Representation Stability in the Partition Lattice

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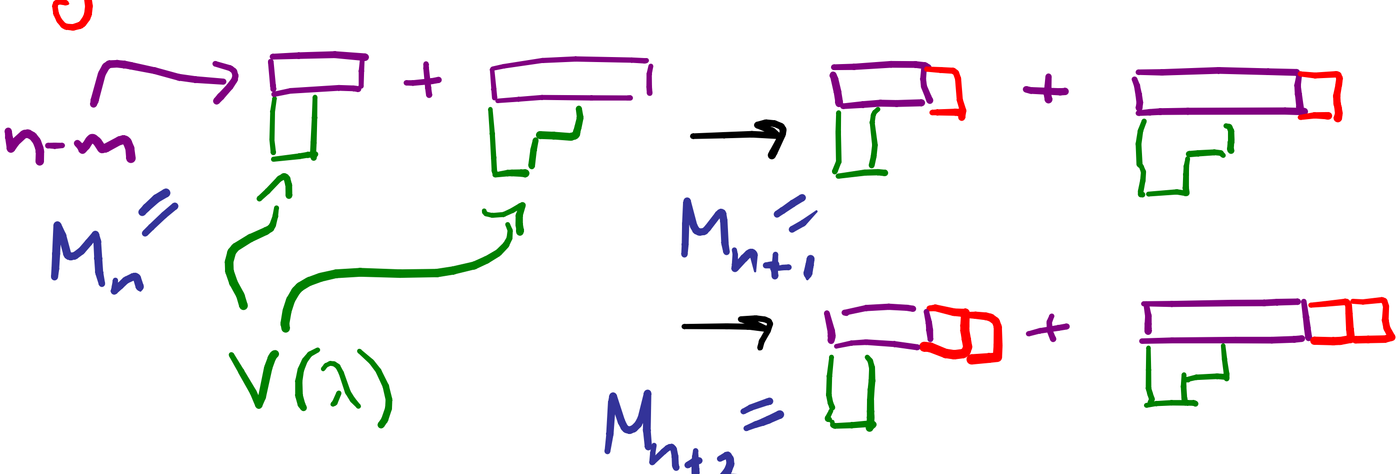
Representation Theoretic Stability

Defn (Church, Farb): A series of S_n-modules $M_1, M_2, \ldots$ stabilizes at $B>0$ if for each $n>B$, we have

$$M_n = \sum_{\lambda \vdash m \le B} c_\lambda V(\lambda) \quad \text{where} \quad V(\lambda) \cong S^{(n-m,\lambda)}$$

