

Crystal Graphs &
SB-labelings

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(joint work with Cristian
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Perspective & Main Goals:

1. Study crystal graphs regarded as posets via poset map to weak Bruhat order, namely via the (right) key map. (w/ Lenart)

Poset Structure for Many Crystals

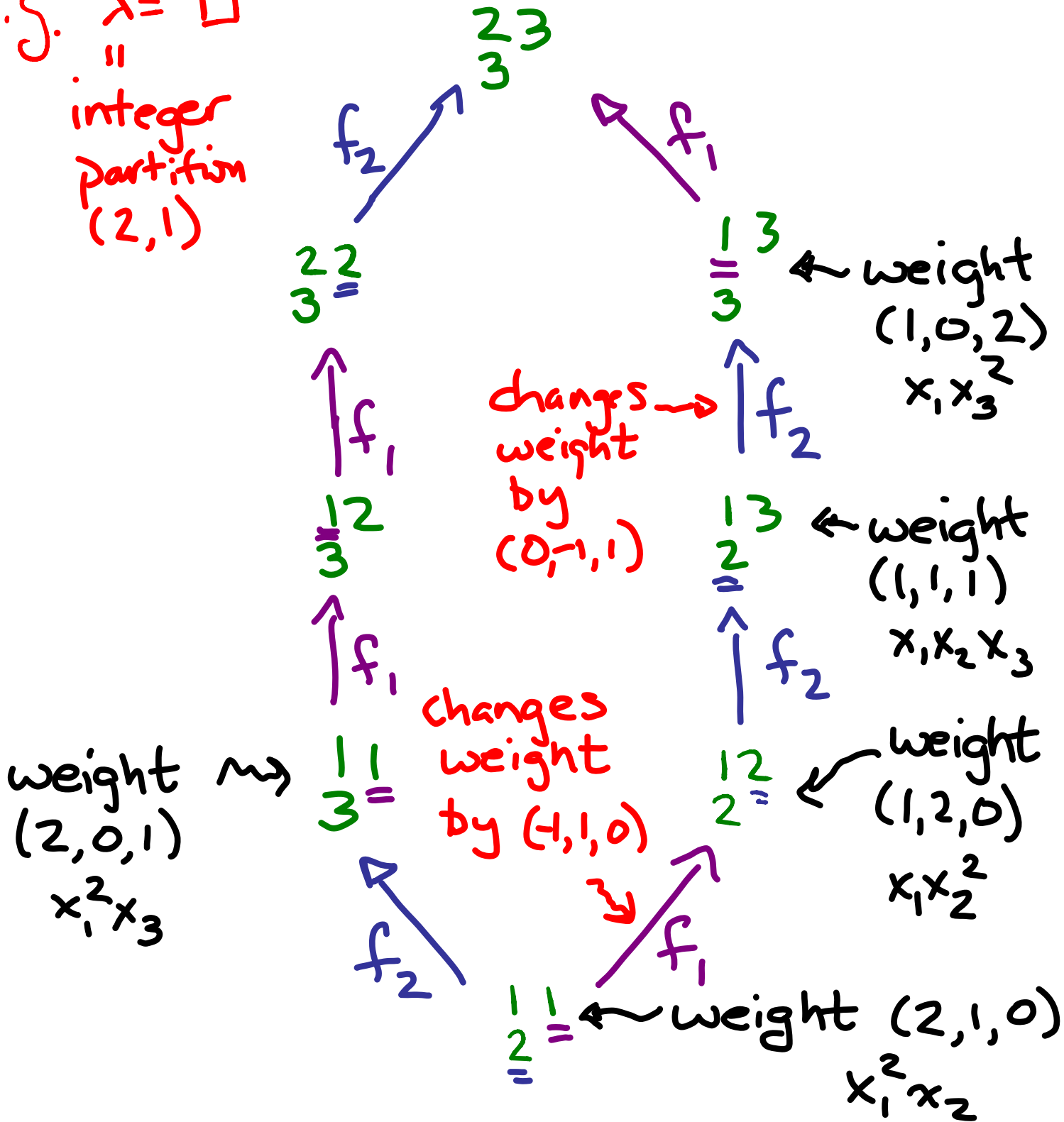
$$u \prec_{\text{crystal}} v \iff u \xrightarrow{f_i} v \text{ for some } i$$

↑ "cover relation" i.e. $u \prec v$ s.t. $\nexists z$ with $u \prec z \prec v$

2. Introduce SB-labelings for finite lattices. Show SB-labeling $\Rightarrow M(u, v) \in \{0, \pm 1\}$ for all $u \leq v$ (w/ Meszáros)
3. Discover surprising new relns amongst crystal operators via theory of SB-labelings (w/ Lenart)

(Type A) Crystals of Highest Weight Rep's (w/ Kashiwara Lowering Operators f_1, f_2, \dots)

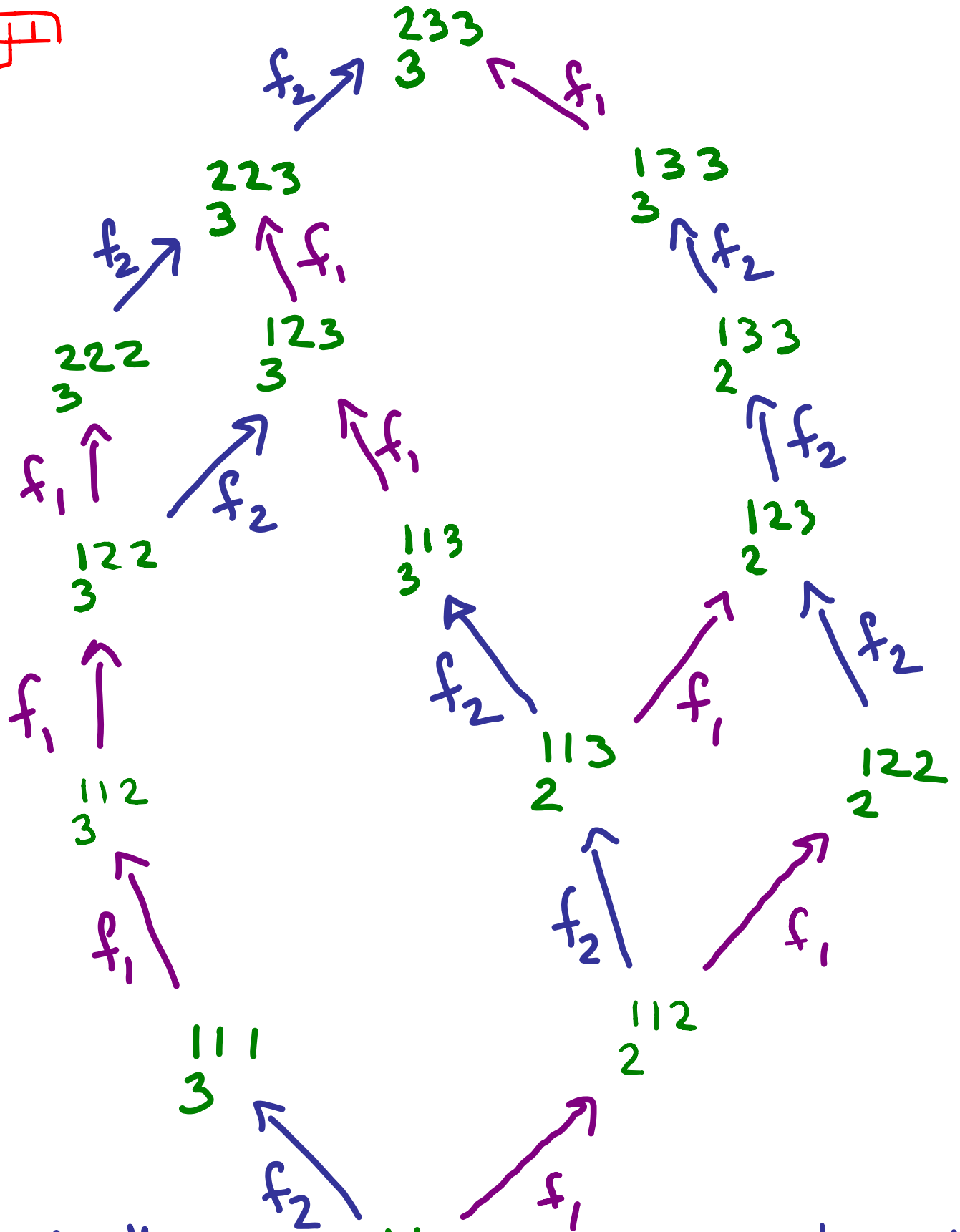
e.g. $\lambda = \begin{array}{|c|} \hline \square \\ \hline \end{array}$
 " integer
 partition
 (2,1)



Motivations for Crystal Graphs

- Study rep'n theory of Kac-Moody algebras (e.g. affine Lie algebras) A by passing to univ. env. alg. $U(A)$
 - \neq quantized algebra w/ parameter q
- $q \rightsquigarrow 1$ yields $U(A)$
- $q \rightsquigarrow 0$ yields alg. w/ same dims of weight spaces in rep'ns, encoded in "crystal graphs"
- poset elts \leftrightarrow basis vectors for the various weight spaces (guaranteed to exist by crystal basis properties)
- cover rel'ns \leftrightarrow crystal (lowering) operators

$$\lambda = \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}$$



"character"
of crystal

$$= x_1^3 x_2 + x_1^2 x_2^2 + \dots = \text{weight} \left(\begin{array}{|c|c|c|} \hline \text{---} & & \\ \hline \text{---} & & \\ \hline \text{---} & & \\ \hline \end{array} \right) + \text{weight} \left(\begin{array}{|c|c|c|} \hline \text{---} & & \\ \hline \text{---} & & \\ \hline \text{---} & & \\ \hline \end{array} \right) + \dots$$

