# Crystal Graphs & SB-labelings

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(joint work with Cristian Lenart & with Karola Mészáros)

# Perspective & Main Goals:

1. Study crystal graphs regarded as posets via poset map to weak Bruhat order, namely via the (right) key map. (w/ Lenart)

   Poset Structure for Many Crystals

   $u <\cdot_{crystal} v \iff u \xrightarrow[f_i]{} v$ for some $i$

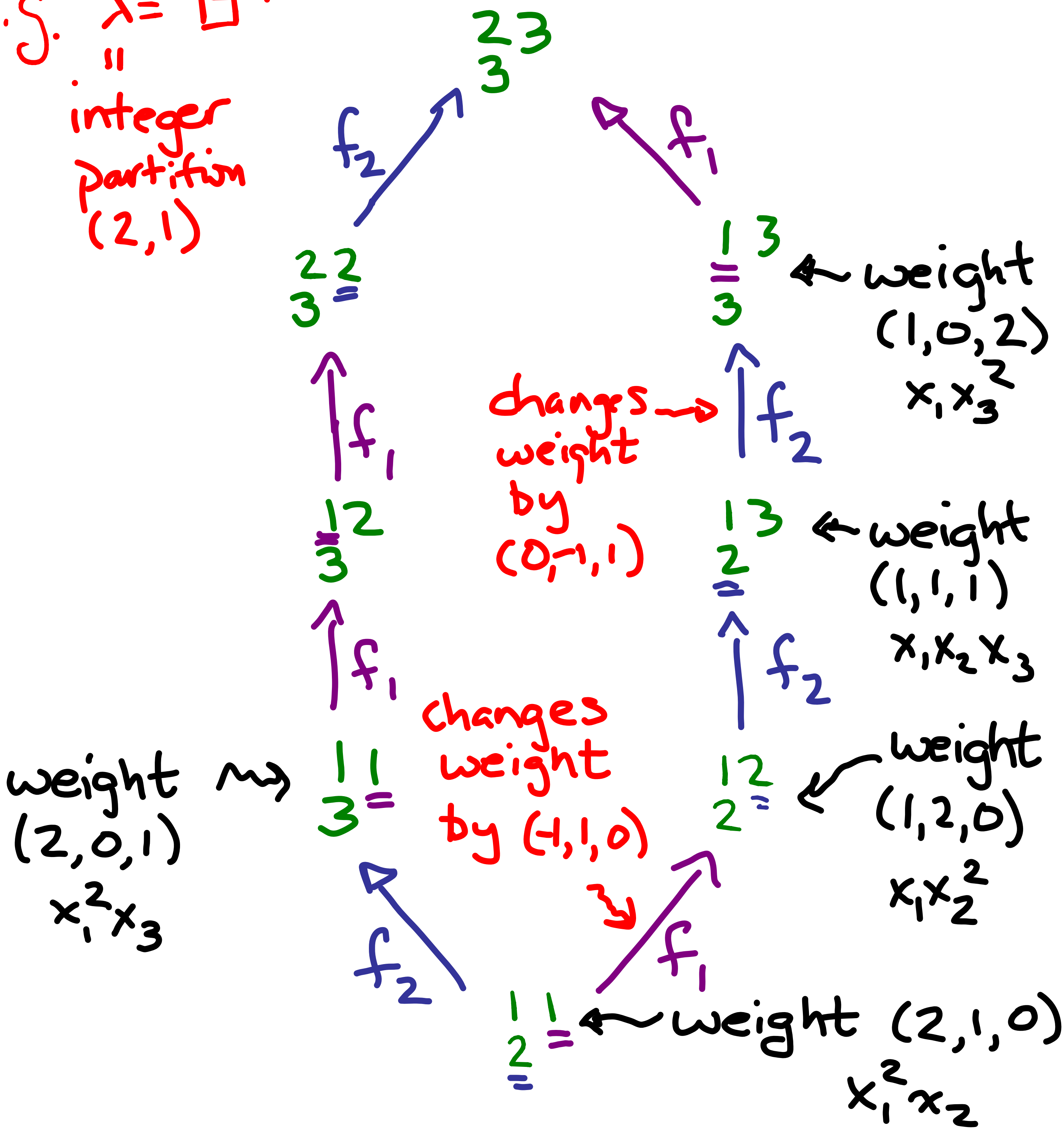
   ["cover relation" i.e. $u < v$ s.t. $\nexists z$ with $u < z < v$