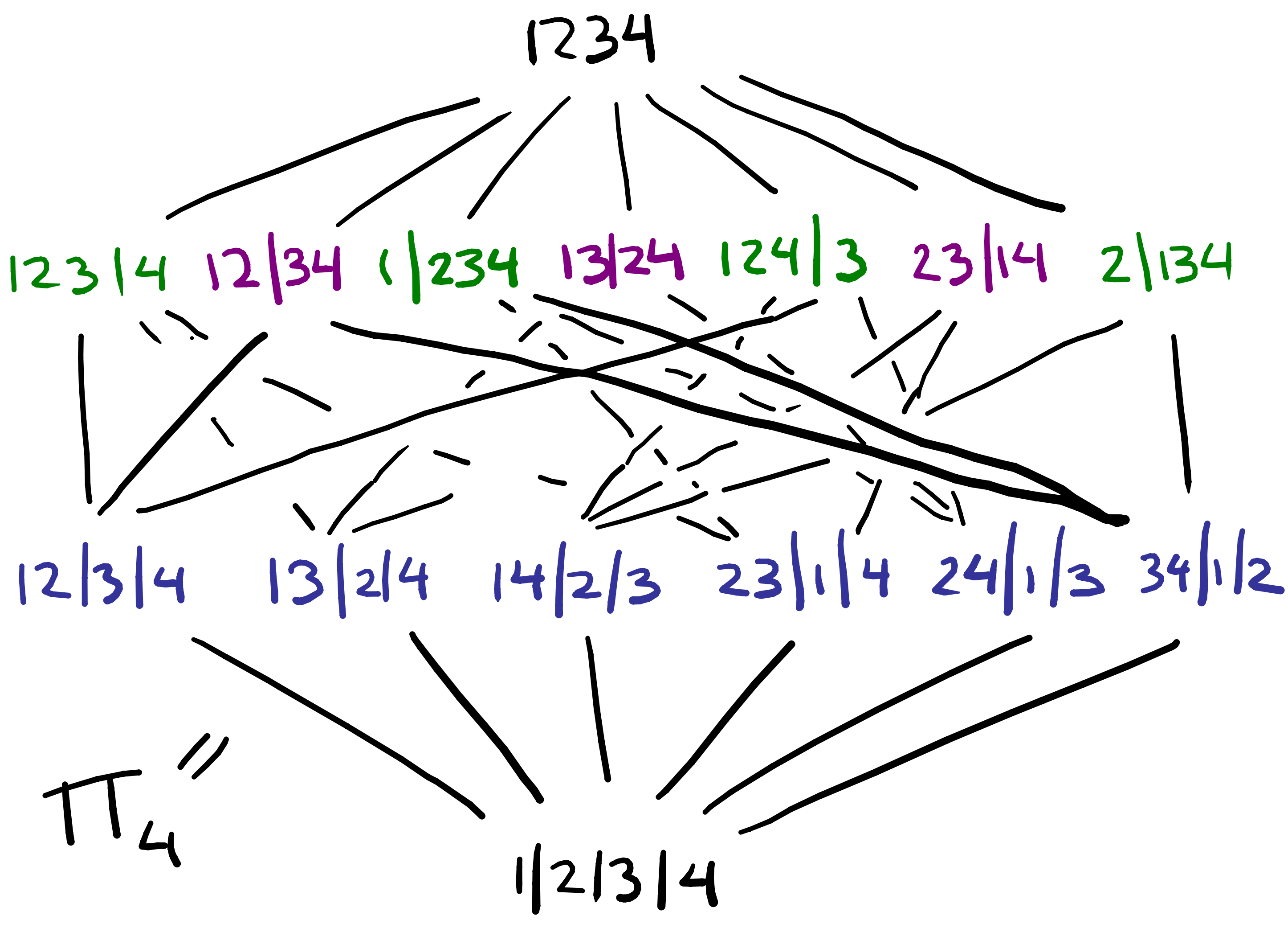
and where c_λ does not depend on n

e.g.



Our Focus: S_n-rep'ns from partition lattice

Partition Lattice Π_n & its S_n-representions



• S_n acts by permuting values

e.g. $(13)[12|3|45] = 32|1|45$

Our Starting Point:

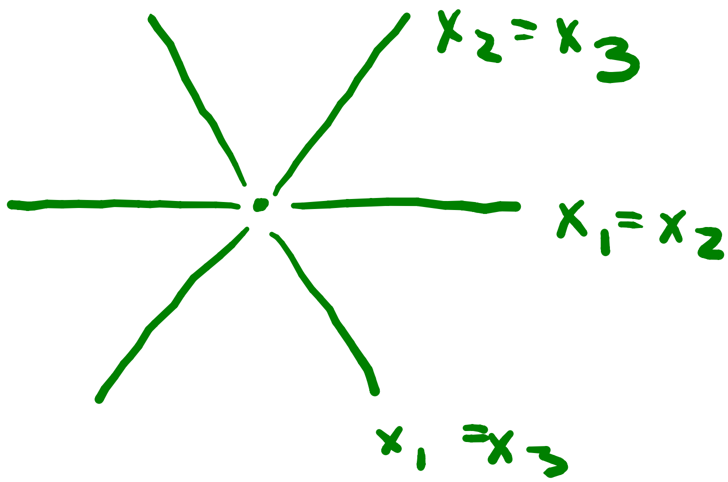
Thm (Church-Farb): $H^i(M_n)$ stabilizes for $n \geq 4i$ where M_n is configuration space of n distinct points in plane & i is held fixed.

