Bruhat Interval Polytopes & Posets Arising as 1-Skeleta of Polytopes

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-partly joint work with Christian Gaetz (Berkeley)

I. Background & Examples

II. Results for Simple Polytopes

- homotopy type of poset intervals
- join = "pseudojoin"

III. Connection to Linear Programming & a Conjecture

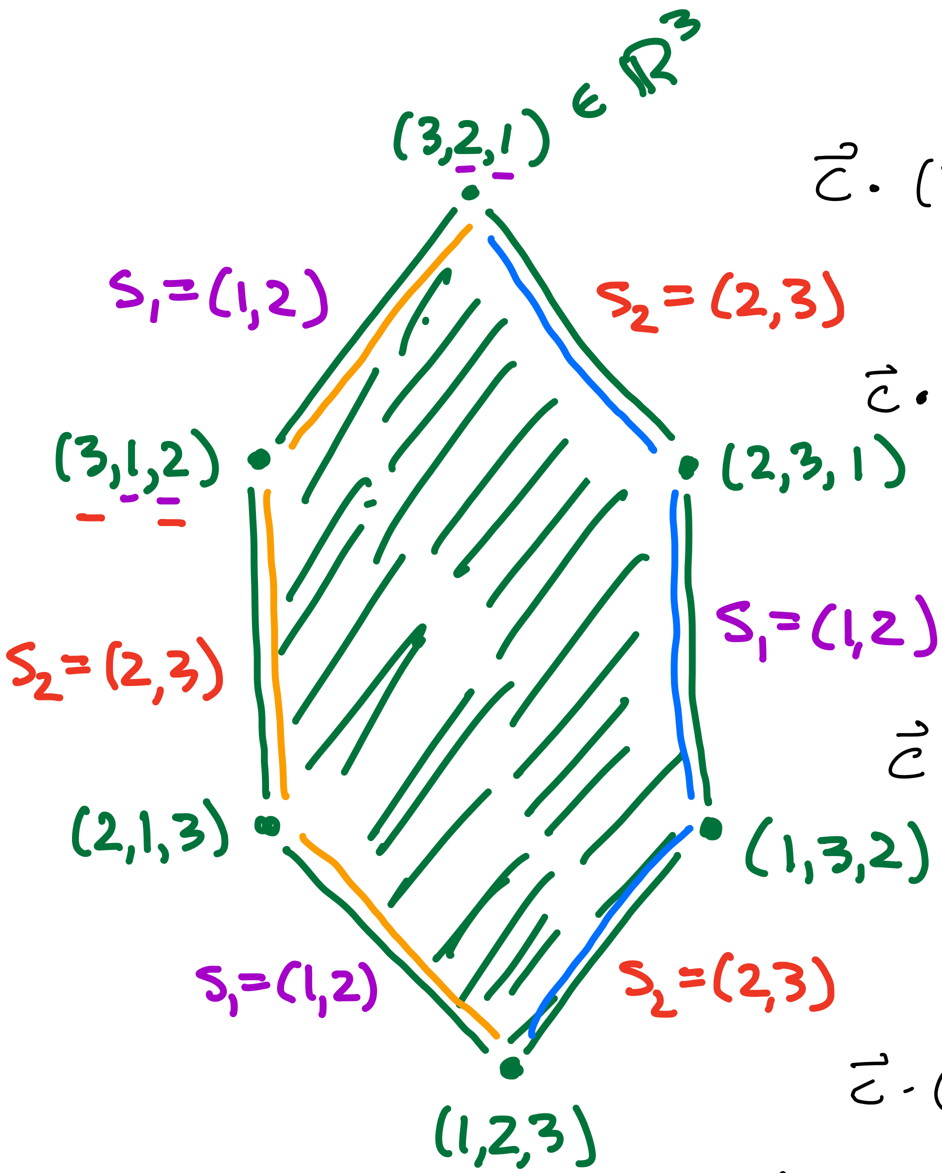
IV. Extension to Bruhat Interval Polytopes & Higher Connectivity

V. Proof Techniques & Open Qns

e.g. Polytope: Permutahedron in $\mathbb{R}^3$

Poset: Right Weak Order

(via cost vector $\vec{c} = (3,2,1)$)



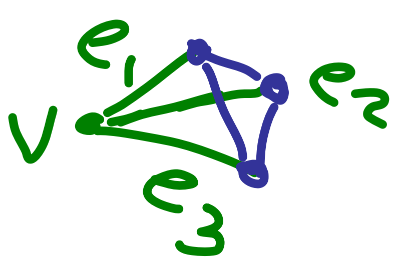
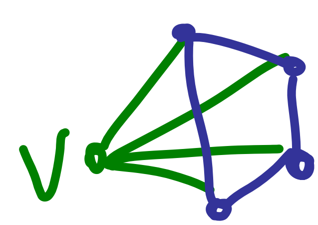
$$3^2+2^2+1^2$$
$$\vec{c} \cdot (3,2,1) = 14$$
$$\vee$$
$$\vec{c} \cdot (2,3,1) = 13$$
$$\vee$$
$$\vec{c} \cdot (1,3,2) = 11$$
$$\vee$$
$$\vec{c} \cdot (1,2,3) = 10$$

Braid move: $s_1 s_2 s_1 \xrightarrow[\text{2-face}]{\text{move across}} s_2 s_1 s_2$

Quick Background on Polytopes

- A **polytope** in $\mathbb{R}^d$ is convex hull of finite # vertices, or equiv'ly a bounded set that is an intersection of half spaces.

- Polytope is **simple** if for each vertex v and each collection $e_1, e_2, \ldots, e_r$ of edges emanating out from v, there is r-dim'l face containing all these edges.

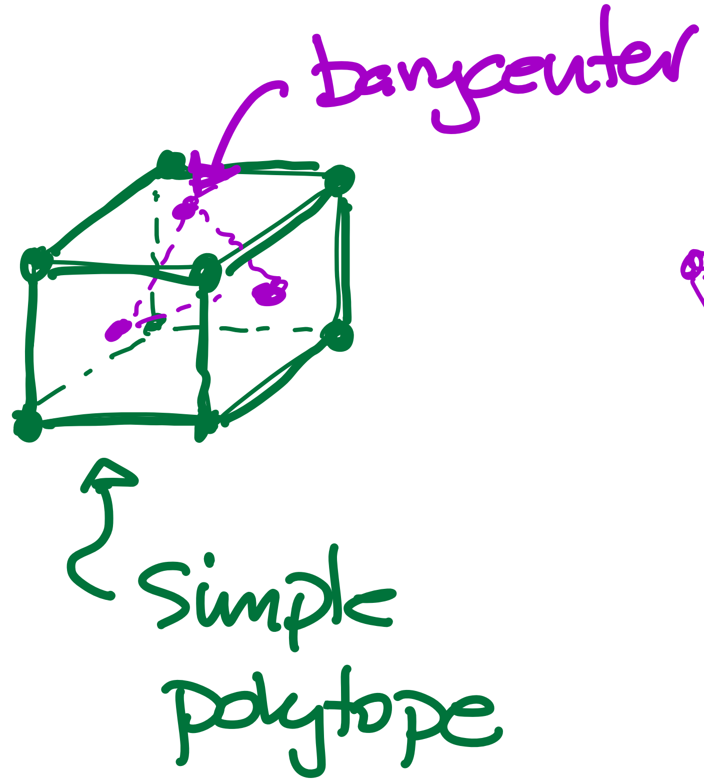
e.g.  but not 

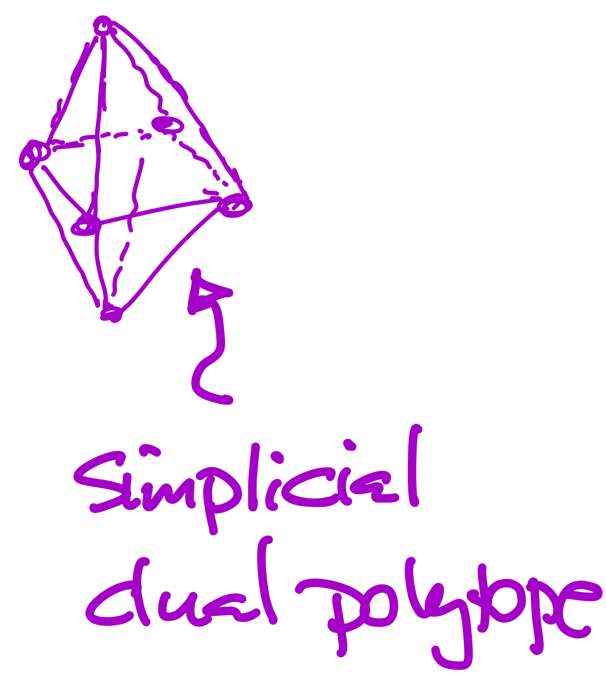
• Equivalently, a polytope P is simple if its "dual polytope" P^* is "simplicial"