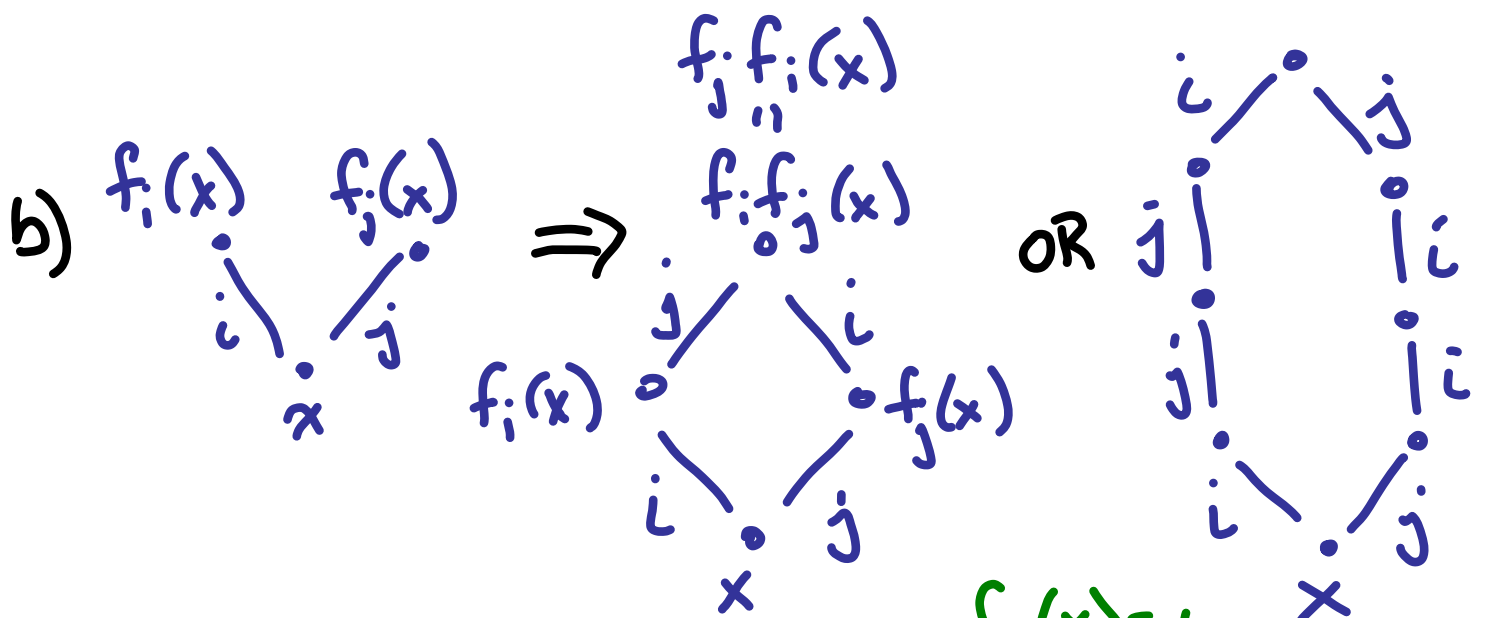
highest wt vector $(3,1,0)$

character of rep'n
"

Stanbridge Crystals: "g-crystals"

(Crystals of highest weight reps
in simply laced case)

a) $\chi_{B(\lambda)}(t) = \sum_{b \in B(\lambda)} t^{\text{wt}(b)}$ = character of irrep $B(\lambda)$
 (e.g. Schur fn's in type A)



c) likewise for e_i, e_j
 "raising operators":

$f_i(x) = y$
 $f_i \uparrow \downarrow e_i$
 $x = \dot{e}_i(y)$

d) rel's depend on location

Type A crystal for highest weight rep'n of shape λ

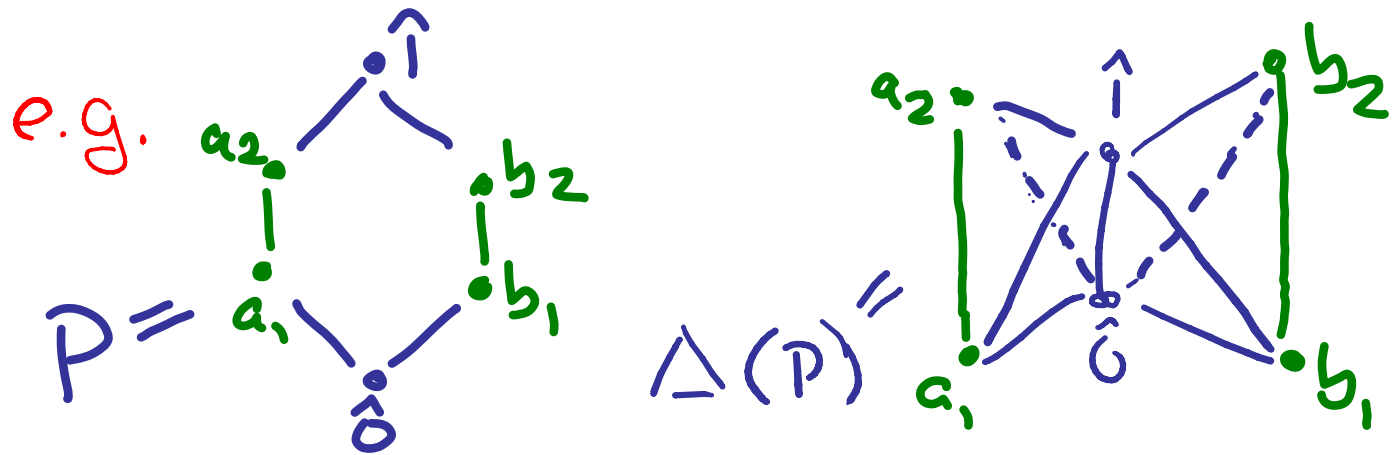
1. $\hat{O} = \begin{matrix} 111 \dots 1 \\ 22 \dots 2 \\ 33 \dots \\ \vdots \end{matrix}$ of shape λ
 ← "highest weight vector"

2. $u \xrightarrow{i} v$ has v obtained from u by incrementing to $i+1$ rightmost i not in "parenthesization pair" with an $i+1$

e.g. $\begin{matrix} 1111444 \\ 2233 \\ \boxed{3}44 \end{matrix} \xrightarrow{f_3} \begin{matrix} 1111444 \\ 2233 \\ \boxed{4}44 \end{matrix}$
 $\begin{matrix} \boxed{3}4433444 \\ \underbrace{\hspace{10em}} \end{matrix} \xrightarrow{\hspace{1em}} \begin{matrix} \boxed{4}4433444 \\ \underbrace{\hspace{10em}} \end{matrix}$

Parenthesization Pairs: Read leftmost column bottom to top, then subsequent columns, ignoring all but i 's & $i+1$'s; pair up consec. $i+1, i$; delete; repeat...

Def'n: The **order complex** (or **nerve**) of a poset P is the abstract simplicial complex $\Delta(P)$ whose i -dim'l faces are the $(i+1)$ -chains $v_0 < v_1 < \dots < v_i$ in P .



Recall: $M_P(u, v) = \hat{\chi}(\Delta(u, v))$
 =
 subposet $\{z \in P \mid u < z < v\}$

(Recursive)

Def'n: $M_P(u, u) = 1$

$$M_P(u, v) = - \sum_{u \leq z < v} M_P(u, z)$$

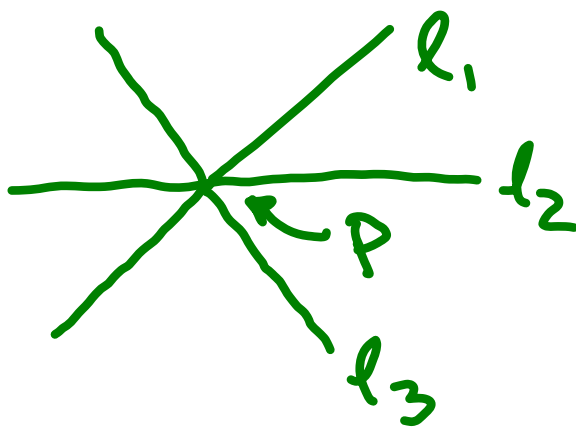
"open interval"
 (u, v)

Intuition for μ (via an Example)

e.g. "counting" points in the \mathbb{R}^2

complement of \rightsquigarrow

yields:

