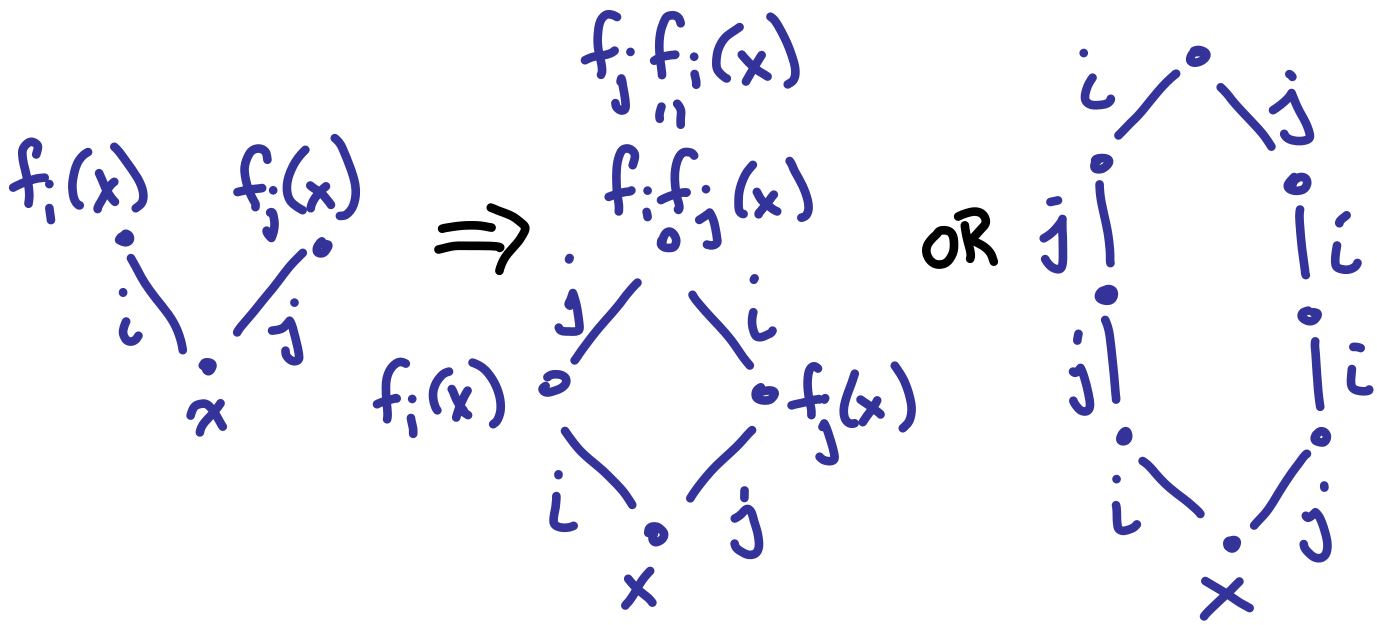
$$\begin{array}{l}11111444\\2233\\\boxed{3}44\end{array} \quad \overset{f_3}{\rightsquigarrow} \quad \begin{array}{l}11111444\\2233\\\boxed{4}44\end{array}$$

$$\boxed{3}4433444 \rightsquigarrow \boxed{4}4433444$$

Parenthesization Pairs: Read leftmost column bottom to top, then subsequent columns, ignoring all but i's & $i+1$'s; pair up consec. $i+1, i$; delete; repeat...

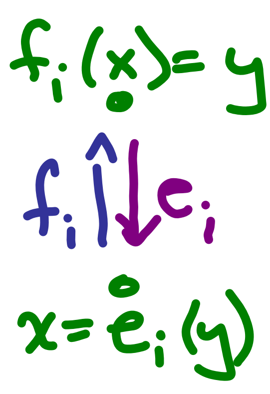
Stembridge: "$\mathfrak{g}$-crystals"

(Crystals of highest weight rep'ns in simply laced case)

a) 

b) likewise for e_i, e_j

"raising operators":



c) reln's depend on location

A Motivation

Given quantized enveloping algebra $U = U^- \otimes_{\mathbb{Q}(v)} U^0 \otimes_{\mathbb{Q}(v)} U^+$, the canonical basis (or crystal basis) B has the remarkable property that each highest weight module V_λ has a basis $\{v_\lambda b \mid v_\lambda b \neq 0\}$, i.e. the elements of crystal poset.