convex hull of barycenters of max'l faces of ∂P

ie ∂P^* is simplicial complex

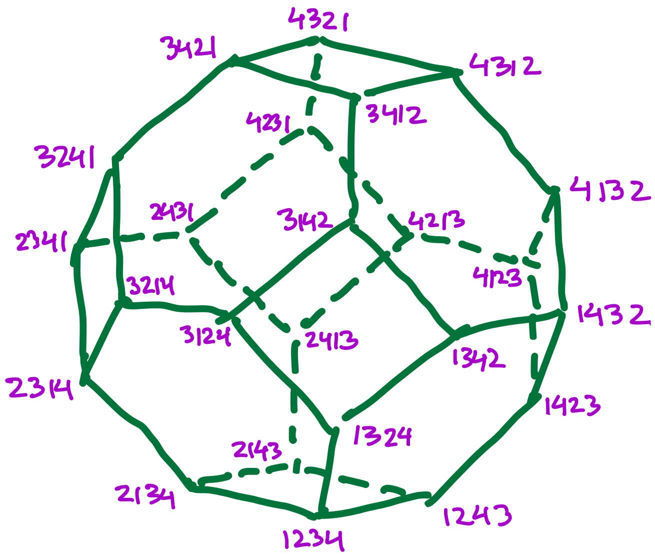
e.g.





Next: Some important examples, with associated posets...

Permutahedron with weak order poset "Hasse diagram" as its 1-skeleton graph



- cost vector $\vec{c} = (4,3,2,1) \in \mathbb{R}^4$ has $\vec{c} \cdot \vec{v}_1 < \vec{c} \cdot \vec{v}_2$ for $\vec{v}_1 < \vec{v}_2$ in weak order

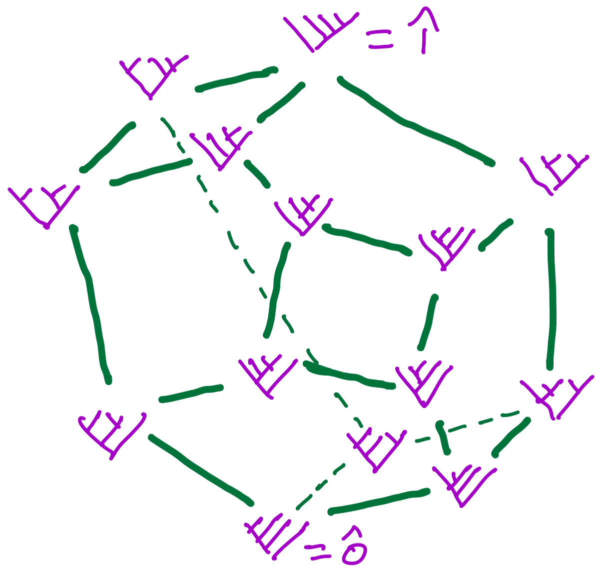
e.g. $\vec{c} \cdot (4,3,2,1) = 4^2 + 3^2 + 2^2 + 1^2$
$\vee$
$\vec{c} \cdot (1,2,3,4) = 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 4$

- graph has up edge $\vec{u} \to \vec{v}$ if $\vec{u}$ obtained from $\vec{v}$ by sorting consec pair of letters

e.g. $1432 \to 4132$

Associahedron with Tamari lattice

Hasse diagram as 1-skeleton digraph (using Loday's realization of Assoc.)



Tamari lattice is poset on binary trees with cover relations $\lessdot$

$((a,b),c) \quad (a,(b,c))$

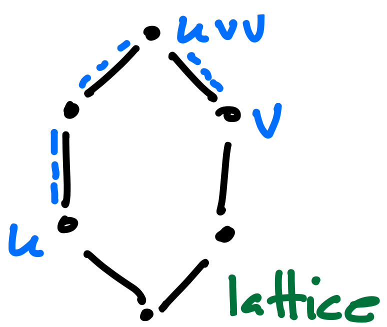
Third Example: Generalized associahedra & Cambrian lattices (cluster algebras of finite type - edges for mutation)

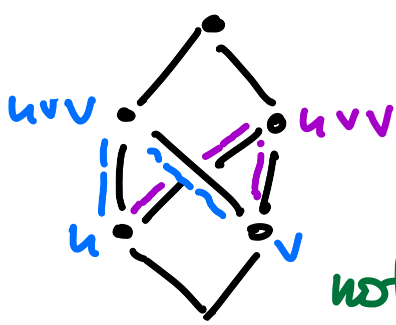
Quick Background on Posets

The Hasse diagram of poset P is directed graph with vertices the elements of P and edges $u \to v$ for $u < v$ s.t. $\nexists\, z \in P$ with $u < z < v$, i.e., "cover relations" $u \lessdot v$

- A poset L is a lattice if for each $u, v \in L$ there exists unique least upper bound $u \vee v$ (called the join) and unique greatest lower bound $u \wedge v$ (called meet)

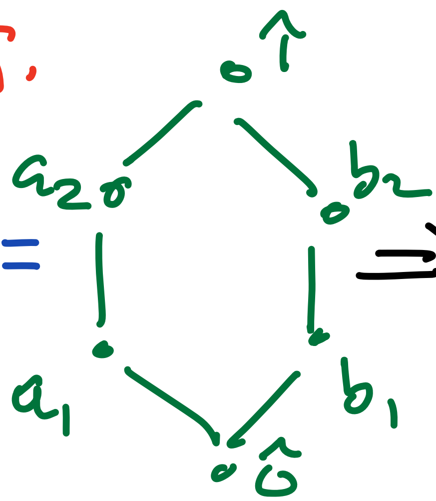
e.g.

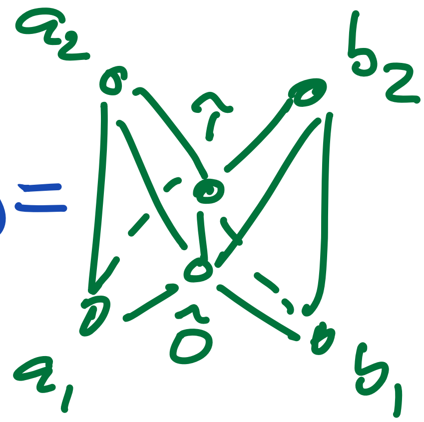




The order complex, denoted $\Delta(P)$, of poset P is simplicial complex whose i-dimensional faces are the $(i+1)$-chains $v_0 < v_1 < \cdots < v_i$ in P.

e.g.

$P =$ 

$\Longrightarrow \Delta(P) =$ 

Thm (Hall): $\mu_P(u,v) = \tilde{\chi}(\Delta(u,v))$

$= -1 + \#\text{vert's} - \cdots$

The atoms of $[u,v]$ in P are the elements a covering u, i.e. with $u \lessdot a$.