Thm (Church-Farb): More generally, letting M_n^d be the configuration space of n distinct points on a connected orientable d-manifold, $H^i(M_n^d)$ stabilizes for
$$\begin{cases} n \geq 4i & \text{if } d = 2 \\ n \geq 2i & \text{if } d > 2 \end{cases}$$

Our First Objective: Sharpen some of these bounds.

Reinterpreting via Subspace Arrangement Complements

- M_n = complement of type A (complex) braid arrt $\{x_i = x_j \mid 1 \le i < j \le n\}$



- Π_n = intersection lattice of A_n

- S_n-module structure for $H^i(M_n)$ will translate (via Goresky-MacPherson formula) to Whitney homology in Π_n.

Church-Farb Method for other Manifolds

- Use Totaro's E_2-page of Leray spectral sequence showing manifold + $H^i(M_n(\mathbb{R}^d))$ determines cohomology of config. space on M.
- More specifically:

$$E_2^{p,(d-1)q} = \bigoplus_{\substack{S \text{ with} \\ |S| = n-q}} H^{q(d-1)}(\underbrace{C_S(\mathbb{R}^d)}_{\text{product of subspace arrangement complements}}) \otimes H^p(M^S)$$

for set partition S with $|S|$ parts &

e.g. for $S = \{1,3\}\{2,4,5\}$

- $C_S(M) := \{\underline{x} \in M^5 \mid x_1 \neq x_3;\ x_2 \neq x_4 \neq x_5\}$
$$= C_{\{1,3\}}(M) \times C_{\{2,4,5\}}(M)$$
- $M^S := \{\underline{x} \in M^5 \mid x_1 = x_3;\ x_2 = x_4 = x_5\}$

& $E_2^{p,q} = 0$ for $d-1 \nmid q$

Motivations from Number Theory:

- Church-Ellenberg-Farb & Matchett/Wood-Vakil, & others:

$$\langle H^i(PConf_n(\mathbb{C})), V\rangle_{S_n} = \dim_{\mathbb{Q}_\ell} H^i_{et}(Conf_n; \mathcal{V})$$

coefs twisted by V

yielding various counting formulas over finite field via "Grothendieck-Lefschetz formula" & counting fixed pts of Frobenius map

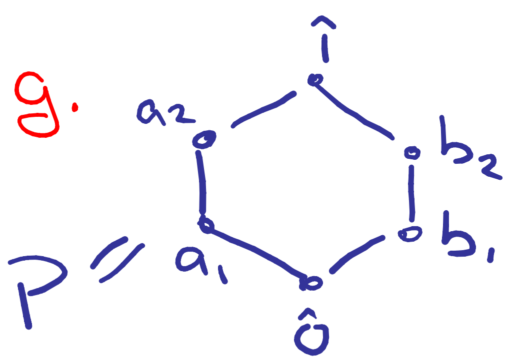
e.g. # $\square$-free degree n polys $= q^n - q^{n-1}$

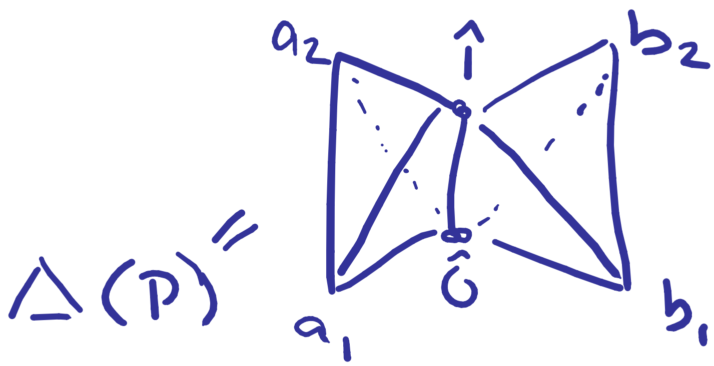
Remark: Applications to number theory focus on $M = \mathbb{R}^2$ case

Question: How does this relate to Björner-Ekedahl results?