Transferring Structure from Weak Order to Crystals via Poset Map Called "Key"

- related to key polynomials of Lascoux & Schutzenberger
- Schubert poly's are positive sum of key poly's

Right key "k" of a "KM"-crystal

$$k\begin{pmatrix} 1\,2\,4 \\ 2\,3 \end{pmatrix} = s_3 s_2 s_1 s_2$$

1   3

$$k\begin{pmatrix} 1\,1\,4 \\ 2\,3 \end{pmatrix} = s_3 s_2 s_1 \qquad k\begin{pmatrix} 1\,2\,3 \\ 2\,3 \end{pmatrix} = s_2 s_1 s_2 = s_1 s_2 s_1$$

2   3   1   2

$$\begin{matrix} 1\,1\,4 \\ 2\,2 \end{matrix} \qquad k\begin{pmatrix} 1\,1\,3 \\ 2\,3 \end{pmatrix} = s_2 s_1 \qquad k\begin{pmatrix} 1\,2\,2 \\ 2\,3 \end{pmatrix} = s_1 s_2$$

3   2   1

$$k = s_3 s_2 s_1 \qquad k\begin{pmatrix} 1\,1\,3 \\ 2\,2 \end{pmatrix} = s_2 s_1 \qquad k\begin{pmatrix} 1\,1\,2 \\ 2\,3 \end{pmatrix} = s_1 s_2$$

2   1

$$k\begin{pmatrix} 1\,1\,2 \\ 2\,2 \end{pmatrix} = s_1 \qquad k\begin{pmatrix} 1\,1\,1 \\ 2\,3 \end{pmatrix} = s_2$$

1   2

$$\begin{matrix} 1\,1\,1 \\ 2\,2 \end{matrix}$$

k: crystal poset $\to$ weak Bruhat order

key $k(\hat{0}) = e$

$$u \le v \Rightarrow k(u) \le k(v)$$

New Algorithm to Calculate Right Key of a KM-Crystal

(1) $key(\hat{0}) = e$

(2) if $\hat{0} \xrightarrow[i]{} a$, then $key(a) = s_i$

(i.e. $\hat{0} <\cdot a$)

(3) if v covers 2 or more elements

then $key(v) = \bigvee_{\{u \mid u \to v\}} key(u)$

(for join taken in weak order)

(4) if $u \xrightarrow[i]{} v$ and v does not cover any other elements, then:

(a) $key(v) = key(u)$ if $\exists\, u' \xrightarrow[i]{} u$

(b) $key(v) = s_i \cdot key(u)$ otherwise

Thm (H.–Lenart): For $\hat{0}$ = highest weight vector in symmet. KM-crystal, $\Delta(\hat{0},u) \simeq$ ball unless $u = \min(R^{-1}(w_0(W_J))$ where $\Delta(\hat{0},u) \simeq S^{|J|-2}$.

Proof: Use Quillen fibre lemma with

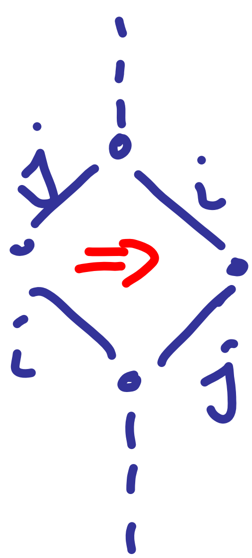
$$f : \text{crystal} \longrightarrow \text{Boolean algebra} \left(\begin{array}{l}\text{poset of} \\ \text{subsets} \\ \text{of } \{1,\dots,n\}\end{array}\right)$$