2. Introduce SB-labelings for finite lattices. Show SB-labeling $\Rightarrow$ $\mu(u,v) \in \{0, \pm 1\}$ for all $u \le v$ (w/ Mészáros)

3. Discover surprising new rel'ns amongst crystal operators via theory of SB-labelings (w/ Lenart)

# (Type A) Crystals of Highest Weight Rep'ns (w/ Kashiwara Lowering Operators $f_1, f_2, \dots$)

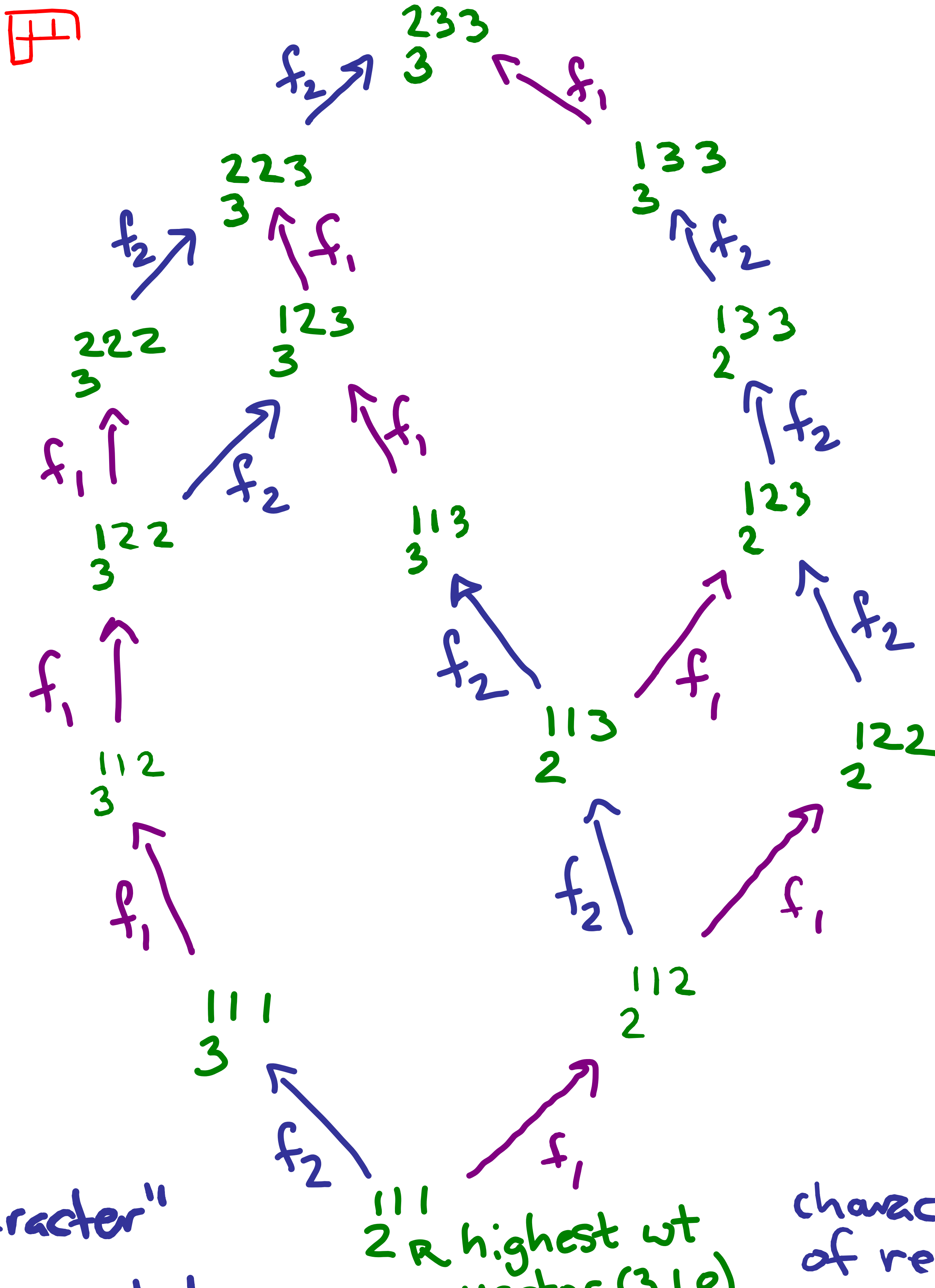
e.g. $\lambda = \square\!\square / \square$ = integer partition $(2,1)$