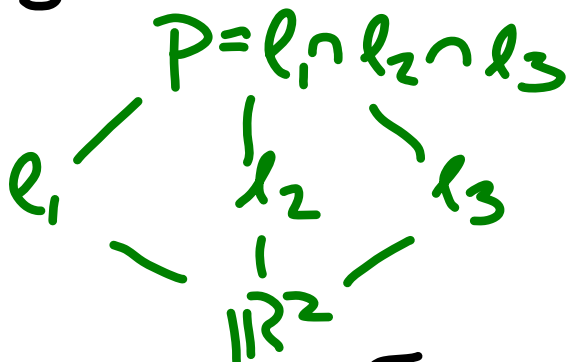


$$\mathbb{R}^2 - l_1 - l_2 - l_3 + 2P$$



- Coefficients $1, -1, -1, -1, 2$ in such inclusion-exclusion counting formula given by Möbius function $\mu(\mathbb{R}^2, -)$

in

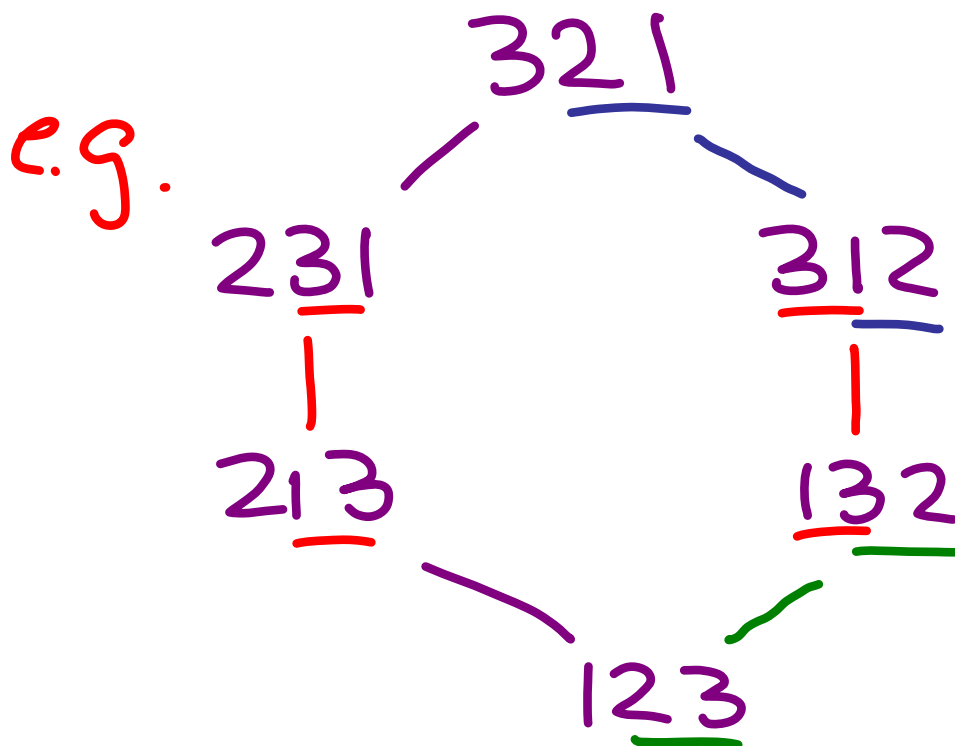


- Working over \mathbb{F}_2 , really do get $\mathbb{F}_2^2 - \mathbb{F}_2 - \mathbb{F}_2 - \mathbb{F}_2 + 2$

Weak Bruhat Order: A Partial Order on Permutations

$u < v$ iff u obtained from v by adjacent transposition

$S_i = (i, i+1)$ sorting letters in consecutive positions i & $i+1$



(captures structure of bubble sorting)

More Formally:

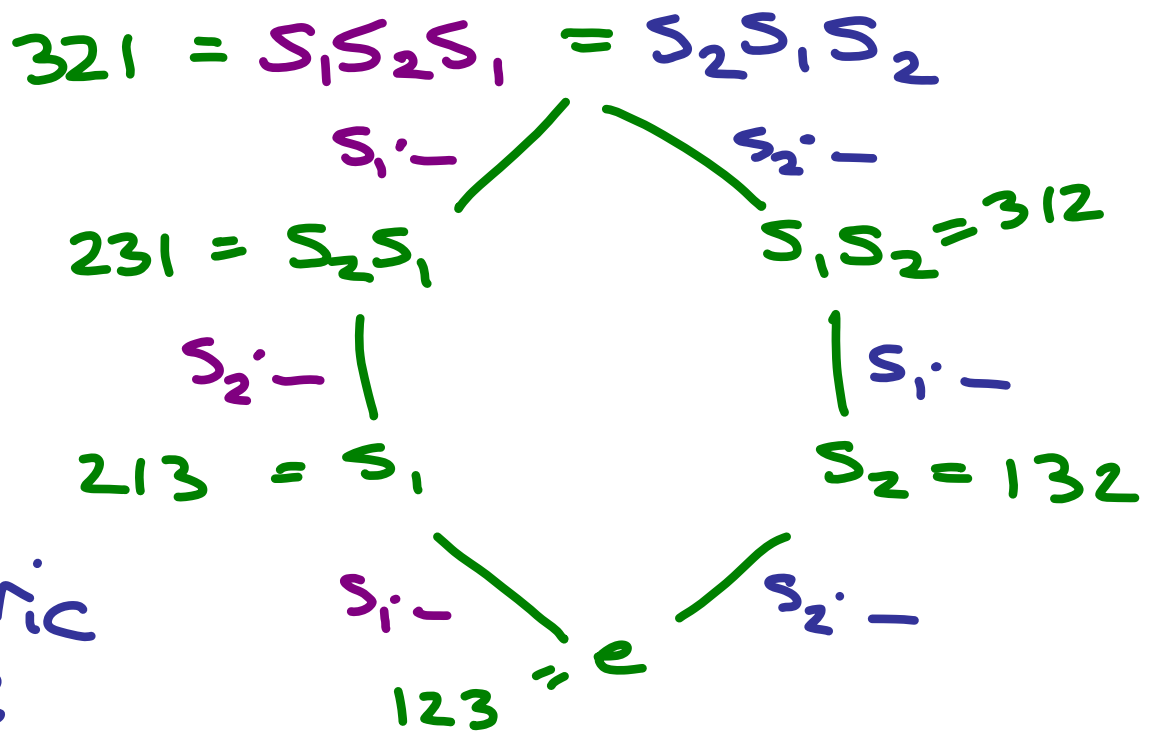
Def'n: The (left) weak order on Coxeter group W is partial order with $u < v \iff v = s_i \cdot u$ for $u, v \in W$ s.t. $\text{length}(v) > \text{length}(u)$ where $\text{length}(u) := \min \{r \mid u = s_{i_1} \dots s_{i_r}\}$ for s_{i_1}, \dots, s_{i_r} "simple reflections"

e.g.

S_3

and other

symmetric groups:



e.g. $W = S_n$
 $S = \{s_1, s_2, \dots, s_{n-1}\}$ for $s_i = (i, i+1)$

with relations:

$$s_i^2 = e \quad \& \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \& \quad s_i s_j = s_j s_i$$

(for $|j-i| > 1$)

"braid relns"

Note:

- Saturated chains from e to w
 \Downarrow
 "reduced expressions" $s_{i_1} \dots s_{i_l(w)}$ for w .
- Likewise saturated chains u to v
 \Downarrow
 reduced expressions for $v u^{-1}$

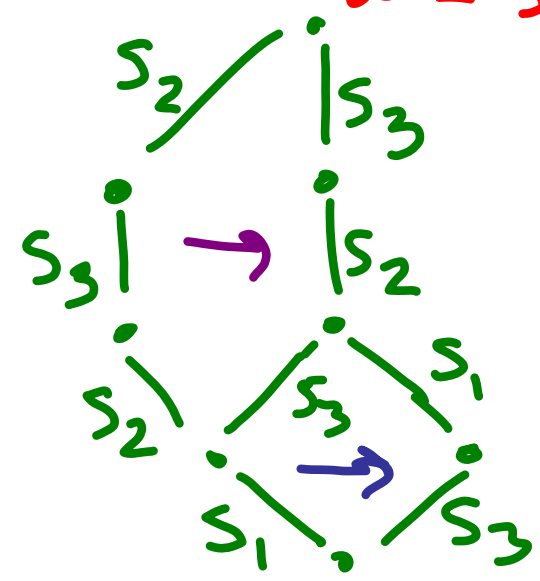
Connectadness under Braid Moves

Thm (see e.g. Björner-Brenti book): Let (W, S) be Coxeter system; let $w \in W$. Then every two reduced expressions for w are connected via braid moves.

e.g. $s_2 s_3 s_2 s_1 \rightarrow s_3 s_2 s_3 s_1$
 $\rightarrow s_3 s_2 s_1 s_3$

$w = s_2 s_3 s_2 s_1 = s_3 s_2 s_3 s_1 = s_3 s_2 s_1 s_3$

left weak order:




Homotopy Type for Weak Order

Thm (Edelman, Björner): Each $\Delta(u,v) \simeq$ ball or sphere.