$$x \longmapsto \max\{S \mid w_0(J_S) \leq_{weak} \text{key}(x)\}$$

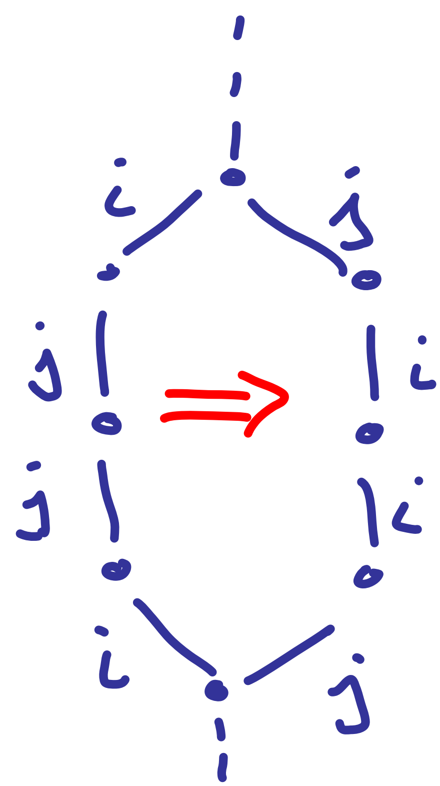
Corollary: $\mu(\hat{0},u) = 0, \pm 1$

highest weight vector

Thm (H.–Lenart): Given any lower interval $(\hat{0}, u)$ in a $\mathfrak{g}$-crystal, then set of saturated chains from $\hat{0}$ to u is connected by "Stembridge moves", namely moves of the form