Question: Do our upcoming results extend to $\mathbb{R}^k$ for $k>2$ via Wachs + Sundaram-Wachs + Sundaram-Welker results on nonpure Whitney homology, on Whitney homology of k-equal arr'ts, & on G-module structure of config. spaces?

Def'n: The **order complex** of a finite poset P is the simplicial complex $\Delta(P)$ whose i-dim'l faces are the $(i+1)$-chains in P.

e.g. 



- Let $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$ e.g. for Π_n

S_n-Representations on Chains (i.e on Faces) & on Homology

- S_n action on set partitions is order-preserving & rank-preserving
- Thus, it induces S_n-action α_S on chains $u_1 < u_2 < \dots < u_j$ with u_r of rank i_r for $1 \le r \le j$ and $S = \{i_1, \dots, i_j\}$, in other words on faces of $\Delta(\overline{\Pi}_n)$ with vertices colored S, where vertices in $\Delta(\overline{\Pi}_n)$ colored by poset rank.

- S_n-action on chains commutes with simplicial boundary map

$$d(u_0 < \dots < u_r) = \sum_{0 \le i \le r} (-1)^i (u_0 < \dots < \hat{u}_i < \dots < u_r)$$

- Thus, S_n-action on i-faces induces S_n-rep'n on ith homology
- But homology of Π_n is concentrated in top degree due to EL-shellability:

<u>Thm</u> (Stanley + Björner): Π_n is supersolvable, hence is EL-shellable.

- Likewise, homology of $\Pi_n^S = \{u \in \Pi_n \mid rk(u) \in S\}$ also concentrated in top degree by:

  Thm (Björner): P graded & EL-shellable $\Rightarrow$ P^S also EL-shellable

- The virtual rep'n $\beta_S := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T$ is actual S_n-rep'n on top homology of $\Pi_n^S := \{u \in \Pi_n \mid rk(u) \in S\}$ (since lower homology vanishes in Π_n^S)

Aside: in analogy to EL-shellability:

  Thm (Björner & Wachs): P graded & CL-shellable $\Rightarrow P^S$ CL-shellable.

Whitney Homology

$WH_i(P) :=$ "i-th Whitney homology of P"

$$= \bigoplus_{rk(u)=i} \tilde{H}_{i-2}(\hat{0}, u) = \bigoplus WH_\lambda(P)$$

λ has n-i blocks

$$WH_\lambda(P) := \bigoplus_{\substack{u \in P \\ type(u)=\lambda}} \tilde{H}_{top}(\hat{0}, u)$$

Thm (Sundaram): $WH_i(P) \cong \beta_{1 \ldots i}(P) + \beta_{1 \ldots i-1}(P)$

Observation: This implies WH_i stabilizes at same bound as $\beta_{1 \ldots i}$

- β_S is subrepresentation of α_S & will stabilize at least as fast as α_S.

(using $\alpha_S = \sum_{T \subseteq S} \beta_T$

& $\beta_S = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T$

Goresky-MacPherson Formula

(for cohomology of subspace arr't)

$$\tilde{H}^i(M_A) \cong \bigoplus_{x \in L_A^{>0}} \tilde{H}_{\mathrm{codim}(x)-2-i}(\hat{0}, x)$$

subspace arr't complement

as groups

intersection lattice

(OS-Algebra = presentation of cohomology ring for complex hyperplane arr't complement)

Plan: Apply to braid arrangement using S_n-equivariant version yielding Whitney homology

Rk: Wachs generalized Whitney homology to nonpure-case.

- Sundaram-Wachs applied to analyze S_n-module structure of k-equal arrangement.