Defn: Given polytope P & cost vector $\vec{c}$, let $G(P,\vec{c})$ = digraph on vertices of P with $u \to v$ $\iff \vec{c}\cdot u < \vec{c}\cdot v$

Thm (H.) If P is simple polytope & $\vec{c}$ is generic cost vector s.t. $G(P,\vec{c})$ is Hasse diagram of lattice L, then $\Delta(u,v) \simeq$ ball or sphere S^{i-2} for each $u < v$. Hence,

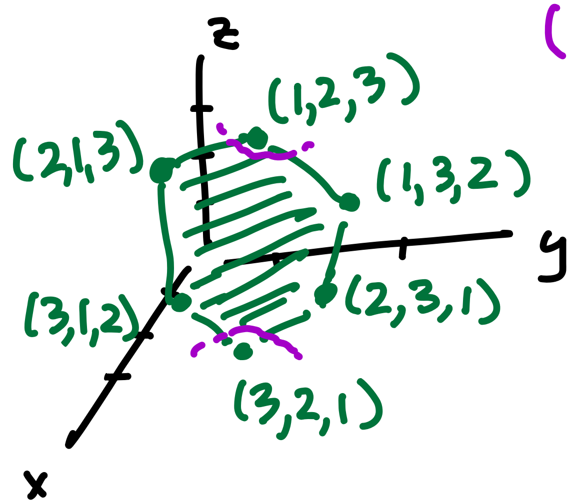
$$\mu_L(u,v) = \begin{cases} 0 & \text{if ball} \\ (-1)^i & \text{if } S^{i-2} \end{cases}$$

e.g. $P \subseteq \mathbb{R}^3$

$\parallel$

$\mathrm{Conv}\big((1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\big)$

$\parallel$

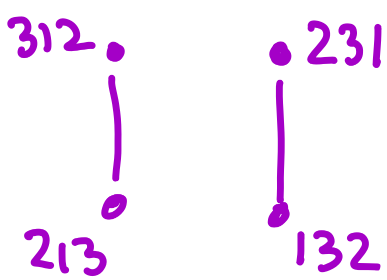


(right weak order)

cost vector

Poset open interval (123, 321)

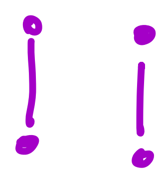
$\parallel$



$\rightsquigarrow$

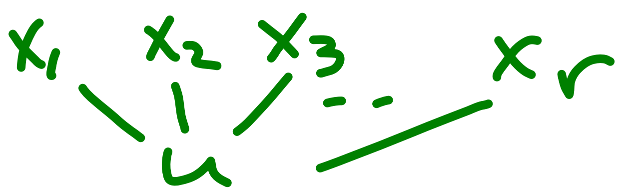
Order complex $\Delta(123, 321)$

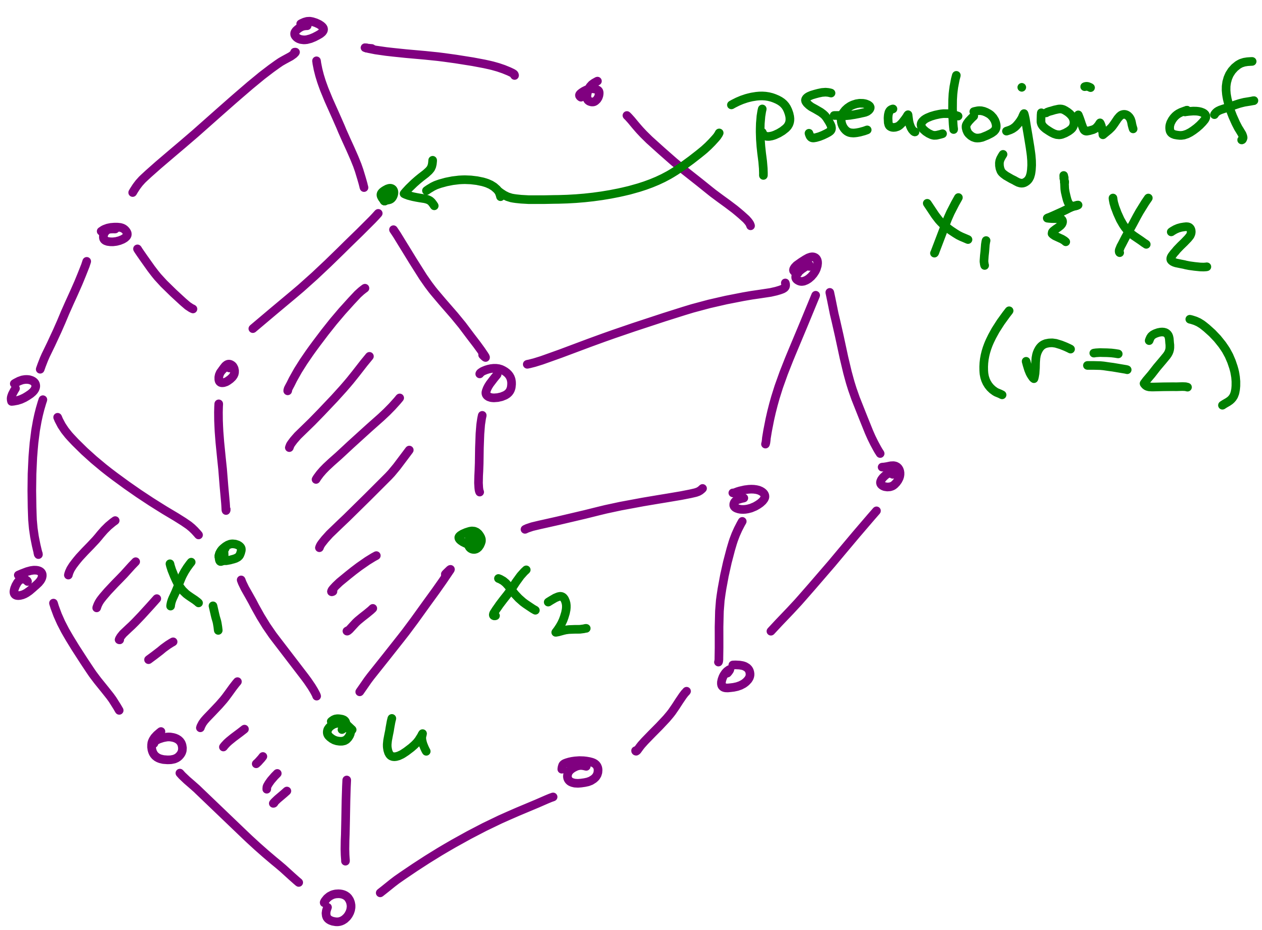
$\parallel$



$\simeq S^0$

Def'n (H.): The pseudo-join of $x_1, x_2, \ldots, x_r$ is the sink of the unique r-face of simple polytope P containing





Note: Since pseudo-join of $x_1, \ldots, x_r$ is an upper bound, there exists directed path from $x_1 \vee \cdots \vee x_r$ to pseudo-join$(x_1, \ldots, x_r)$