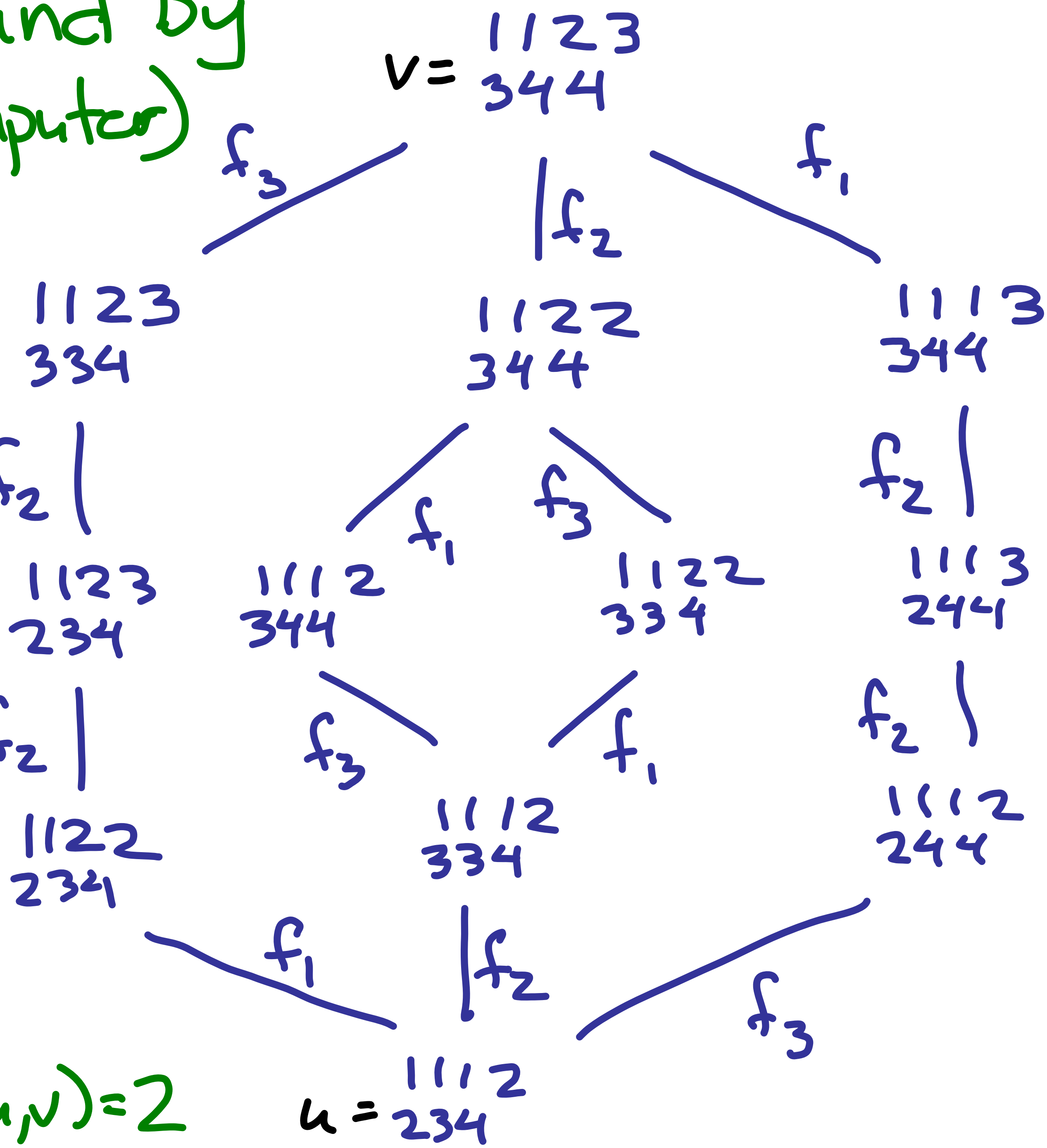
and



Note: Likewise in doubly-laced case via "Sternberg moves".

H.-Lenart: New Reln's

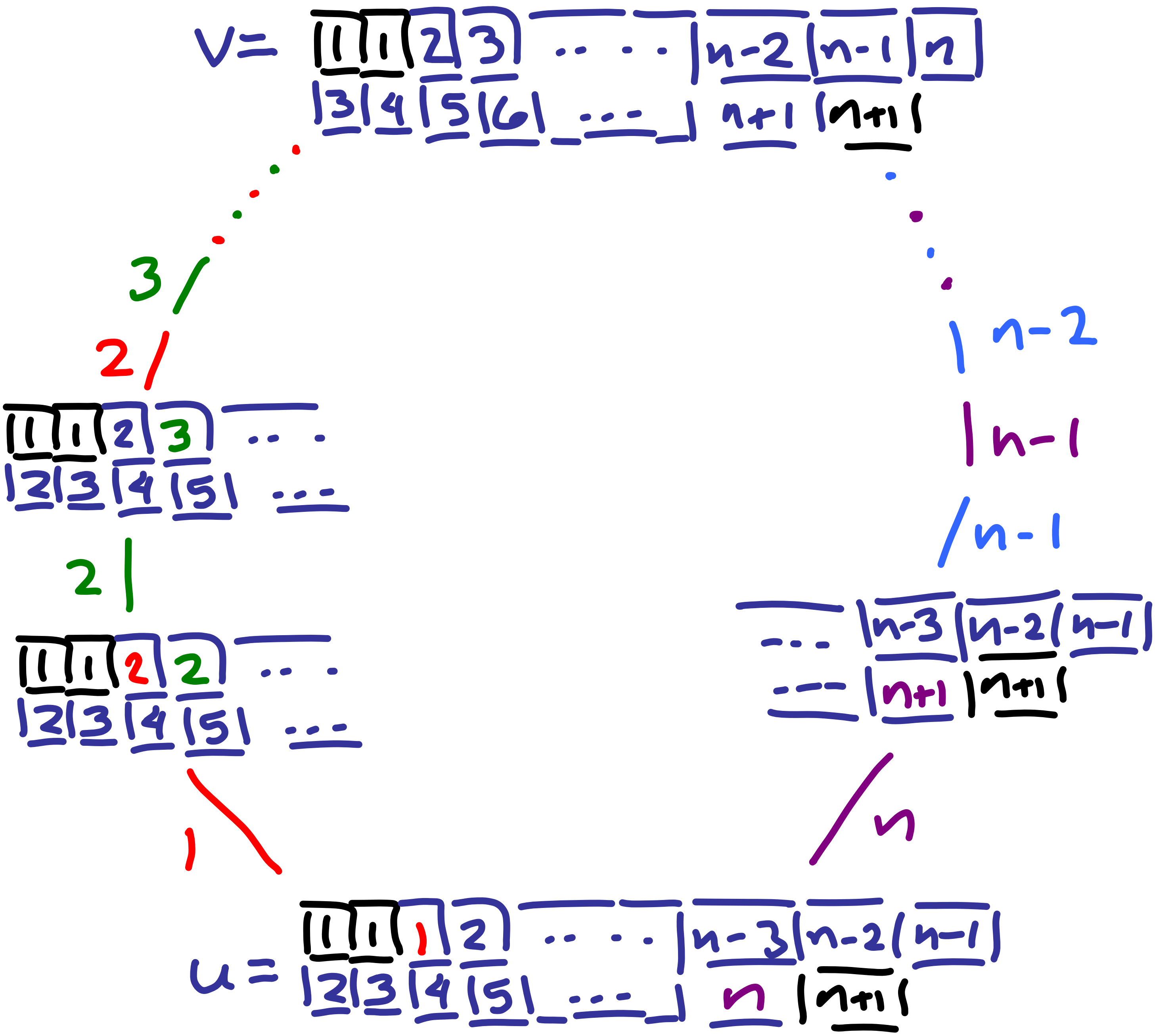
(found by computer)