# Motivations for Crystal Graphs

- Study rep'n theory of Kac-Moody algebras (e.g. affine Lie algebras) $A$ by passing to univ. env. alg. $\mathcal{U}(A)$ & quantized algebra w/ parameter $q$
- $q \rightsquigarrow 1$ yields $\mathcal{U}(A)$
- $q \rightsquigarrow 0$ yields alg. w/ same dim's of weight spaces in rep'ns, encoded in "crystal graphs"
- poset elts $\leftrightsquigarrow$ basis vectors for the various weight spaces (guaranteed to exist by crystal basis properties)
- cover rel'ns $\leftrightsquigarrow$ crystal (lowering operators)

$\lambda =$




"character" of crystal $= x_1^3 x_2 + x_1^2 x_2^2 + \ldots = \text{weight}\begin{pmatrix} 1\,1\,1 \\ 2 \end{pmatrix} + \text{weight}\begin{pmatrix} 1\,1\,2 \\ 2 \end{pmatrix} + \ldots$

character of rep'n

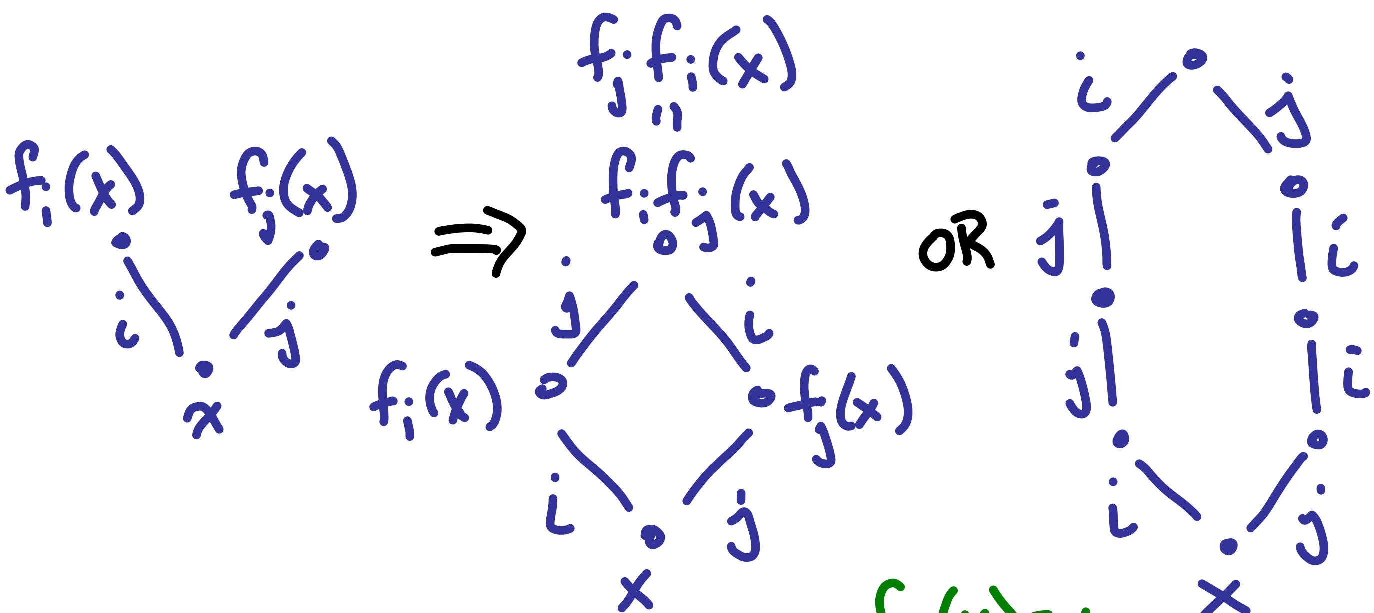
||

# Stembridge Crystals: "$\mathfrak{g}$-crystals"

(Crystals of highest weight rep'ns in simply laced case)