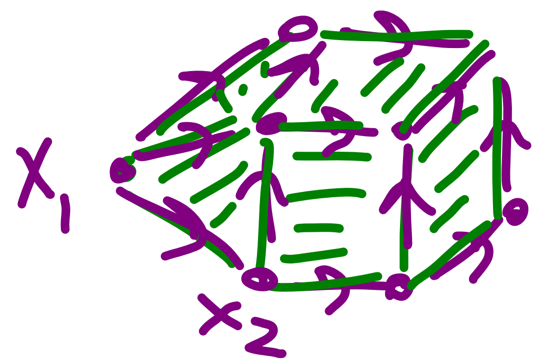
Thm (H.): Let P be a simple polytope and $\vec{c}$ be generic cost vector with $G(P, \vec{c})$ Hasse diagram of finite lattice. Then

$$\text{pseudo-join}(x_1, x_2, \ldots, x_r) = x_1 \vee \cdots \vee x_r$$

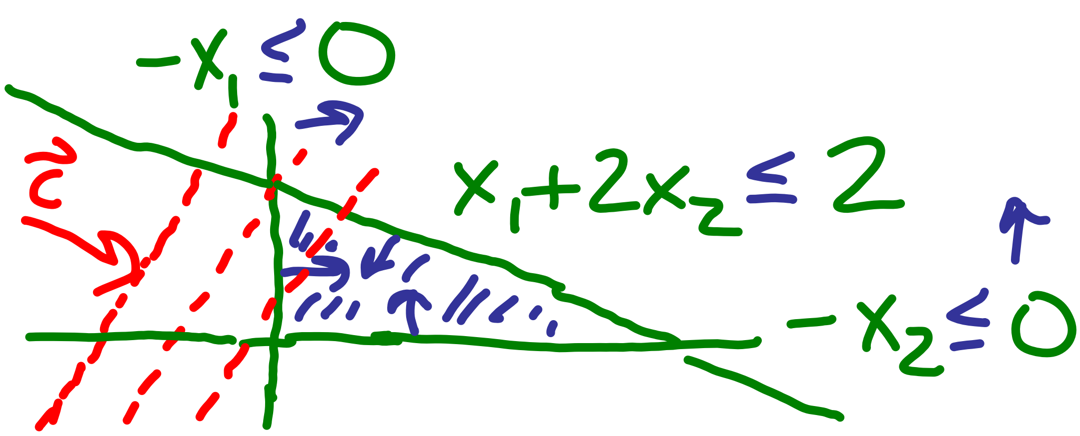
Pf: induction on r with $r = 2$ base case especially tricky part.

Linear Programming

- Given a matrix A & vectors $\vec{b}, \vec{c}$ seek $\max\{\vec{c}\cdot\vec{x} \mid A\vec{x} \le \vec{b}\}$
- $\{\vec{x} \mid A\vec{x} \le \vec{b}\}$ is polytope P if set is bounded



e.g. $\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}} \le \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{\vec{b}}$



Solving Linear Programs via Simplex Method

- Define $G(P,\vec{c}) :=$ directed graph on 1-skeleton of P, i.e. on vertex-edge graph of P, with

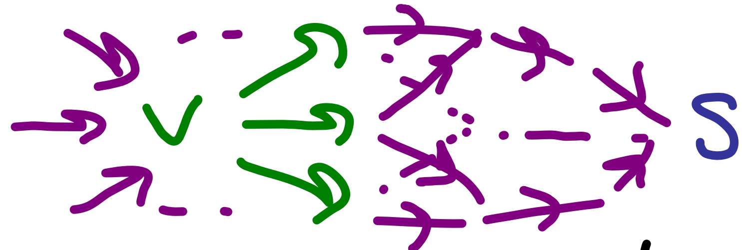
$$x_1 \to x_2 \iff \vec{c}\cdot\vec{x}_1 < \vec{c}\cdot\vec{x}_2$$

- $\max\{\vec{c}\cdot\vec{x} \mid A\vec{x} \leq \vec{b}\} =$ sink of $G(P,\vec{c})$

Simplex Method: walk from some vertex $v \in G(P,\vec{c})$ following arrows $v \to v_1 \to v_2 \to \ldots \to s$ to sink s

- also may walk backwards to source of $G(P,\vec{c})$ to minimize $\vec{c}\cdot\vec{x}$

Pivot Rule: method to choose which out arrow



to follow from v towards sink s.

Key Questions:

1. What is typical complexity of simplex method (path length)?

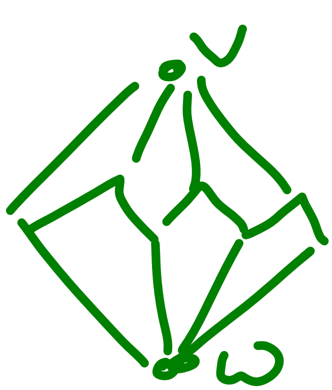
2. What is worst case? (i.e. diameter of $G(P, \vec{c})$)

Today: simple polytopes where $G(P, \vec{c})$ is Hasse diagram

Hirsch Conjecture: For d-dim'l polytopes with n facets (max'l faces), diameter of 1-skeleton graph, denoted $\Delta(d,n)$, satisfies $\Delta(d,n) \le n-d$.

Francisco Santos: After many decades eluding many people, he constructed counterexamples

("spindles" := polytopes with vertices v, w s.t. each facet includes v or w.)



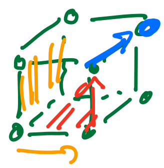
43-dim's, 86 facets, diam ≥ 44

Nonrevisiting Path Conjecture

- For each u, v in polytope P, there exists path u to v that does not revisit any facets (max'l faces)

Nonrevis. Path Conj. $\Rightarrow$ Hirsch Conj.

- Any nonrevisiting path u to v leaves a facet at each step but is still incident to at least d facets when it reaches d, so



$$\#\text{steps} \leq \#\text{facets} - d$$

Upshot: Both conjectures are false in general. (So is "monotone Hirsch Conj.")

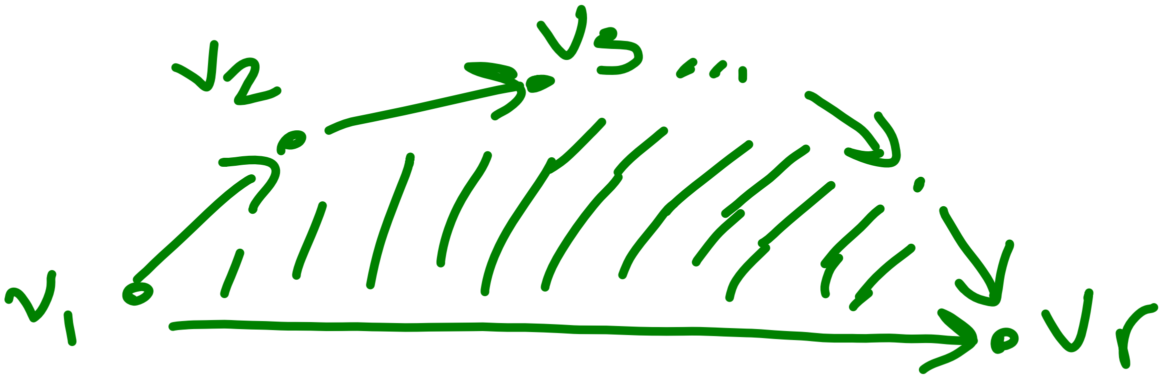
Conjecture (H.): If P is simple & $G(P,\vec{c})$ is Hasse diagram of lattice, then no directed path in $G(P,\vec{c})$ ever revisits any facet (equiv'ly any face) it has left.

Known cases: 3-polytopes; spindles w/ source & sink the "distinguished vertices"

Consequence of Conjecture: Under these conditions, no directed path has length greater than # facets − dim (P), so monotone Hirsch Conjecture holds.

Thm (Dominik Preuß): Under hypotheses of Hersh Conjecture, monotone Hirsch conj. holds.