$\mu_P(u,v)=2$

& not connected by "Stembridge moves"

Arbitrarily High Rank Disconnected Open Intervals



label sequences: 1, 2, 2, 3, 3, 4, 4, ..., n-1, n-1, n

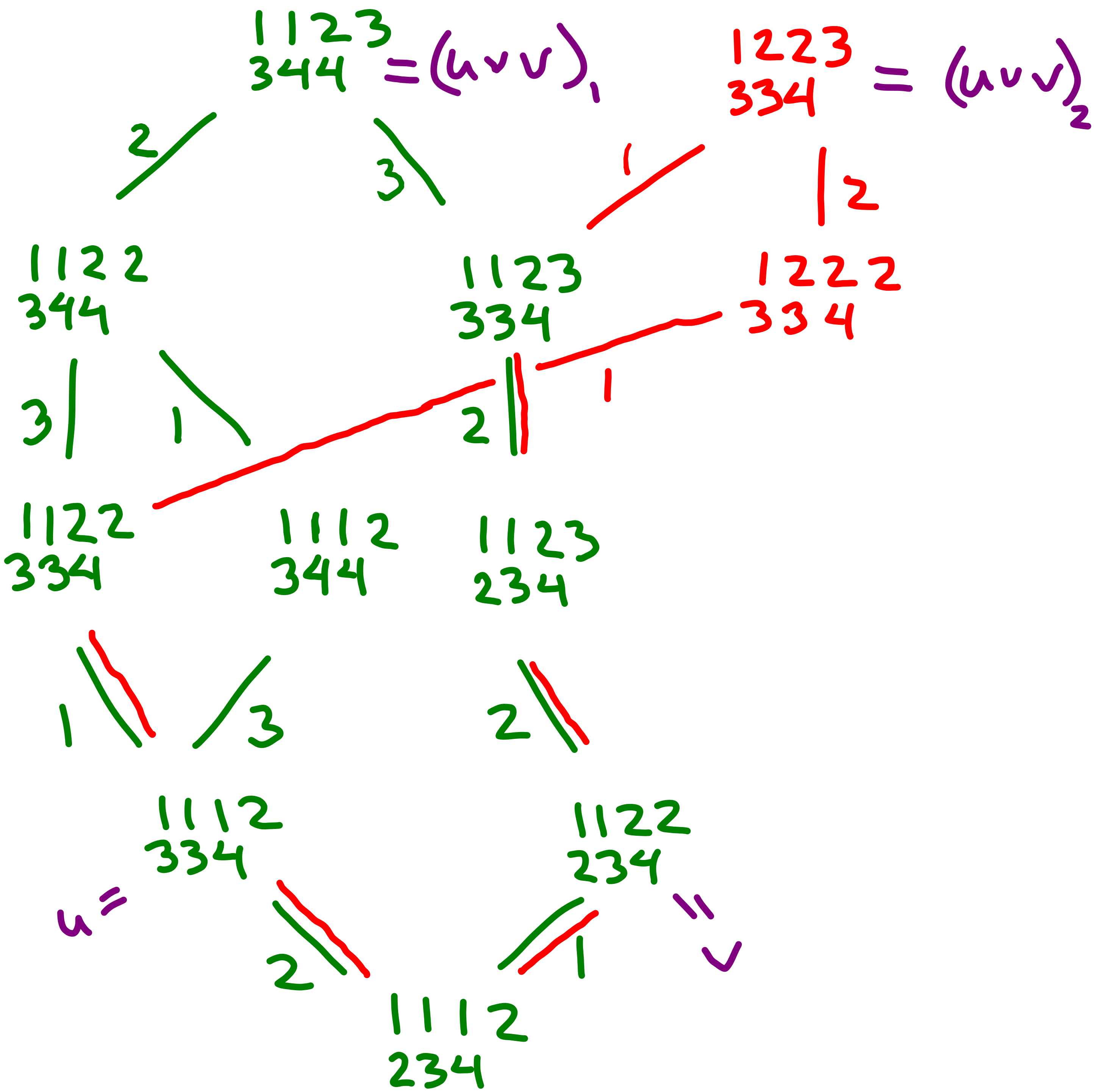
& n, n-1, n-1, ..., 2, 2, 1 in distinct components