G-Equivariant Enrichment of Goresky-MacPherson Formula

Thm (Sundaram-Welker): Let A be a G-arrangement of $\mathbb{C}$-linear subspaces in $\mathbb{C}^n$ for G a finite subgroup of $GL_n(\mathbb{C})$. Then

$$\tilde{H}^i(M_A) \cong_G \bigoplus_{x \in (L_A^{>0})/G} \mathrm{Ind}_{\mathrm{Stab}(x)}^{G} \tilde{H}_{\mathrm{codim}(x)-i-2}(\hat{0}, x)$$

(in our case) $= WH_i(L_{A_n}) = WH_i(\Pi_n)$

Note: Config. space of n distinct points in $\mathbb{R}^{2d}$ gives generating subspaces of real codim $2d$ & $\tilde{H}^i(M_A) = 0$ unless $i = (2d-1)\mathrm{rk}(x)$ for some x, i.e. unless $2d-1$ divides i.

Upshot for Stability:

- $\beta_{1,\ldots,i}(\Pi_n)$ stabilizes at $B>0$

$\iff WH_i(\Pi_n)$ stabilizes at $B>0$

$\iff H^i(M_n)$ stabilizes at $B>0$

Thm (H-Reiner): $H^i(M_n)$ stabilizes at $3i+1$. More generally, $H^i(M_n^{2d})$ for M_n^{2d} = config. space of n distinct pts in $\mathbb{R}^{2d}$ stabilizes for $n \geq 3\frac{i}{2d-1}+1$.

Thm (H-Reiner): $\beta_S(\Pi_n)$ stabilizes at $n \geq 4\max(S)$ for any fixed S.

Past Results on Π_n:

Thm (Hanlon-Stanley): $\Pi_n \cong \mathrm{sgn} \otimes (\zeta_n \uparrow_{C_n}^{S_n})$

Method: Calculate $\mu_{\Pi_n^g}(\hat{0}, \hat{1}) = \chi_{\Pi_n}(g)$

Thm (Joyal): $\mathrm{lie}_n \cong \zeta_n \uparrow_{C_n}^{S_n}$

Thm (Barcelo): Explicit S_n-equivt bijection yielding $\Pi_n \cong \mathrm{sgn} \otimes \mathrm{lie}_n$

Thm (Kraskiewicz & Weyman):

$$\mathrm{lie}_n \cong \bigoplus_{\substack{T \text{ SYT} \\ \text{w/ } \mathrm{maj}(T) \equiv 1 \pmod n}} S^{\lambda(T)}$$

★ **Key Fact for Stability:** $u \in \Pi_n$ of rank i has at most $2i$ letters in nontrivial blocks

Open Question: Is there a "Michelle Wachs style" homology basis for Π_n explaining:

Thm (Kraskiewicz & Weyman):

$$\Pi_n \cong \bigoplus_{\substack{T \text{ SYT} \\ \text{w/ } maj(T) \equiv 1 \pmod{n}}} S^{\lambda(T)^{\text{transpose}}}$$

Suggested Step 1:

- Read "A basis for the homology of the d-divisible partition lattice" by M. Wachs

Suggested Step 2:

- Read "On the (co)homology of the partition lattice & the free Lie algebra" by M. Wachs

Thm (Sundaram):

S_j-rep'n on top homology of Π_j

$$ch(WH_\lambda) = \prod_{j \text{ odd}} h_{m_j}[\pi_j] \prod_{j \text{ even}} e_{m_j}[\pi_j]$$

$$= \Big(\underbrace{h_{m_1}}_{j=1 \text{ part}}\Big)\underbrace{\Big(\prod_{\substack{j \text{ odd} \\ j>1}} h_{m_j}[\pi_j]\Big)\Big(\prod_{j \text{ even}} e_{m_j}[\pi_j]\Big)}_{\text{"}\widehat{Wh_\lambda}\text{" has degree} \leq 2i \text{ by } \star}$$

where ch = Frobenius characteristic isomorphism from ring of S^n-repn's to ring of symmetric functions

$h_n :=$ complete symmetric fn

$$= \sum_{1 \leq i_1 \leq i_2 \leq \cdots} x_{i_1} x_{i_2} \cdots x_{i_n} = ch(\text{triv})$$

$e_n :=$ elementary symmetric fn

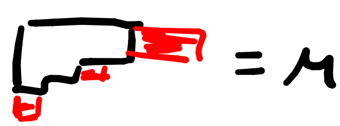
$$= \sum_{1 \leq i_1 < i_2 < \cdots} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_n} = ch(\text{sgn})$$

Thm (H-Reiner): Holding i fixed & letting n grow, $\beta_{1-i}(\Pi_n)$, $WH_i(\Pi_n)$ & $H^i(M_n)$ stabilize as S_n-rep'ns at $n \geq 3i+1$.