A proof: Use "poset map" [a map f s.t. $u \leq v \Rightarrow f(u) \leq f(v)$] to reduce to $\Delta(\bar{B}_n) = \text{scl}(\partial \Delta_{n-1}) \simeq S^{n-2}$
poset of proper subsets of $\{1, 2, \dots, n\}$

via $f(u) = \max \{ T \mid \omega_0(\omega_T) \leq u \}$
longest element parabolic subgroup

! use "Quillen fiber lemma" (a.k.a. "Quillen Theorem A")

Recall Quillen fibre lemma: Given a poset map $f: P \rightarrow Q$ s.t. $g \in Q \Rightarrow \Delta(\{p \in P \mid f(p) \leq g\})$ is contractible, then $\Delta(P) \simeq \Delta(Q)$. 
(\neq dual version w/ $f(p) \geq g$)

Some Motivations & History for

Topology of Poset Order Complexes

1. Applications to finite group theory
(Quillen, Aschbacher, Brown, Shaveshlan, etc.)
2. Applic's to commutative algebra e.g. via isomorphism of poset order complex to bar complex (for particular posets)
e.g. $u < v < w \mapsto [u^{-1}v \mid v^{-1}w]$ for weak order

Outline for Remainder of Talk

II. New Def'n for (Right) Key Map (to weak order) (w/ Lenart)

III. Analogy w/ Weak Order for Lower Intervals in Crystals (w/ Lenart)

IV. Non-Analogous Results w/ Weak Order for Arbitrary Intervals (w/ Lenart)

- Arbitrarily large Möbius functions
- Arbitrarily high degree non-redundant "relations" amongst crystal operators

V. SB-labelings (w/ Mešzáros)

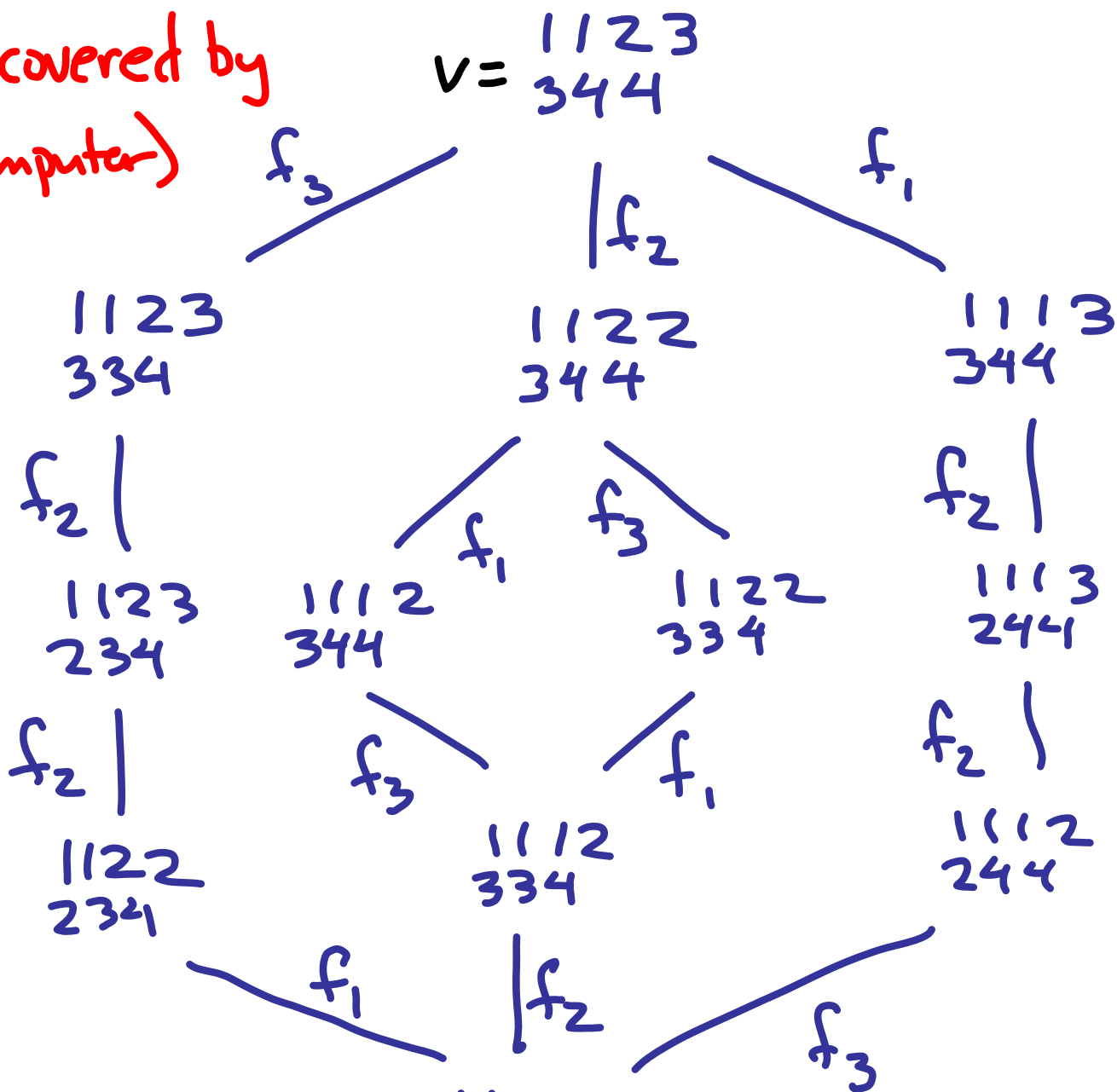
Applic: $M(u, v) \neq 0, \pm 1 \Rightarrow$ rel'n within $[u, v]$ not generated by Stembridge local rel'ns (w/ Lenart)

VI. More Examples, etc. (time permitting)

Examples with Unexpected(?) Structure

"Base Case":

(discovered by computer)



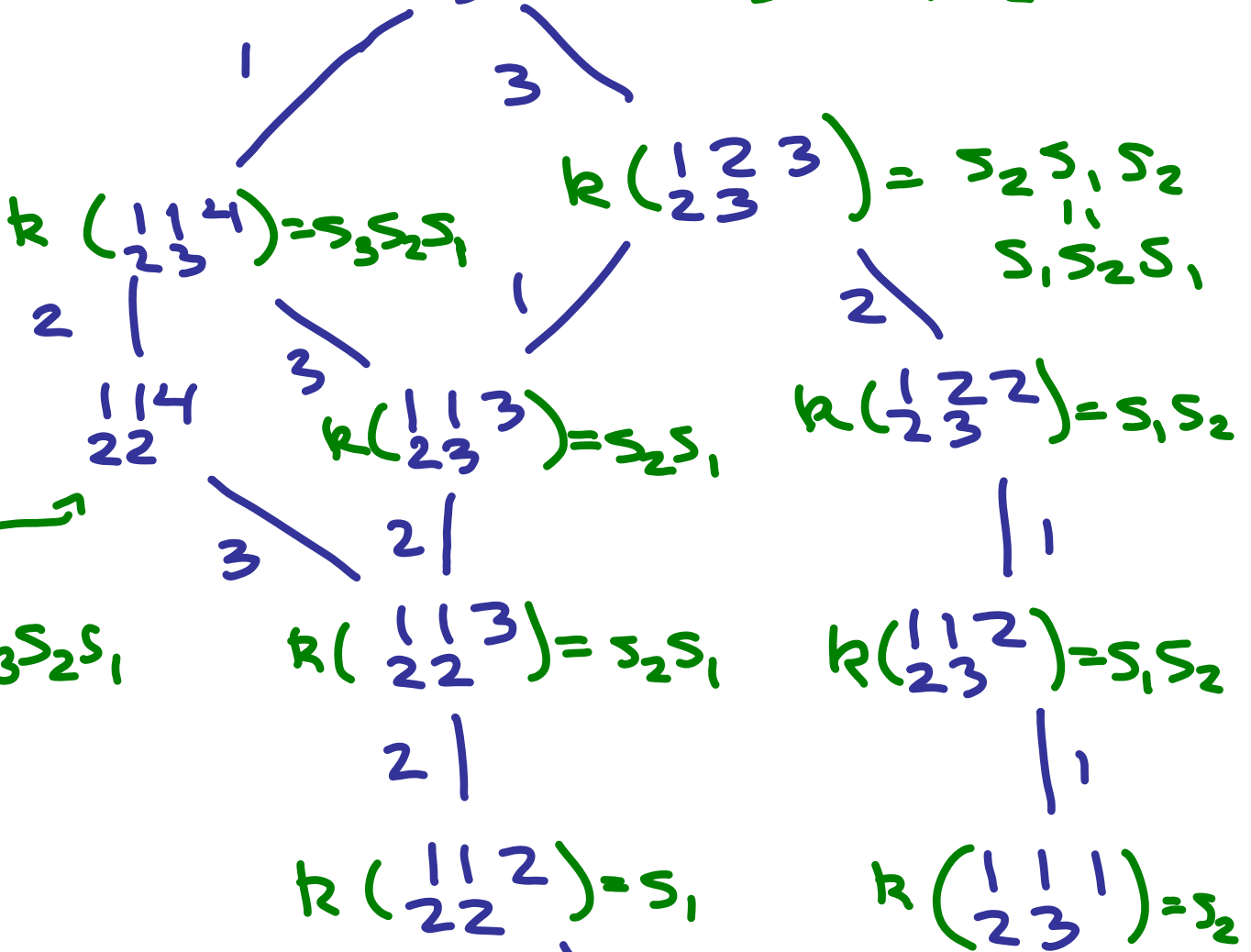
$M_p(u, v) = 2$

$u = \begin{matrix} 1112 \\ 234 \end{matrix}$

‡ not connected by "Steinbridge moves"

II. Right key "k" of a "KM"-crystal (related to Lascoux-Schützenberger key polynomials)

$$k \left(\begin{smallmatrix} 124 \\ 23 \end{smallmatrix} \right) = s_3 s_2 s_1 s_2$$



k: crystal poset \rightarrow weak Bruhat order

$$u \leq v \Rightarrow k(u) \leq k(v)$$

$\begin{smallmatrix} 111 \\ 22 \end{smallmatrix}$
 key $k(\hat{0}) = e$

New Algorithm to Calculate Right Key of a KM-Crystal

- (1) $\text{key}(\hat{\sigma}) = e$
- (2) if $\hat{\sigma} \xrightarrow{i} a$, then $\text{key}(a) = s_i$
(i.e. $\hat{\sigma} \leftarrow a$)
- (3) if v covers 2 or more elements
then $\text{key}(v) = \bigvee_{\{u \mid u \rightarrow v\}} \text{key}(u)$
(for join taken in weak order)
- (4) if $u \xrightarrow{i} v$ and v does not cover
any other elements, then:
 - (a) $\text{key}(v) = \text{key}(u)$ if $\exists u' \xrightarrow{i} u$
 - (b) $\text{key}(v) = s_i \cdot \text{key}(u)$ otherwise

Key Polynomials \neq right / left key

(see Lascoux-Schutzenberger \neq e.g.
Reiner-Shimozono)

- Motivations:
- (1) Schubert poly. G_w is positive sum of "key polynomials"
 - (2) Key polynomial records character for Demazure module
 - (3) The (closely related) right \neq left key maps determine smallest Demazure modules containing a given crystal element
 - (4) Key maps give poset maps from KM-crystal to weak order enabling transfer of properties.