Def'n (H.): $G(P,\vec{c})$ has the Hasse diagram property if it is Hasse diagram of finite poset, i.e. $v_1 \to v_2 \to \cdots \to v_r$ for $r \geq 3$ directed path precludes $v_1 \to v_r$



Note: precludes d-simplices as faces for $d \geq 2$

Note: Equivalent to non-revisiting of 1-dim'l faces

Note: Poset often not "atomic"

Important Non-Examples:

"Klee-Minty Cubes"

e.g. $n=3$

$\vec{c}=(0,0,1)$

- path visits all vertices!
- first polytopes exhibiting inefficiency of simplex method

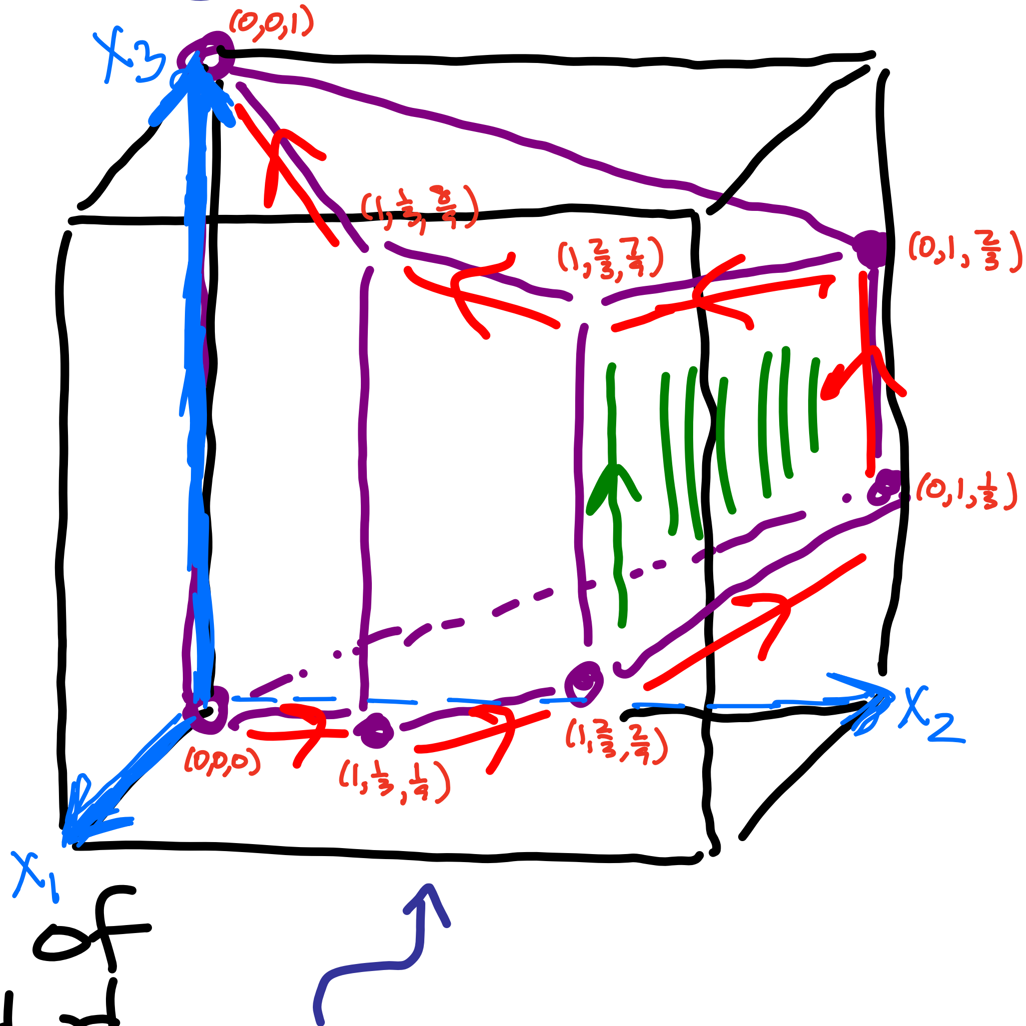


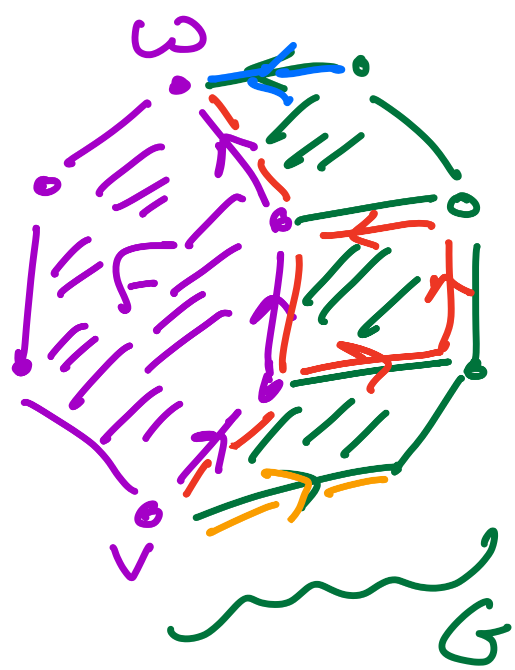
Figure modelled after one in Gärtner-Henk-Ziegler paper

n-dimensional Klee-Minty cube

$$:= \{(x_1,\dots,x_n) \mid 0 \leq x_1 \leq 1 ; \ \varepsilon x_{i-1} \leq x_i \leq 1-\varepsilon x_{i-1} \text{ for } i>1 ; \ 0<\varepsilon<\tfrac{1}{2}\}$$

Evidence for Conjecture: Nonrevisiting Dimension Propagation Lemma (H.):

If P is simple polytope with faces $F \subseteq G$ with $\dim G = \dim F + 1$ and $G(P, \vec{c})$ is Hasse diagram, then $v, w \in F$ cannot have directed path P_F v to w in F, outward edge v to $G \setminus F$ & inward edge $G \setminus F$ to w



Other Examples w/ Hasse Diagram Property

- Hypercube $\rightsquigarrow$ Lattice of subsets of $\{1, \ldots, n\}$
- Permutahedron $\rightsquigarrow$ Weak order
- Associahedron $\rightsquigarrow$ Tamari Lattice
- Generalized Associahedra $\rightsquigarrow$ Cambrian Lattices

Next: Directionally Simple Polytopes

- Bruhat Interval Polytopes (image of moment map on nonneg part of Schubert variety)

Bruhat Interval Polytopes

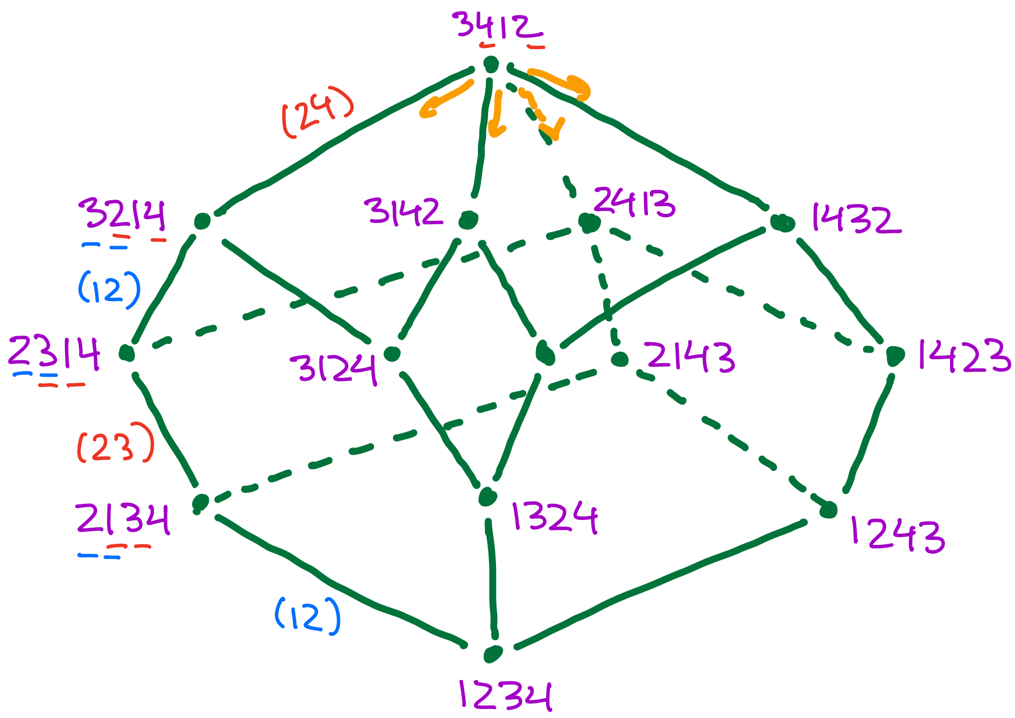
For $u <_{\text{Bruhat}} v$, let

$$Q_{u,v} := \mathrm{Conv}\left(\{z \in \text{Bruhat} \mid u \leq z \leq v\}\right)$$

$$Q_v := Q_{e,v}$$

$P_v :=$ poset on 1-skeleton of Q_v

$P_{3412} =$



- not simple polytope e.g. have 4 edges downward from 3412 (top)

A polytope P with cost vector $\vec{c}$ s.t. $G(P,\vec{c})$ is Hasse diagram of poset is directionally simple if $\forall u \in P$ and all collections $a_1, \ldots, a_r$ of poset elements covering u, there exists r-face of P containing $u, a_1, \ldots, a_r$

Thm (Goetz): Q_w is directionally simple w.r.t. cost vector consistent with Bruhat order.