Thm (H.-Lenart) There exist $u<v$ s.t. $M(u,v)=2^j$ for all positive integers j.

Thm (H.-Lenart): $M(u,v) \neq 0, \pm 1$ in q-crystal $\Rightarrow$ relation within $[u,v]$ not implied by Stembridge local relations.

Appendix: a few slides with extra details...

Non-Lattice Example:



Examples with $\mathcal{M}(u,v) = 2^j$

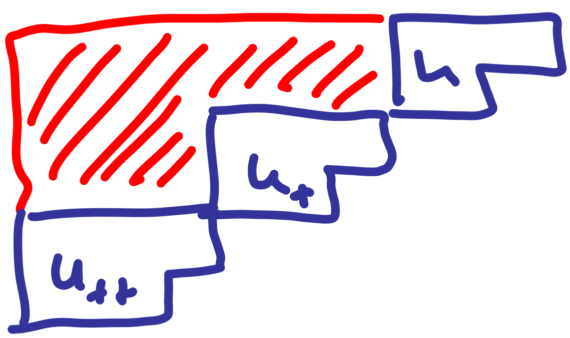
$j=1$: $u = \begin{array}{l} 1112 \\ 234 \end{array}$ $\qquad v = \begin{array}{l} 1123 \\ 344 \end{array}$

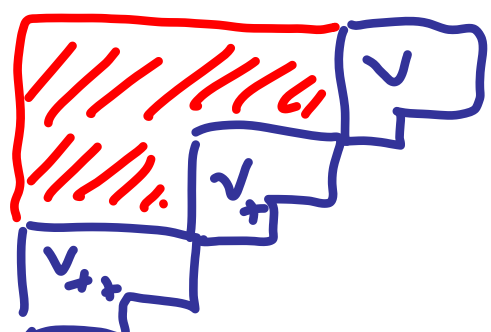
$j=2$: $u^{(2)} = \begin{array}{l} 1111\,\overbrace{1112}^{u} \\ 2222\,234 \\ 6667 \\ 789 \end{array}$ $\qquad v^{(2)} = \begin{array}{l} 1111\,\overbrace{1123}^{v} \\ 2222\,344 \\ 6678 \\ 899 \end{array}$

$u_+ := u+5 = \begin{array}{l} 6667 \\ 789 \end{array}$ $\qquad v_+ := v+5 = \begin{array}{l} 6678 \\ 899 \end{array}$

$$[u^{(2)}, v^{(2)}] \cong [u,v] \times [u,v]$$

so $\mathcal{M}(u^{(2)}, v^{(2)}) = 2^2$

$u^{(3)} =$ 

$v^{(3)} =$ 

$$[u^{(k)}, v^{(k)}] \cong \underbrace{[u,v] \times \cdots \times [u,v]}_{k\text{-fold}} \qquad \mathcal{M} = 2^k$$

Crystals

A **crystal** B of type Φ is a nonempty set B with raising & lowering operators e_i, f_i & fns

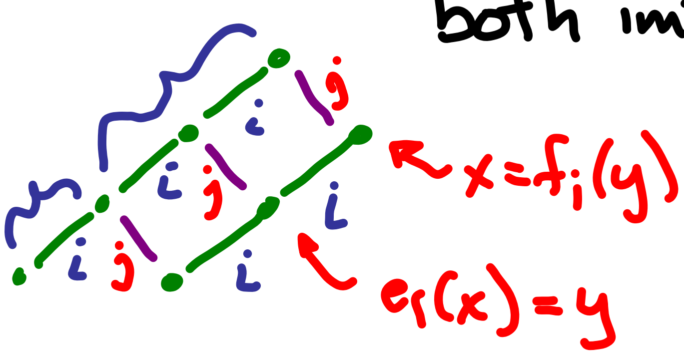
$$\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \cup \{-\infty\}$$

$$\mathrm{wt} : B \longrightarrow \Lambda = \text{weight lattice of type } \Phi$$

s.t.