Idea: $\widehat{WH}_i = \bigoplus_{\lambda \vdash n} \widehat{WH}_\lambda(\Pi_n)$ stabilizes at $n=2i$ since at most $2i$ letters in nontrivial blocks. Obtain upper bound of $i+1$ on length of 1st row in S^λ in $\widehat{WH}_i$. Pieri Rule says multiplying $ch(\widehat{WH}_i) = \bigoplus c_\lambda s_\lambda$ by h_{λ_1+j} is stable.

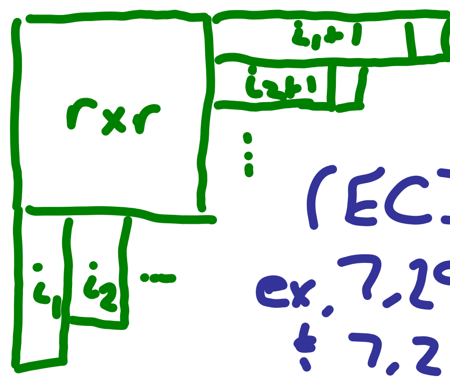
Pieri Rule:

$$h_n s_\lambda = \sum s_\mu$$



Key Lemma: Each S^λ appearing in $\widehat{WH}_i$ has $\lambda_1 \le i+1$.

Pf: $\text{type}(u) = (m_1, m_2, \ldots)$ for $\text{rk}(u) = i$, then $i + \sum_{j>1} m_j \le 2i$ letters in nontrivial blocks & $\sum_{j>1} m_j \le i$.



- $e_{m_2}[h_2] = \sum s_\lambda$ (ECII, ex. 7.29b & 7.28e)

  $\Rightarrow \lambda_1 \le m_2 + 1$ for each s_λ in it.

- $\langle 1, \pi_n \rangle = 0$ for $n > 2$

  $\Rightarrow$ each block B_j with $|B_j| > 2$ adds at most $|B_j| - 1$ to $\log_{x_1}(x_1^a)$.

- Multiplying contributions gives bound of $m_2 + 1 + \sum_{j>2} m_j \le i+1$. $\blacksquare$

Facts about Symmetric Functions

- $S^\lambda \longleftrightarrow$ schur fn $s_\lambda = \sum_{\substack{T \text{ SSYT} \\ \text{shape } \lambda}} x^T$

$$T = \begin{array}{|c|c|c|c|} \multicolumn{4}{c}{\overbrace{\hphantom{1 \quad 1 \quad 2 \quad 2}}^{\lambda_1}} \\ \hline 1 & 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} \rightsquigarrow \underbrace{x_1^2 x_2^2 x_3 x_4}_{x^T \,=}$$

$$\Rightarrow \boxed{s_\lambda \text{ must include some monomial divisible by } x_1^{\lambda_1}.}$$

- Wreath product of rep'ns $\longleftrightarrow$ plethysm of symmetric fns

$$\Rightarrow \boxed{\begin{array}{l} f \text{ includes } x_1^a \;\&\; g \text{ includes } x_1^b \\ \text{then } f[g] \text{ includes } x_1^{ab}. \end{array}}$$

<u>Thm</u> (Hersh-Reiner): $\beta_S(\Pi_n)$ for any fixed S stabilizes for $n \geq 4\max S$.

<u>Idea</u>: Show $ch(\alpha_S(\Pi_n))$ has upper bound of $2\max(S)$ on length of 1st row in "h-free" part of symmetric fn f (analogue of $\widehat{WH}_i(n)$).