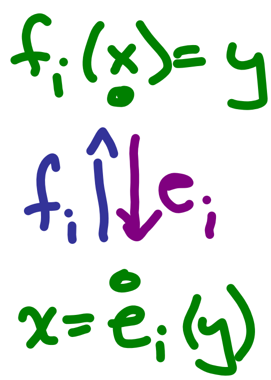
a) $\chi_{B(\lambda)}(t) = \sum_{b \in B(\lambda)} t^{wt(b)}$ = character of irrep $B(\lambda)$

(e.g. schur fn's in type A)

b) 


c) likewise for $e_i, e_j$

"raising operators":




d) reln's depend on location

# Type A crystal for highest weight rep'n of shape $\lambda$

1. $\hat{O} = \begin{array}{l} 111\ldots 1 \\ 22\ldots 2 \\ 33\ldots \\ \vdots \end{array}$ of shape $\lambda$ ← "highest weight vector"

2. $u \xrightarrow{i} v$ has $v$ obtained from $u$ by incrementing to $i+1$ rightmost $i$ not in "parenthesization pair" with an $i+1$

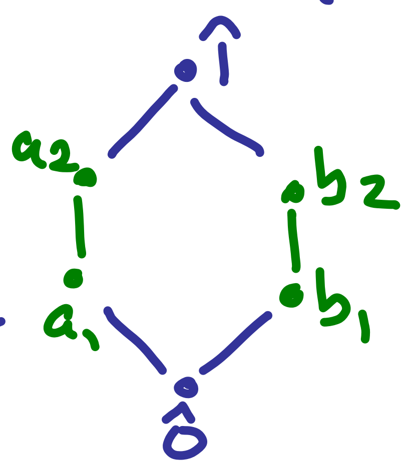
e.g. $\begin{array}{l} 1111444 \\ 2233 \\ \boxed{3}44 \end{array} \overset{f_3}{\rightsquigarrow} \begin{array}{l} 1111444 \\ 2233 \\ \boxed{4}44 \end{array}$

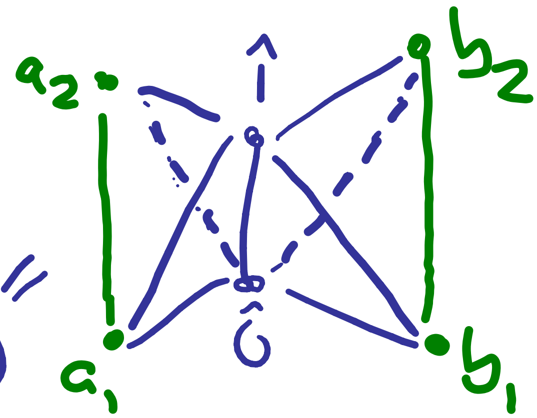
$\boxed{3}4433444 \rightsquigarrow \boxed{4}4433444$

Parenthesization Pairs: Read leftmost column bottom to top, then subsequent columns, ignoring all but $i$'s & $i+1$'s; pair up consec. $i+1, i$; delete; repeat...

**Defn:** The order complex (or nerve) of a poset $P$ is the abstract simplicial complex $\Delta(P)$ whose $i$-dim'l faces are the $(i+1)$-chains $v_0 < v_1 < \ldots < v_i$ in $P$.

e.g.

$P =$ 


$\Delta(P) =$ 


**Recall:** $\mu_P(u,v) = \tilde{\chi}(\Delta(\underbrace{u,v}))$

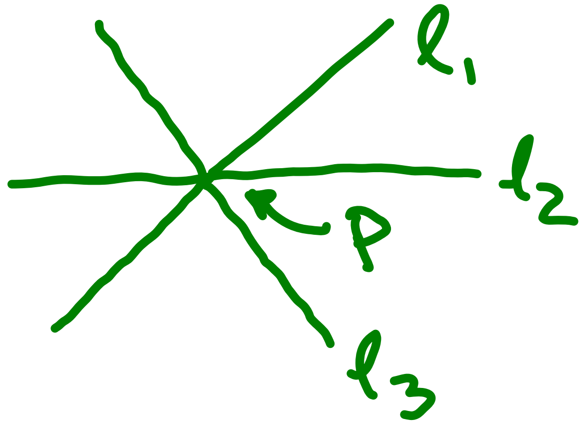
$\underbrace{u,v} =$ subposet $\underbrace{\{z \in P \mid u < z < v\}}_{\text{"open interval" } (u,v)}$