Thm (Goetz): P_w is lattice.

Qn (Stanley): What is $\mu_{P_\omega}(u,v)$?

Thm (Gaetz-H.): $\mu_{P_\omega}(u,v) = 0$ or $(-1)^a$ where $a = \#$atoms of $[u,v]$ depending whether $[u,v]$ is face of P_ω. Moreover, $\Delta(u,v) \simeq$ ball or S^{a-2}.

Method: Generalize results from simple to "directionally simple" polytopes.

Thm (Gaetz-H): If P is directionally simple polytope & $G(P,\vec{c})$ is Hasse diagram of lattice L, then:

(a) $\mathrm{psj}(a_1, \ldots, a_i) = a_1 \vee \ldots \vee a_i$ for $u \in L$ covered by $a_1, \ldots, a_i$

(b) $\Delta(u,v) \simeq$ ball or sphere $\forall u < v$

Bridge Polytopes & Plabic Graphs
(& a Result on Higher Connectivity)

Lauren Williams:

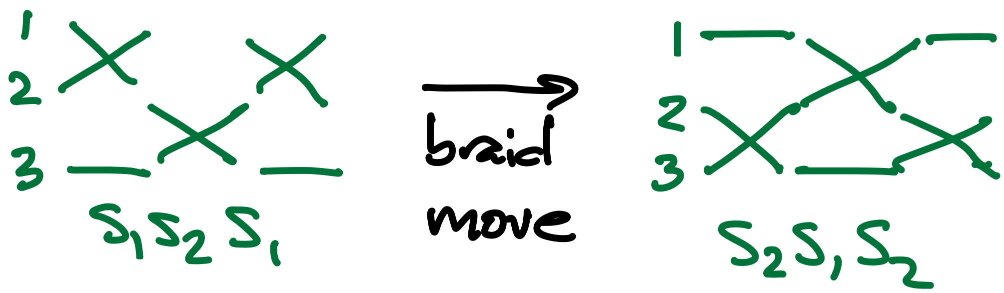
- "Bridge polytopes" all isomorphic to Bruhat interval polytopes Q_w for $w \in S_n$ a Grassmannian perm. (i.e. perm w/ at most one descent)
- Maximal chains in such $P_w \longleftrightarrow$ BCFW bridge decompositions of w
- Moves across 2-faces $\longleftrightarrow$ series of Postnikov moves on plabic graphs

Motivation: Postnikov theory of plabic graphs w/ equiv classes (under moves) labeling positroid strata in cell decomp of $Gr(k,n)_{\geq}$

Matsumoto's Theorem: Any two reduced expressions for same permutation connected by braid moves & commutation moves

e.g. $s_3 s_2 s_3 s_1 s_4 s_2 \to s_2 s_3 s_2 s_1 s_4 s_2$

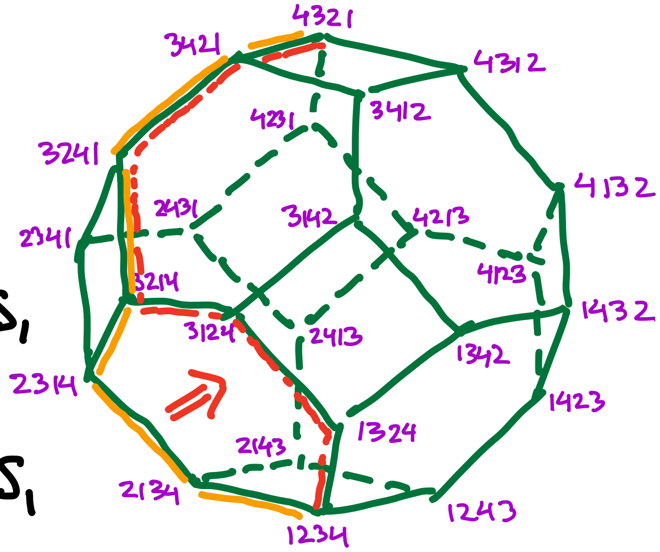
where $s_i = (i, i+1)$



Polytope Version

- reduced expressions $\leftrightarrow$ max chains in weak order
- braid moves & commutation moves $\leftrightarrow$ flips across 2-faces of permutahedron

e.g.



$s_1 s_2 s_1 s_3 s_2 s_1$

$\Downarrow$

$s_2 s_1 s_2 s_3 s_2 s_1$

Thm (Athanasiadis-Edelman-Reiner): Graph of reduced expressions for $w \in S_n$ connected by braid & commutation moves is at least $(a-1)$-connected where $a = \#\text{atoms of } [e, w]$.

Idea: Prove more generally for "flip graphs" of simple polytopes

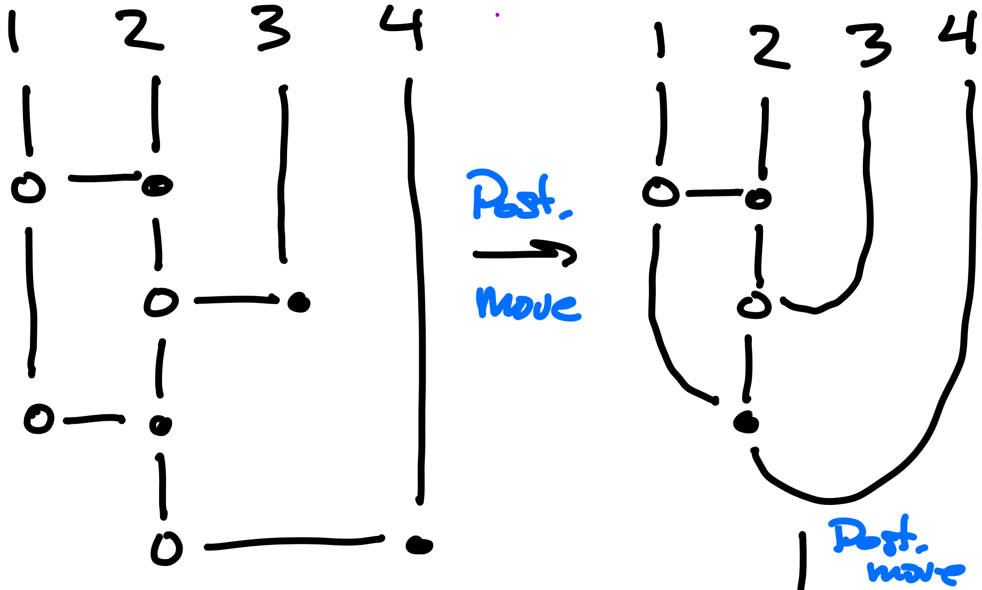
Thm (Gaetz-H.): Extension of AER to directionally simple polytopes.