III. Analogy with Weak Order for Lower Intervals in Crystals

Thm (H.-Lenart) Given u in a
symmetrizable Kac-Moody type crystal
"KM-crystal", then $M(\hat{\sigma}, u) = 0, \pm 1$ with
 $M(\hat{\sigma}, u) = 0$ unless $\text{key}(u) = \omega_0(J)$
for some parabolic subgroup W_J
with $u = \min\{z \mid z \in \text{key}^{-1}(\omega_0(J))\}$
in which case $M(\hat{\sigma}, u) = (-1)^{|J|}$.
Moreover, $\Delta(\hat{\sigma}, u) \simeq S^{|J|-2}$ or ball.

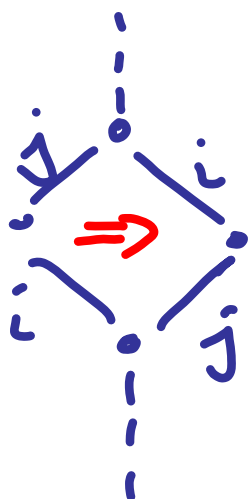
Proof: Quillen fibre lemma via

$f: \text{crystal} \rightarrow \text{Boolean algebra}$

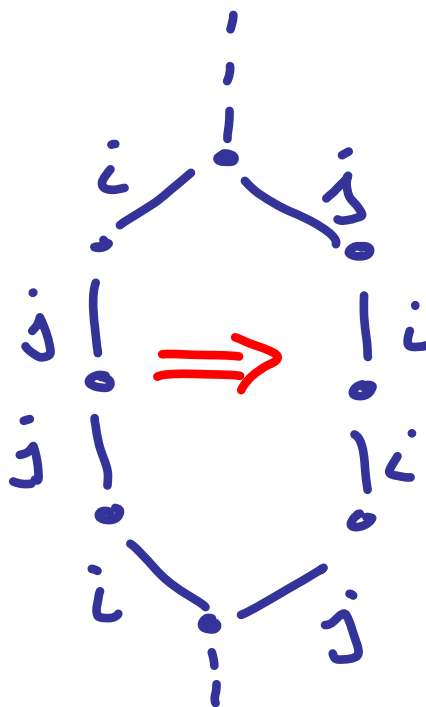
$x \mapsto \max\{S \mid \omega_0(J_S) \leq_{\text{weak}} \text{key}(x)\}$

Thm (H.-Lenart): Given any lower interval $(\hat{0}, u)$ in a γ -crystal, then set of saturated chains from $\hat{0}$ to u is connected by "Stambridge moves", namely moves of the

form

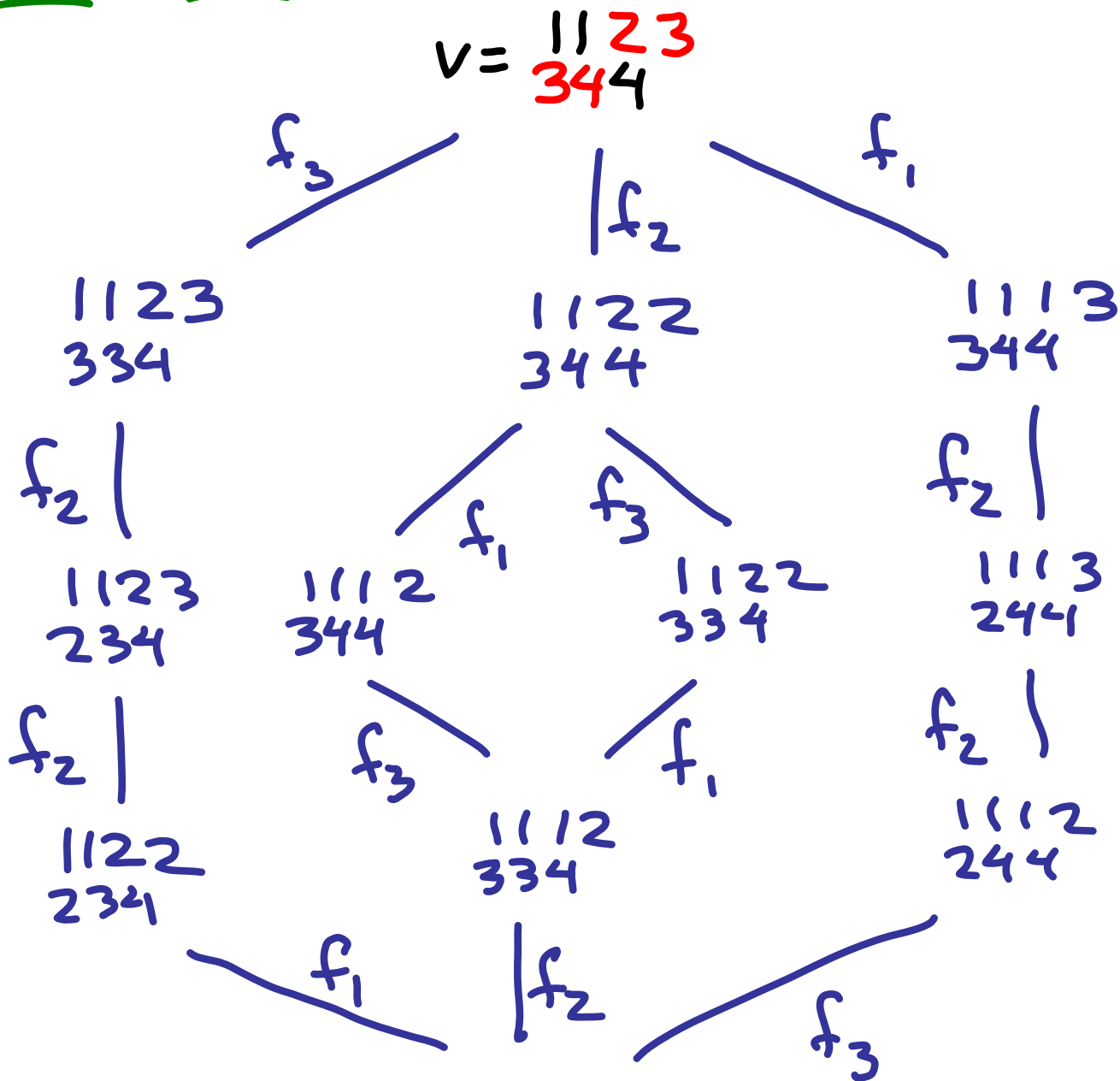


and



Note: Likewise in doubly-laced case via "Stamberg moves".