**(Recursive) Defn:**

$$\mu_P(u,u) = 1$$

$$\mu_P(u,v) = -\sum_{u \leq z < v} \mu_P(u,z)$$

# Intuition for $\mu$ (via an Example)

e.g. "counting" points in the $\mathbb{R}^2$ complement of  yields:

$$\mathbb{R}^2 - \ell_1 - \ell_2 - \ell_3 + 2p$$

- Coefficients $1, -1, -1, -1, 2$ in such inclusion-exclusion counting formula given by Möbius function $\mu(\mathbb{R}^2, \_)$ in

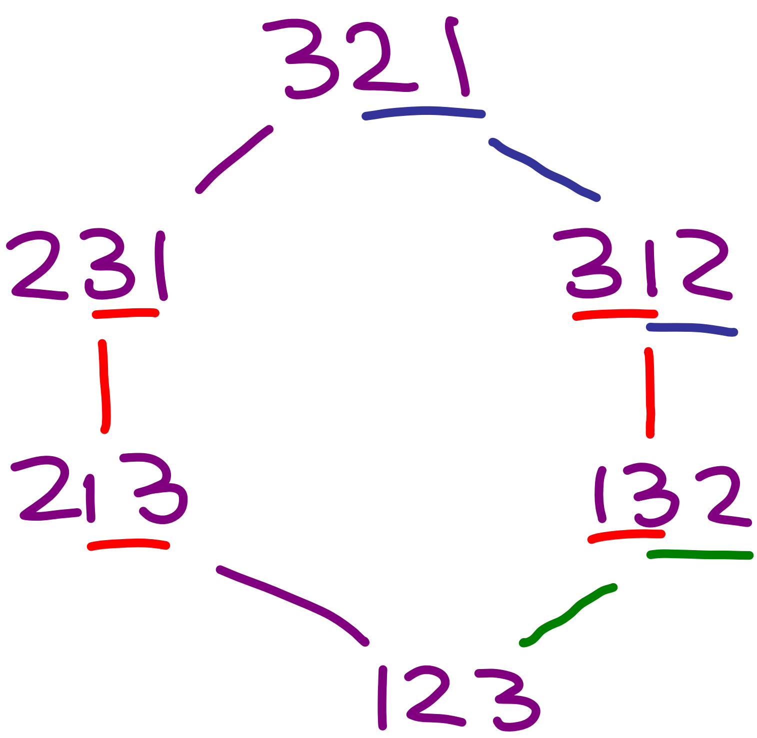
$$\begin{tikzcd}
 & p = \ell_1 \cap \ell_2 \cap \ell_3 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\ell_1 \arrow[dr, dash] & \ell_2 \arrow[d, dash] & \ell_3 \arrow[dl, dash] \\
 & \mathbb{R}^2 &
\end{tikzcd}$$

- Working over $\mathbb{F}_q$, really do get $q^2 - q - q - q + 2$

# Weak Bruhat Order: A Partial Order on Permutations

$u \lessdot v$ iff $u$ obtained from $v$ by adjacent transposition $s_i = (i, i+1)$ sorting letters in consecutive positions $i$ & $i+1$

e.g.




(captures structure of bubble sorting)

## More Formally:

**Defn:** The (left) weak order on Coxeter group $W$ is partial order with $u \lessdot v \iff v = s_i u$ for $u, v \in W$ s.t. $\text{length}(v) > \text{length}(u)$ where $\text{length}(u) := \min\{r \mid u = s_{i_1} \cdots s_{i_r}\}$ for $s_{i_1}, \ldots, s_{i_r}$ "simple reflections"

e.g. $S_3$ and other symmetric groups:

$$
\begin{tikzcd}
 & 321 = s_1 s_2 s_1 = s_2 s_1 s_2 & \\
231 = s_2 s_1 \arrow[ur, dash, "s_1 \cdot \_"] & & s_1 s_2 = 312 \arrow[ul, dash, "s_2 \cdot \_"'] \\
213 = s_1 \arrow[u, dash, "s_2 \cdot \_"] & & s_2 = 132 \arrow[u, dash, "s_1 \cdot \_"'] \\
 & 123 = e \arrow[ul, dash, "s_1 \cdot \_"] \arrow[ur, dash, "s_2 \cdot \_"'] &
\end{tikzcd}
$$

e.g. $W = S_n$

$S = \{s_1, s_2, \dots, s_{n-1}\}$ for $s_i = (i, i+1)$

with relations:

$$s_i^2 = e \;\&\; \underbrace{s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \;\&\; s_i s_j = s_j s_i}_{\text{"braid reln's"}}$$

(for $|j-i|>1$)

Note:

- Saturated chains from $e$ to $w$

  $\Updownarrow$

  "reduced expressions" $s_{i_1} \cdots s_{i_{\ell(w)}}$ for $w$.