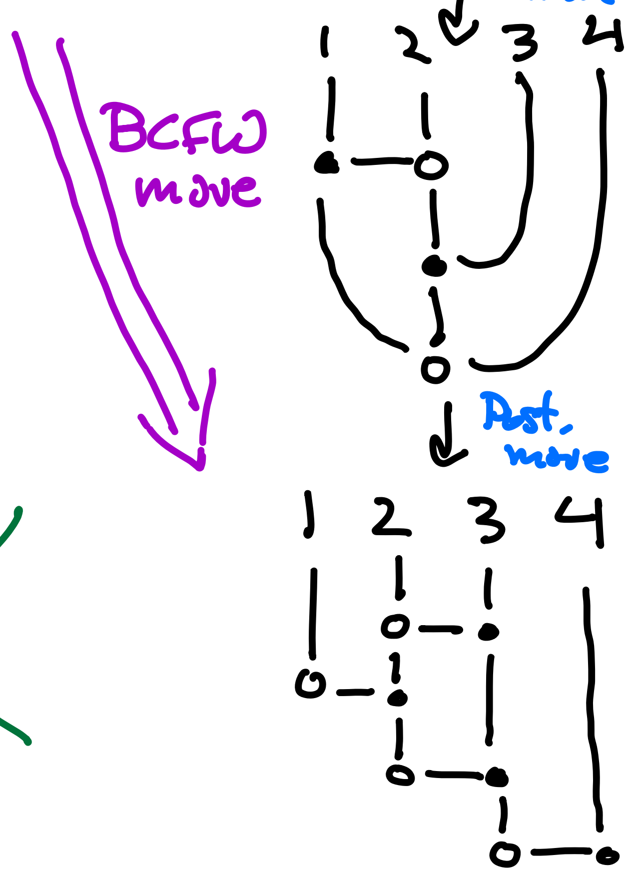
Thm (Gaetz-H.) For Bruhat interval polytopes, the graph whose vertices are poset max'l chains & whose edges are moves across 2-faces is at least $(a-1)$-connected.

Cor (Gaetz-H.) The graph of BCFW bridge decompositions of plabic graphs with edges given by extended Postnikov moves is at least $(a-1)$-connected where $a = \#$ atoms of P_w.

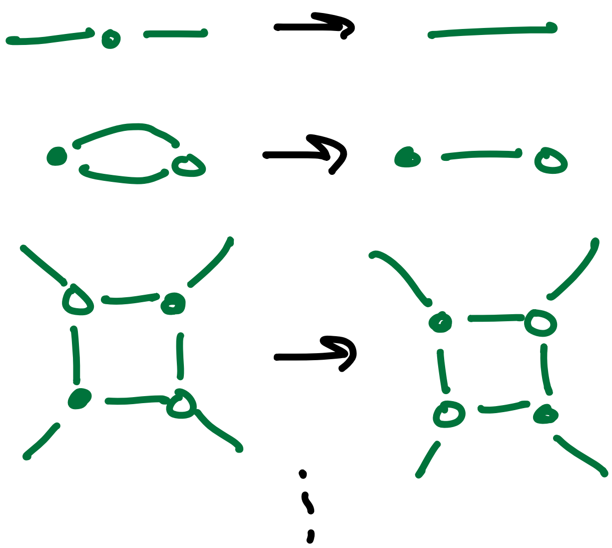
BCFW Bridge Decomposition "Move"

e.g.





Postnikov moves:



Thm (H.): For P a simple polytope & $\vec{c}$ a generic cost vector such that $G(P,\vec{c})$ is the Hasse diagram of a finite lattice L,

$$\Delta(\underbrace{u,v}_{\{z\in L \mid u<z<v\}}) \simeq \text{ball or sphere } S^{|A(u,v)|-2}$$

$A(u,v) := \{a \in L \mid u \lessdot a < v\}$ = "atoms" of $[u,v]$

Idea: Use poset map

$$f(x) = \bigvee_{\substack{a_i \in A(u,v) \\ a_i \le x}} a_i = \text{pseudojoin of } \{a_i \mid a_i \in A(u,v),\ a_i \le x\}$$

by "join = pseudo-join" theorem

then use distinctness of pseudojoins so image = Boolean lattice w/ or without maximal element

Quillen Fiber Lemma (a.k.a. Quillen Theorem A): Given a poset map $f: P \to Q$ s.t. $q \in Q$ implies $\Delta(f^{-1}_{\leq}(q)) = \Delta(\{p \in P \mid f(p) \leq q\})$ is contractible, then $\Delta(P) \simeq \Delta(Q)$.

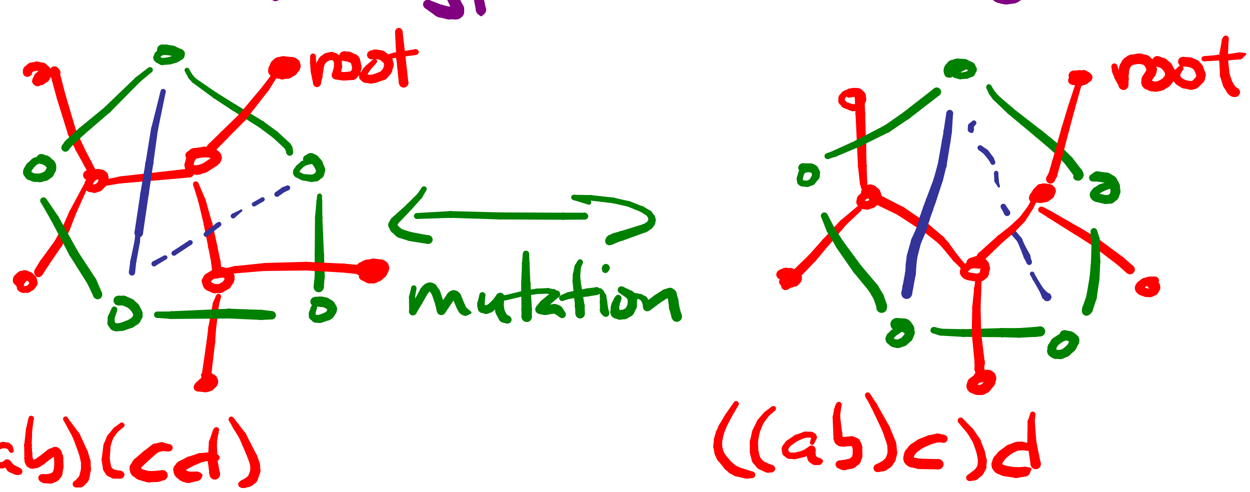
Remark: Used extensively in finite group theory (to characterize groups G via subgroup lattice $L(G)$) & in topological combinatorics.

e.g. **Thm** (Shareshian):
G solvable $\iff L(G)$ shellable

Thm (Stanley):
G supersolv. $\iff L(G)$ supersolv.

Applications: Unified proof for

- permutahedra ⇝ weak order
- associahedra ⇝ Tamari lattice
  (a.k.a
  Stasheff polytopes)
- Bruhat interval polytopes
- generalized associahedra ⇝ Cambrian lattices
  (related to
  cluster algebras
  of finite type & mutation)



Some Further Questions

<u>Qn 1</u>: Does P simple + $G(P,\vec{c})$ Hasse diagram of lattice $\Rightarrow$ no directed path can revisit face it has departed? (If not, variations?)