IV. Non-analogy to Weak Order for "Base Case": Arbitrary Intervals



$M_p(u, v) = 2$

$u = \begin{matrix} 1112 \\ 234 \end{matrix}$

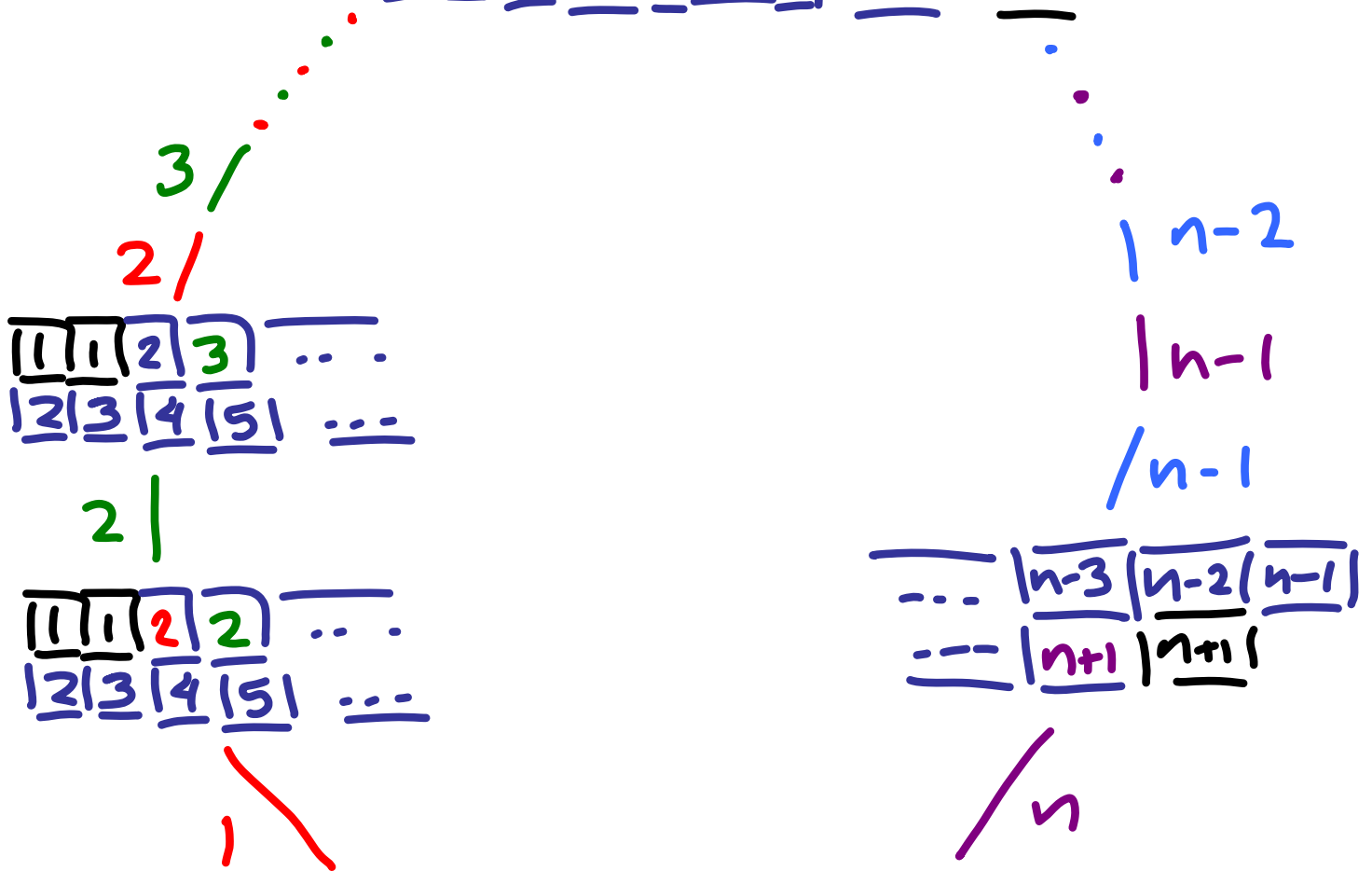
‡ not connected by "Steinbridge moves"

Arbitrarily High Rank

Disconnected Open Intervals

$$v = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline \end{array} \dots \begin{array}{|c|c|c|} \hline n-2 & n-1 & n \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 3 & 4 & 5 & 6 \\ \hline \end{array} \dots \begin{array}{|c|c|} \hline n+1 & n+1 \\ \hline \end{array}$$



$$u = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline \end{array} \dots \begin{array}{|c|c|c|} \hline n-3 & n-2 & n-1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 5 \\ \hline \end{array} \dots \begin{array}{|c|c|} \hline n & n+1 \\ \hline \end{array}$$

label sequences: $1, 2, 2, 3, 3, 4, 4, \dots, n-1, n-1, n$
 $\neq n, n+1, n+1, \dots, 2, 2, 1$ in distinct components

Consequence: Arbitrarily high degree relations $f_{i_1} \dots f_{i_d}(u) = f_{j_1} \dots f_{j_d}(u)$ amongst crystal operators applied to u not implied by any lower degree relations.

Systematic Method to Discover Such Unexpected Relations?

- Möbius functions, using the theory of SB-labelings.

Thm (H.-Lenart): There exist $u < v$ in type A g -crystals with $M(u, v) = 2^j$ for every positive integer j (hence arbitrarily large).

V. SB-labelings (w/ Mészáros)

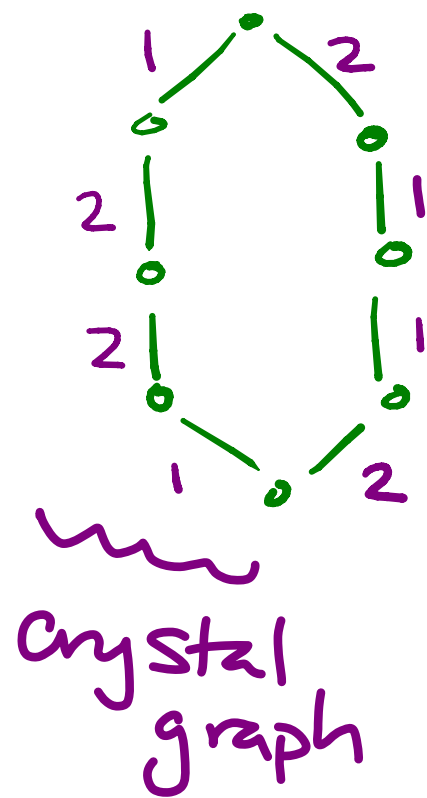
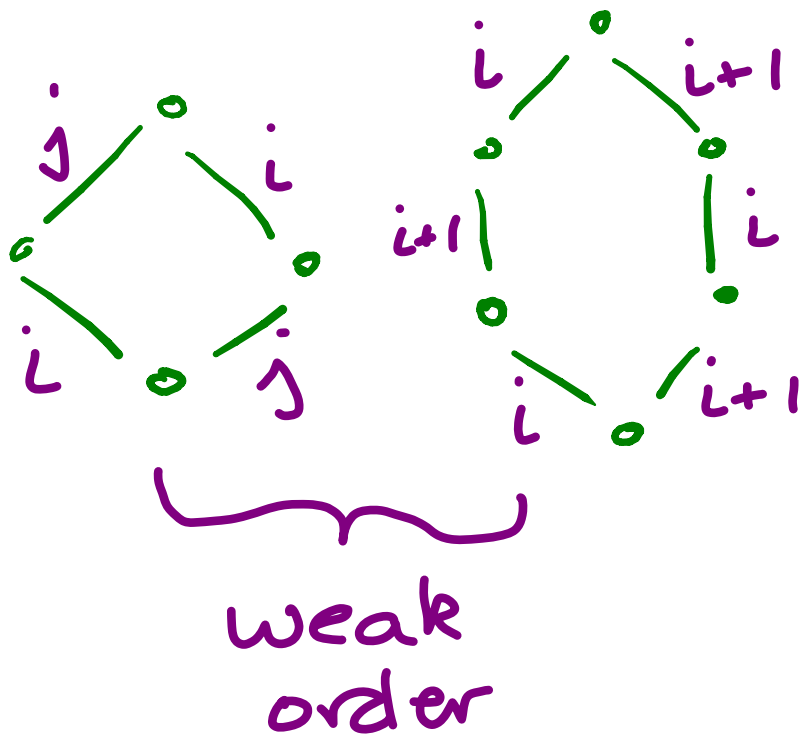
Def'n: Let λ be an edge labeling of a finite lattice L s.t. all $u \prec v \neq u \prec w$ in L with $v \neq w$ meet the conditions:

(1) $\lambda(u, v) \neq \lambda(u, w)$ for $v \neq w$

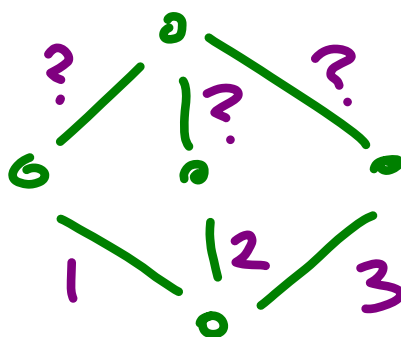
(2) Every saturated chain from u to $v \vee w$ uses each of the labels $\lambda(u, v) \neq \lambda(u, w)$ at least once, but does not use any other label.

Then λ is an **SB-labeling** (index 2 formulation).

e.g.



Non-Example:



Thm (H.-Meszáros): If finite lattice L has labeling λ that is SB-labeling, then $M(u, v) = 0, \pm 1$ for $u, v \in L$. $\Delta(u, v) \simeq$ ball or sphere.

Plan for crystals: Find $M(u, v) \neq 0, \pm 1$ & use that edge coloring is nearly SB-labeling to deduce properties of such intervals $[u, v]$

Examples (of Lattices with SB-labelings)

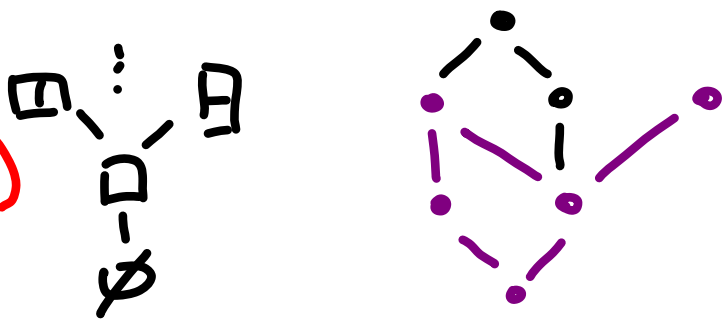
1. Finite distributive lattices

• Let $\lambda(S \leftarrow \cdot S \cup \{i\}) = i$ (for $L = J(P)$)

(including Young's lattice intervals)