- This gives stability bound $n \geq 4\max S$ for $\alpha_S(n)$ & likewise for $\alpha_T(n)$ for $T \subseteq S$.
- Deduce same bound for $\beta_S(n)$ using that $\beta_S(n) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T(n)$

Wiltshire-Gordon Conjectures & Related Results

Defn (Wiltshire-Gordon):

$$V_n^k = \bigoplus_{\substack{|\lambda|=n \\ \ell(\lambda)=n-k \\ \lambda \text{ has no parts of size } 1}} WH_\lambda(\Pi_n)$$

Thm (H-Reiner):

$$\mathrm{Ind}\left(\mathrm{Res}\left(V_n^k\right) \oplus V_{n-1}^k\right) \cong \mathrm{Res}\left(V_{n+1}^{k+1}\right)$$

(conjectured by Wiltshire-Gordon)

Idea: Symmetric fns & generating fns

Qn: Pf by clever homology bases?

Thm (H-Reiner):

$$V_n = ch\left(\bigoplus_k V_n^k\right) \cong \bigoplus S^{\lambda(T)}$$

T is "Whitney generating" SYT

where T is **Whitney generating** if

- 1 and 2 both appear in 1st row
- if 3 in 1st row, then 1st "ascent" $k>2$ is even (or there is no ascent & n is even)
- if 3 & 4 in 2nd row, then 1st ascent odd (or none exists & n is odd)

ascent := i such that i+1 in weakly higher row

Idea: Both sides satisfy same recurrence.

Qn: Refined formula for individual k?

Thm (H.-Reiner): $\langle 1, \beta_S(\Pi_n) \rangle$ stabilizes for $n \geq 2\max S - \binom{|S|-1}{2}$

Idea: Use partitioning for $\Delta(\Pi_n)/S_n$ from (H., 2003) and consequent combinatorial interpret. for $\langle 1, \beta_S(\Pi_n) \rangle$.

$$\langle 1, \beta_S(\Pi_n) \rangle = h_S(\Delta(\Pi_n)/S_n)$$

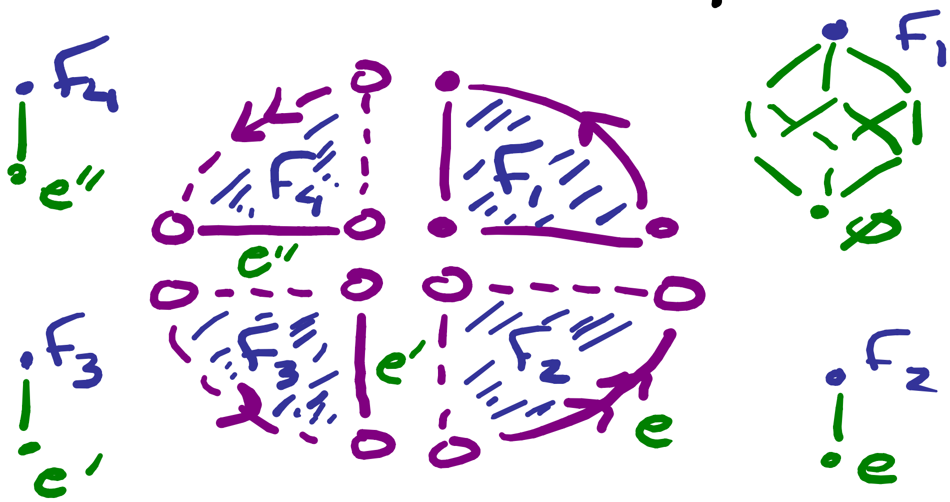
- Injection $\varphi_n : \left\{ \begin{array}{c} \text{facets} \\ \text{contrib.} \\ \text{to } b_S(n) \end{array} \right\} \to \left\{ \begin{array}{c} \text{facets} \\ \text{contrib} \\ \text{to} \\ b_S(n+1) \end{array} \right\}$ eventually also a surjection.

Rk: This is sharp for $S = \{i\}$ but not for every single choice of S.

e.g. (Hanlon): $\langle 1, \beta_{\{1,\dots,i\}}(\Pi_n) \rangle = 0$ for $n > 2$.

Partitioning: Δ is partitionable if face poset $F(\Delta)$ decomposes into disjoint union of boolean algebras w/ facets as top elements

e.g.