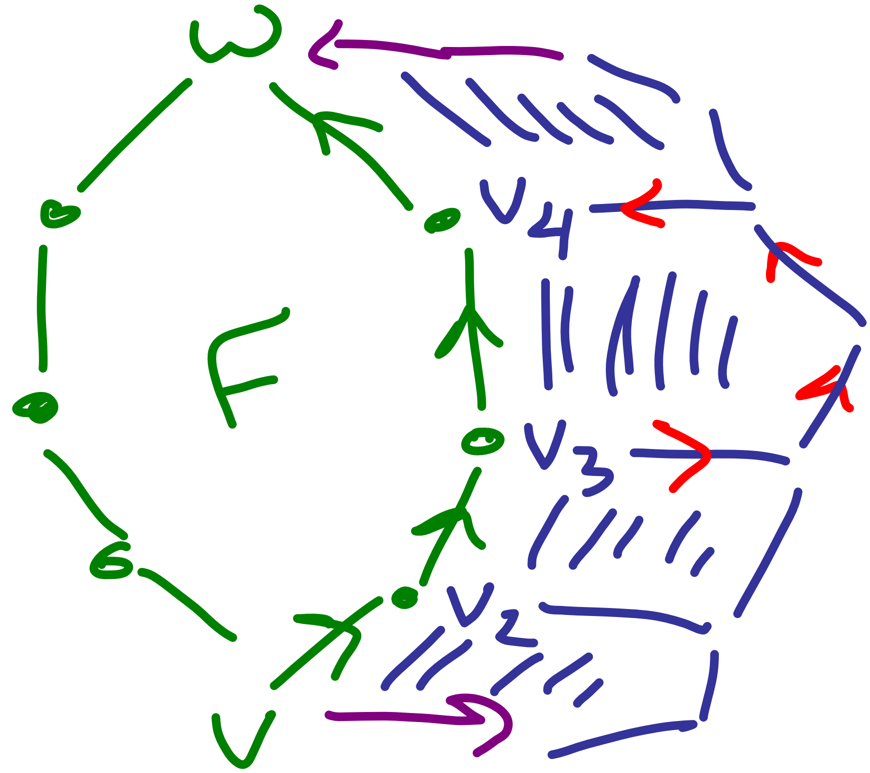
<u>Qn 2</u>: Variations on hypotheses? Non-simple polytopes? Non-lattices?

<u>Qn 3</u>: Structure of posets of joins/pseudo-joins for non-simple polytopes?

<u>Qn 4</u>: Use theory of fiber polytopes & coherent paths to explain/generalize?

Thanks!

Corollary: Monotonicity of out-degrees & partic. outward directions.



} violate Hasse diagram (at switch from out to in)

Corollary: For each face $F \subseteq P$ with $\hat{0} \in F$ or $\hat{1} \in F$, directed paths cannot revisit F after departing from it.

Lemma (H.): For P a simple polytope & $\vec{c}$ generic cost vector s.t. $G(P,\vec{c})$ is Hasse diagram of poset L, let $S,T \subseteq \{a_1,\dots,a_n\}$ be distinct sets of atoms. Then $\text{pseudojoin}(S) \neq \text{pseudojoin}(T)$.

For L a lattice, this also holds for atoms in each interval $[u,v] \subseteq L$.

Cor: Subposet of pseudojoins is isomorphic to Boolean lattice of subsets of $\{a_1,\dots,a_n\}$.

Idea for Distinctness of Pseudo-Joins

(1) Reduce to $S \subsetneq T$ with $|T| = |S| + 1$

- $S_1 \subsetneq S_2 \subsetneq S_3$ with $psj(S_1) = psj(S_3)$ $\Rightarrow psj(S_2) = psj(S_3)$
- $S_1 \not\subseteq S_2$ & $S_2 \not\subseteq S_1$ then use $S_1 \cap S_2 \subsetneq S_1$ with $psj(S_1 \cap S_2) = psj(S_1)$

(2) Use codim one nonrevisiting lemma

(3) For $[u,v]$, use that v is an upper bound for the atoms of $[u,v]$ w/ l.u.b. in $[u,v]$

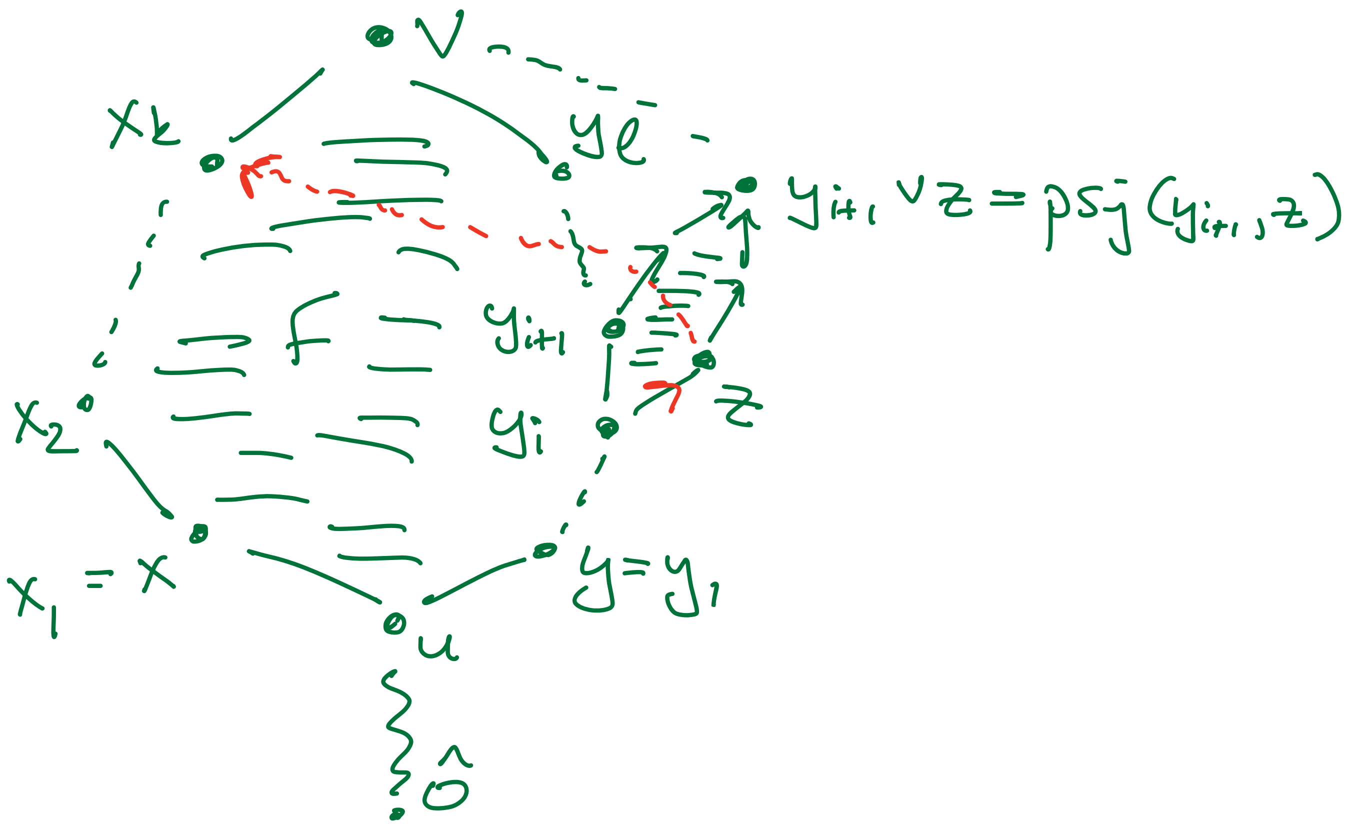
Theorem (Dominik Preuß): If P is simple polytope & $G(P,\vec{c})$ is Hasse diagram of lattice L, then $(P,\vec{c})$ satisfies monotone Hirsch conjecture.

Idea: (1) A vertex $v \in P$ is the unique source of a facet $\Longleftrightarrow$ $v=\vec{0}$ or v is a join irred. of L.

(2) Each cover relation $u_i \lessdot u_{i+1}$ in L has $u_{i+1} = u_i \vee j$ for a join irred. j

(3) Length of longest saturated chain $\leq$ # join irreducibles = # facets $-\dim(P)$

Idea for "join = pseudo-join" 2-dim'l case



- let $d(u)$ = length of longest path $\hat{0}$ to u
- choose 2-face f with source u with $d(u)$ as large as possible s.t. f has vertices x, y both covering u s.t.

  $x \vee y \neq$ pseudo-join of x & y
- There exists x_i or y_i with edge exiting & revisiting f. Choose i as large as possible
- prove x_{i+1} or y_{i+1} also has property $\Rightarrow\Leftarrow$