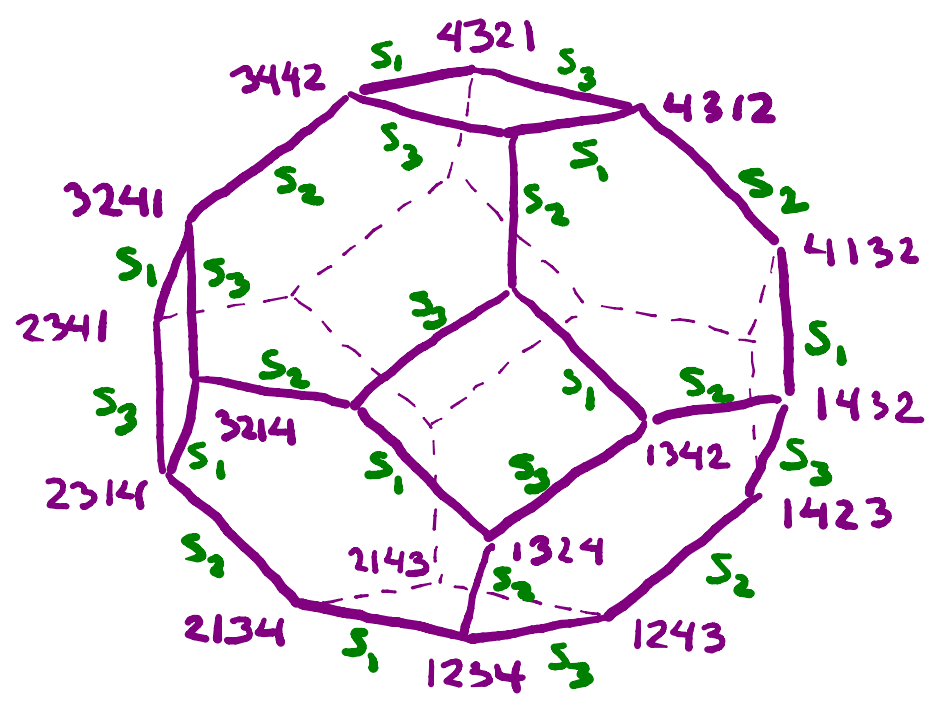
"order ideals"

"poset of order ideals in P"



2. Weak Bruhat Order

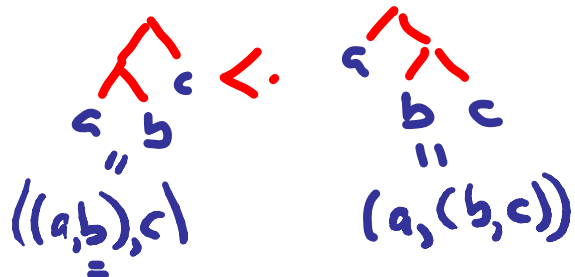
Idea: Use $\lambda(u \leftarrow \cdot s_i u) = s_i$



3. Tamari lattice (1-skeleton of Stasheff polytope/associahedron)

• poset of binary

trees w/n leaves



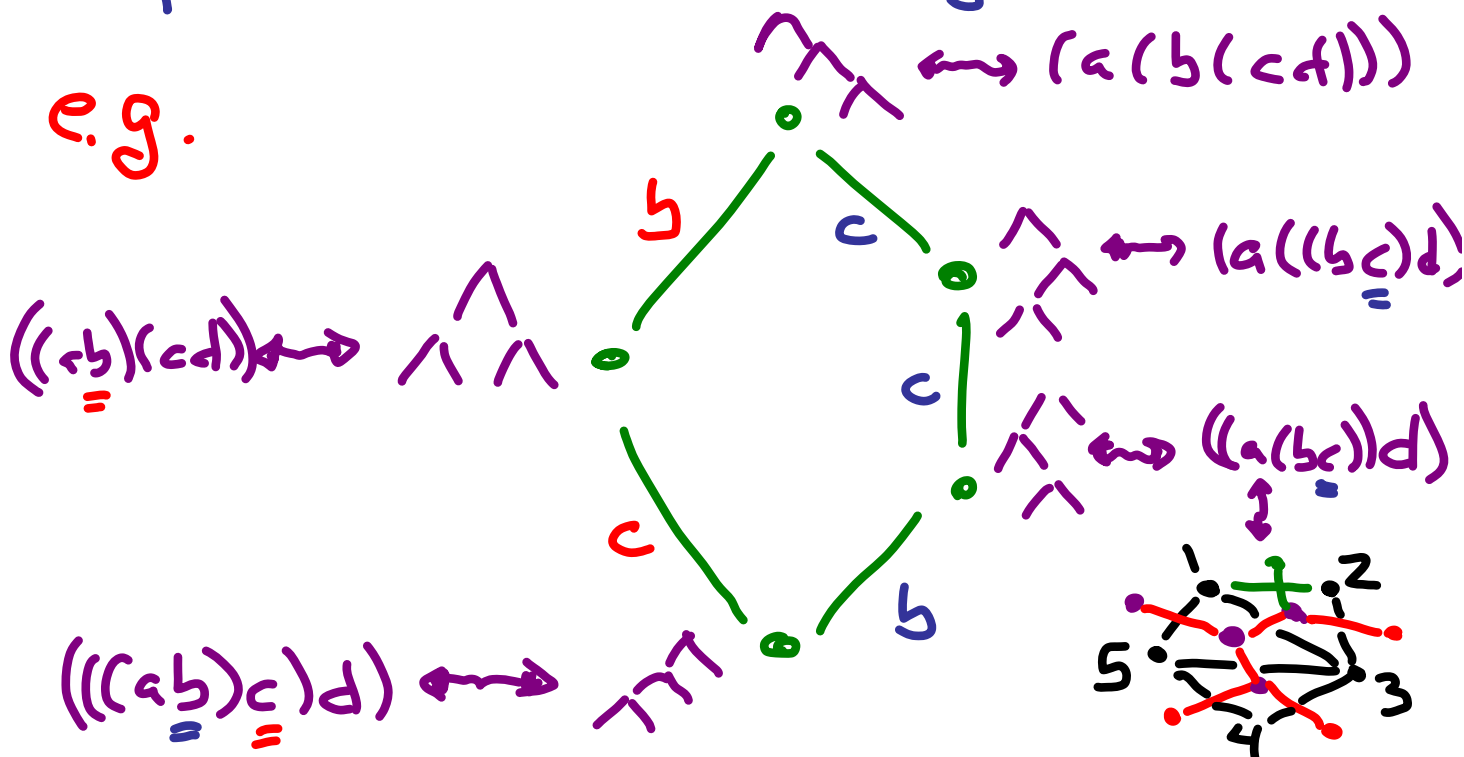
(or parenthesizations or triangulations)

• $\lambda(u \leftarrow v) :=$ letter to immediate left of right parenthesis

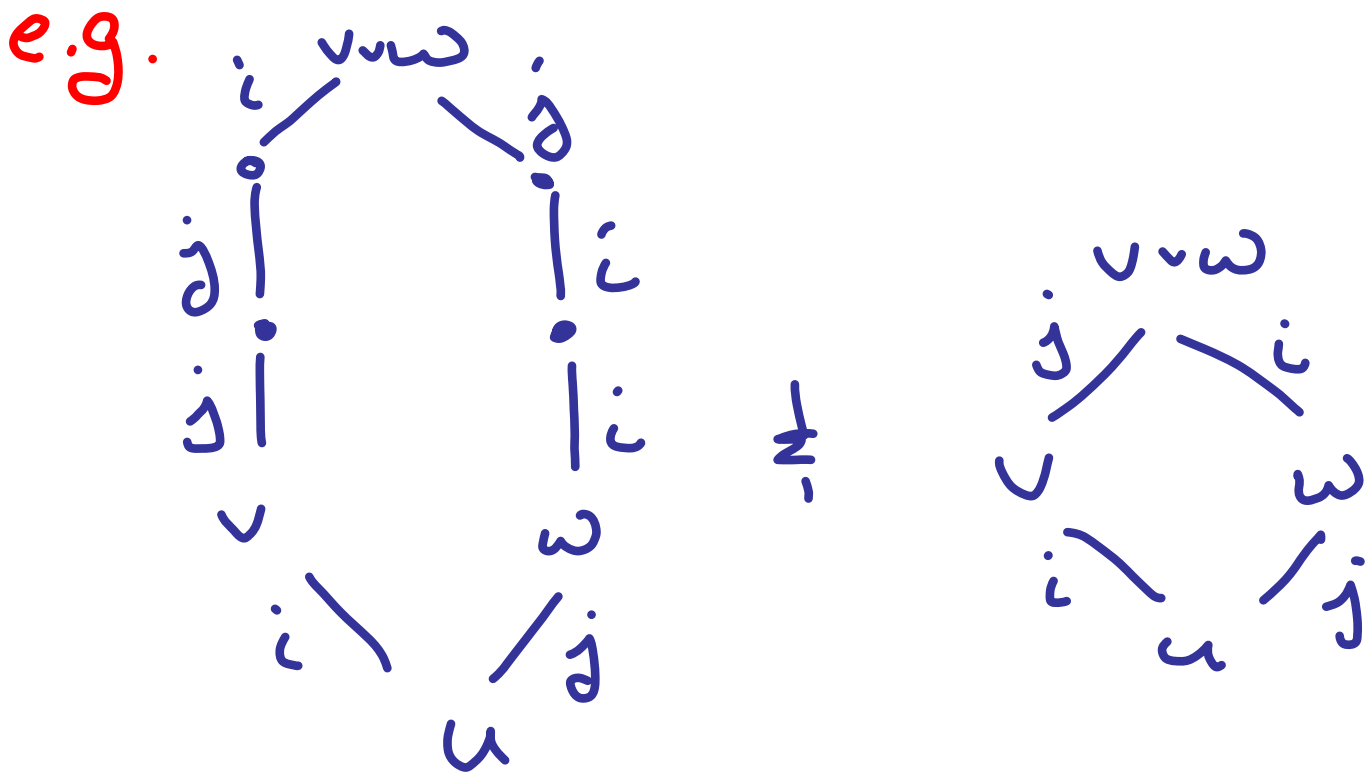
being moved (e.g. $\lambda(\wedge_b \leftarrow \wedge_b) = b$)

(nonpure lex. shellable - Björner & Wachs)

e.g.