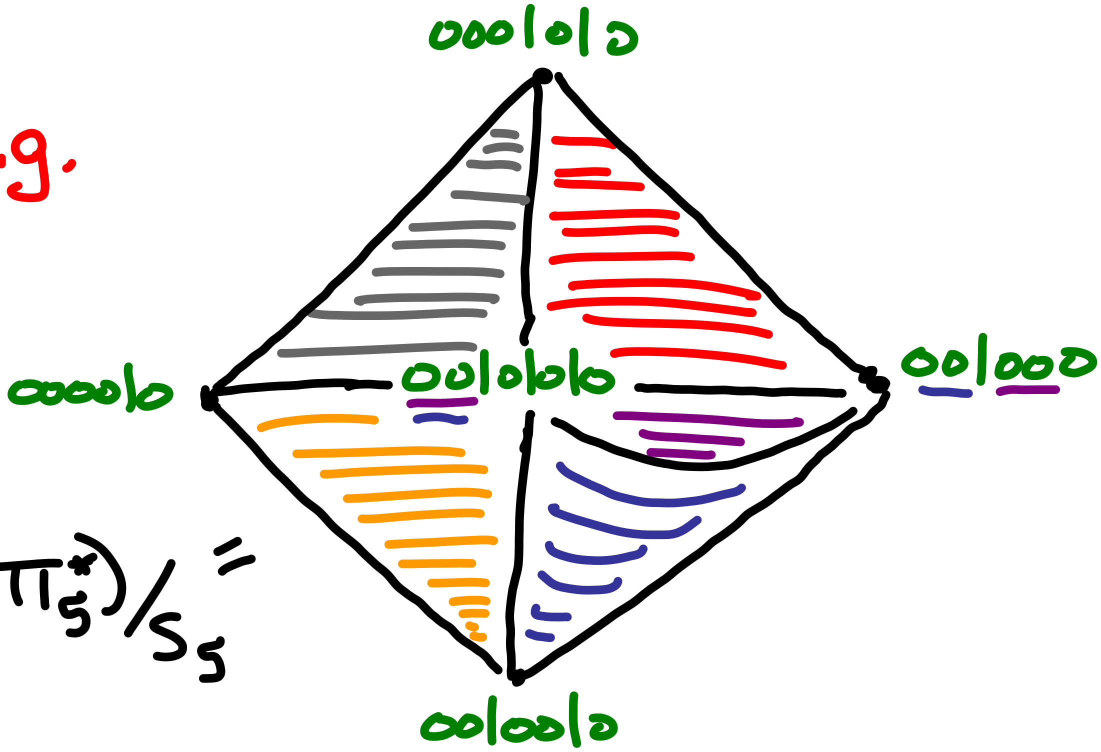


- $h_S(\Delta(\Pi_n)/S_n) = \langle 1, \beta_S(\Pi_n)\rangle$ as # saturated chain orbits with "topological descent set" S.
- $\Delta(\Pi_n)/S_n$ "almost" shellable but $lk_\Delta(F) \cong \mathbb{R}P^n$ for some F

Depicting Faces & a Chain Labeling

e.g.

$\Delta(\Pi_5^*)/S_5 =$



- Label saturated chain S_n-orbits in Π_n^* with sequence of separator insertions positions each as far left as possible

e.g. $\lambda(0\underset{3}{|}00\underset{1}{|}00\underset{2}{|}00) = 351\ldots$

- $0|0|00|000|000|00000|00000$

$\varphi_n = \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \leq n \cdot \max S - 1$

Some Further Questions

1. (Farb) How fast does the multiplicity of any particular $V(\lambda)$ stabilize within M_n?

2. (H-Reiner) How fast does the multiplicity of $V(\lambda)$ stabilize within $\beta_S(\Pi_n)$ as S is held fixed & n grows?

(Note: Qn 1 is special case of Qn 2 with $S = \{1, 2, \ldots, i\}$)

3. (Farb) What rep'ns do we get after stabilization occurs?