(A1) $x, y \in B$, then $e_i(x) = y \iff x = f_i(y)$

both implying $\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$



& $\varepsilon_i(y) = \varepsilon_i(x) - 1$

$\varphi_i(y) = \varphi_i(x) + 1$

(A2) $\varphi_i(x) - \varepsilon_i(x) = \langle \mathrm{wt}(x), \alpha_i^\vee \rangle$

SB-Labeling (General Index Formulation)

- Given a finite lattice L with atoms $A(L)$, an edge-labeling with label set S is a lower SB-labeling if:

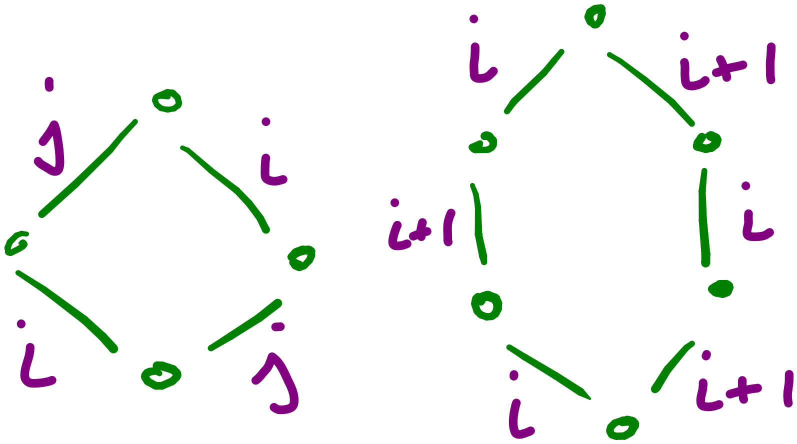
  (1) $A(L) \subseteq S$ and $\lambda(\hat{0}, a) = a$ for each $a \in A(L)$

  (2) If $x \in L$ satisfies $x = \underbrace{a_{i_1} \vee \ldots \vee a_{i_r}}_{\text{join of atoms}}$ then all saturated chains M on $[\hat{0}, x]$ use exactly the labels $\{a_{i_1}, \ldots, a_{i_r}\}$ each with positive multiplicity.

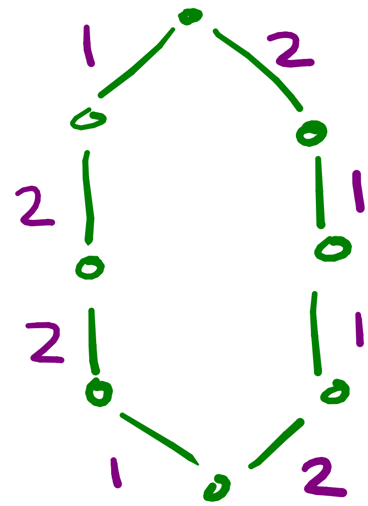
- If these conditions are met for every interval $[u, v]$ then λ is an SB-labeling.

"Sphere" or "Ball"

e.g.

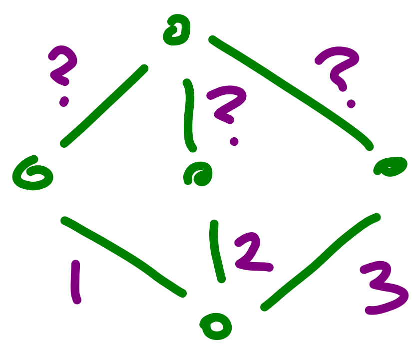


weak order



crystal graph (later)

Non-Example:



Thm (H.-Meszáros): If finite lattice L has labeling λ that is SB-labeling, then $\mu(u,v) = 0, \pm 1$ for $u, v \in L$.