Observation (H.-Lenart): If g -crystals were a lattice s.t. Stembridge local upper bounds were the least upper bounds for every $v \neq w$ s.t. $u \leftarrow v$ & $u \leftarrow w$, then edge coloring would be SB-labeling.



But $M(u, v) \neq 0, \pm 1$ precludes this!

Thm (H-Lenart): If g -crystal has $M(x,y) \neq 0, \pm 1$, then there exists rel'n amongst crystal operators within $[x,y]$ not implied by Stembridge local rel'ns.

VI. Further Examples, etc.

Non-lattice crystal

$$\begin{matrix} 1123 \\ 344 \end{matrix} = (uvv)_1$$

$$\begin{matrix} 1223 \\ 334 \end{matrix} = (uvv)_2$$

2 /

3 \

1 /

1 2

$$\begin{matrix} 1122 \\ 344 \end{matrix}$$

$$\begin{matrix} 1123 \\ 334 \end{matrix}$$

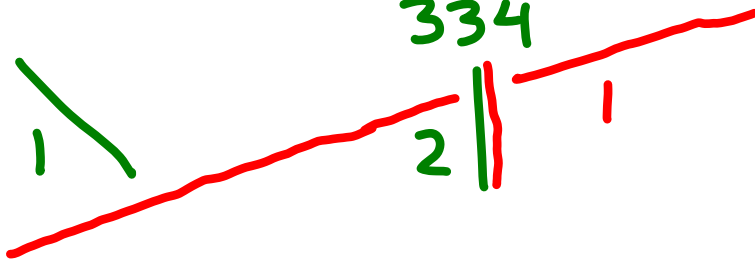
$$\begin{matrix} 1222 \\ 334 \end{matrix}$$

3 |

1 \

2 ||

1



$$\begin{matrix} 1122 \\ 334 \end{matrix}$$

$$\begin{matrix} 1112 \\ 344 \end{matrix}$$

$$\begin{matrix} 1123 \\ 234 \end{matrix}$$

1 \ / 3

2 \ /

$$\begin{matrix} 1112 \\ 334 \end{matrix}$$

$$\begin{matrix} 1122 \\ 234 \end{matrix}$$

u =

2 \ /

1 \ /

v =

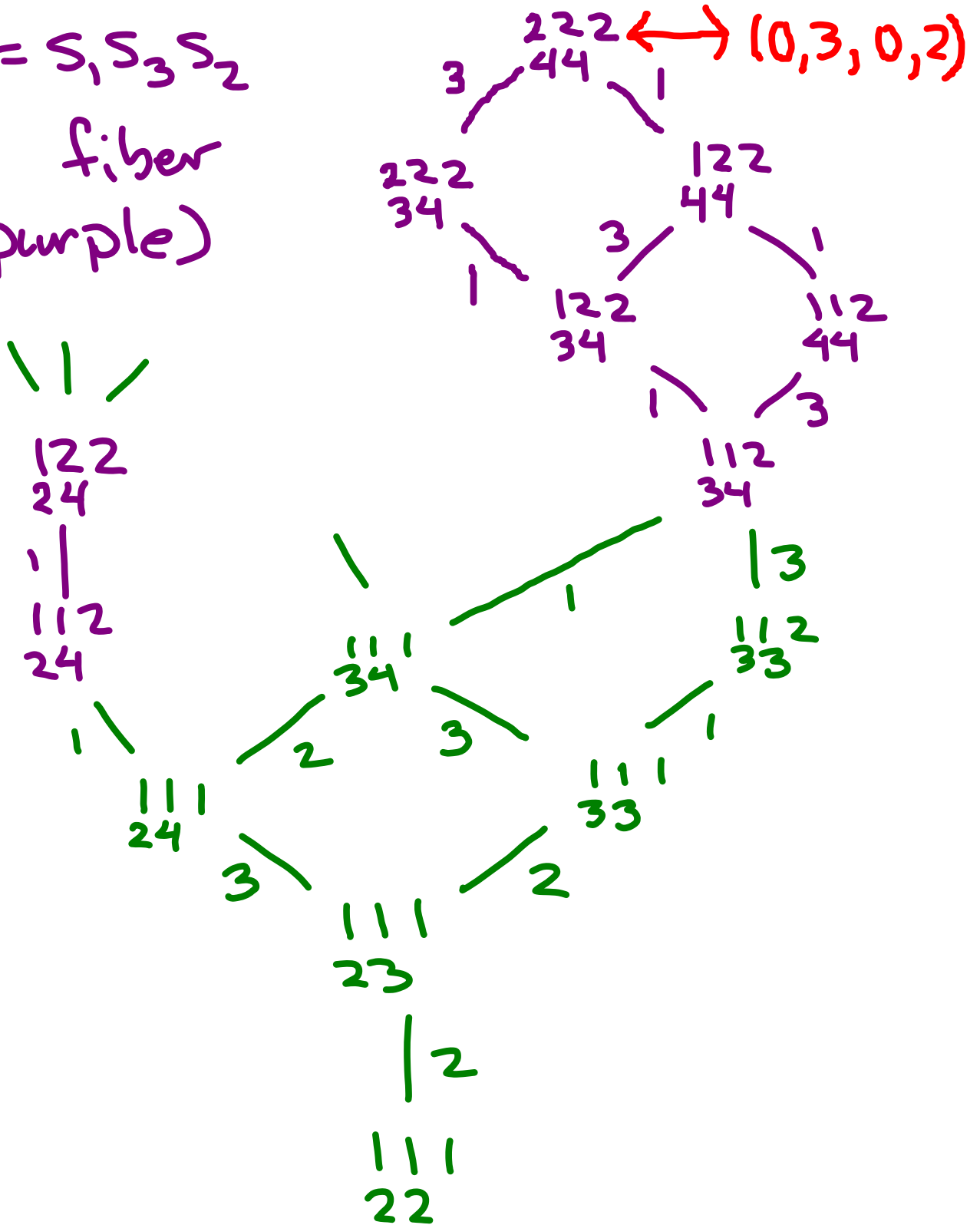
$$\begin{matrix} 1112 \\ 234 \end{matrix}$$

Fiber with Multiple Minimal Elements

Key = $S_1 S_3 S_2$

fiber

(in purple)



Examples with $M(u,v) = 2^j$

$j=1: u = \begin{matrix} 1112 \\ 234 \end{matrix}$

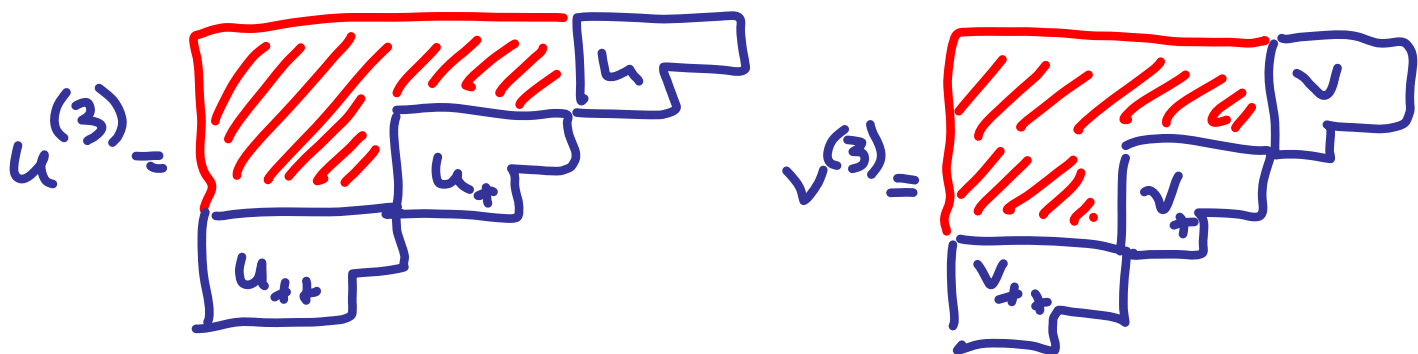
$v = \begin{matrix} 1123 \\ 344 \end{matrix}$

$j=2: u^{(2)} = \begin{matrix} 1111 & \boxed{1112} \\ 2222 & \boxed{234} \\ \boxed{6667} & \\ \boxed{789} & \end{matrix}$ $v^{(2)} = \begin{matrix} 1111 & \boxed{1123} \\ 2222 & \boxed{344} \\ \boxed{6678} & \\ \boxed{899} & \end{matrix}$

$u_+ := u+S = \begin{matrix} \boxed{6667} \\ \boxed{789} \end{matrix}$ $v_+ := v+S = \begin{matrix} \boxed{6678} \\ \boxed{899} \end{matrix}$

$[u^{(2)}, v^{(2)}] \cong [u, v] \times [u, v]$

so $M(u^{(2)}, v^{(2)}) = 2^2$



$[u^{(k)}, v^{(k)}] \cong \underbrace{[u, v] \times \dots \times [u, v]}_{k\text{-fold}} \quad M = 2^k$

A Crystal that is not a Poset