- Likewise saturated chains $u$ to $v$

  $\Updownarrow$

  reduced expressions for $vu^{-1}$

# Connectedness under Braid Moves

**Thm** (see e.g. Björner-Brenti book): Let $(W,S)$ be Coxeter system & let $w \in W$. Then every two reduced expressions for $w$ are connected via braid moves.

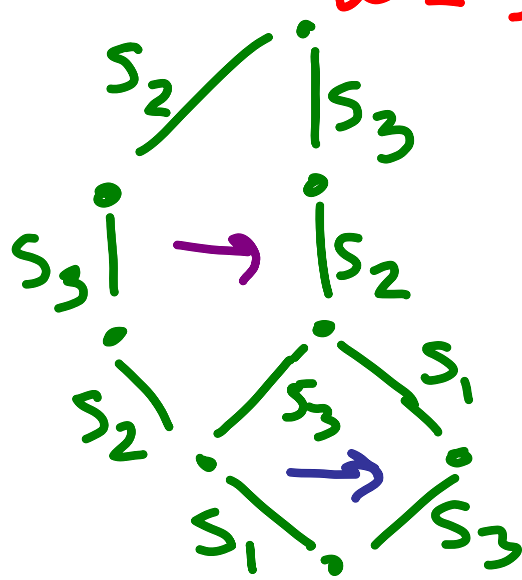
e.g. $s_2 s_3 s_2 s_1 \to s_3 s_2 s_3 s_1$

$\to s_3 s_2 s_1 s_3$

$w = s_2 s_3 s_2 s_1 = s_3 s_2 s_3 s_1$

$= s_3 s_2 s_1 s_3$

left weak order:



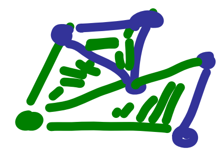
# Homotopy Types for Weak Order

**Thm** (Edelman, Björner): Each $\Delta(u,v) \simeq$ ball or sphere.

**A proof**: Use "poset map" [a map $f$ s.t. $u \le v \Rightarrow f(u) \le f(v)$] to reduce to $\Delta(\underbrace{\bar{B}_n}_{\text{poset of proper subsets of } \{1,2,\ldots,n\}}) = \mathrm{sd}(\partial \Delta_{n-1}) \simeq S^{n-2}$

via $f(u) = \max\{T \mid \underbrace{w_0}_{\text{longest element}}(\underbrace{W_T}_{\text{parabolic subgp}}) \le u\}$

& use "Quillen fiber lemma" (a.k.a. "Quillen Theorem A")

Recall Quillen fibre lemma: Given a poset map $f: P \to Q$ s.t. $q \in Q \Rightarrow$ $\Delta(\{p \in P \mid f(p) \leq q\})$ is contractible, then $\Delta(P) \simeq \Delta(Q)$. 

(& dual version w/ $f(p) \geq q$)

## Some Motivations & History for Topology of Poset Order Complexes

1. Applications to finite group theory (Quillen, Aschbacher, Brown, Shareshian, etc.)
2. Applic's to commutative algebra e.g. via isomorphism of poset order complex to bar complex (for particular posets) e.g. $u < v < w \rightsquigarrow [u^{-1}v \mid v^{-1}w]$ for weak order

# Outline for Remainder of Talk

II. New Def'n for (Right) Key Map (to weak order) (w/ Lenart)

III. Analogy w/ Weak Order for Lower Intervals in Crystals (w/ Lenart)

IV. Non-Analogous Results w/ Weak Order for Arbitrary Intervals (w/ Lenart)

- Arbitrarily large Möbius functions
- Arbitrarily high degree non-redundant "relations" amongst crystal operators

V. SB-labelings (w/ Mészáros)

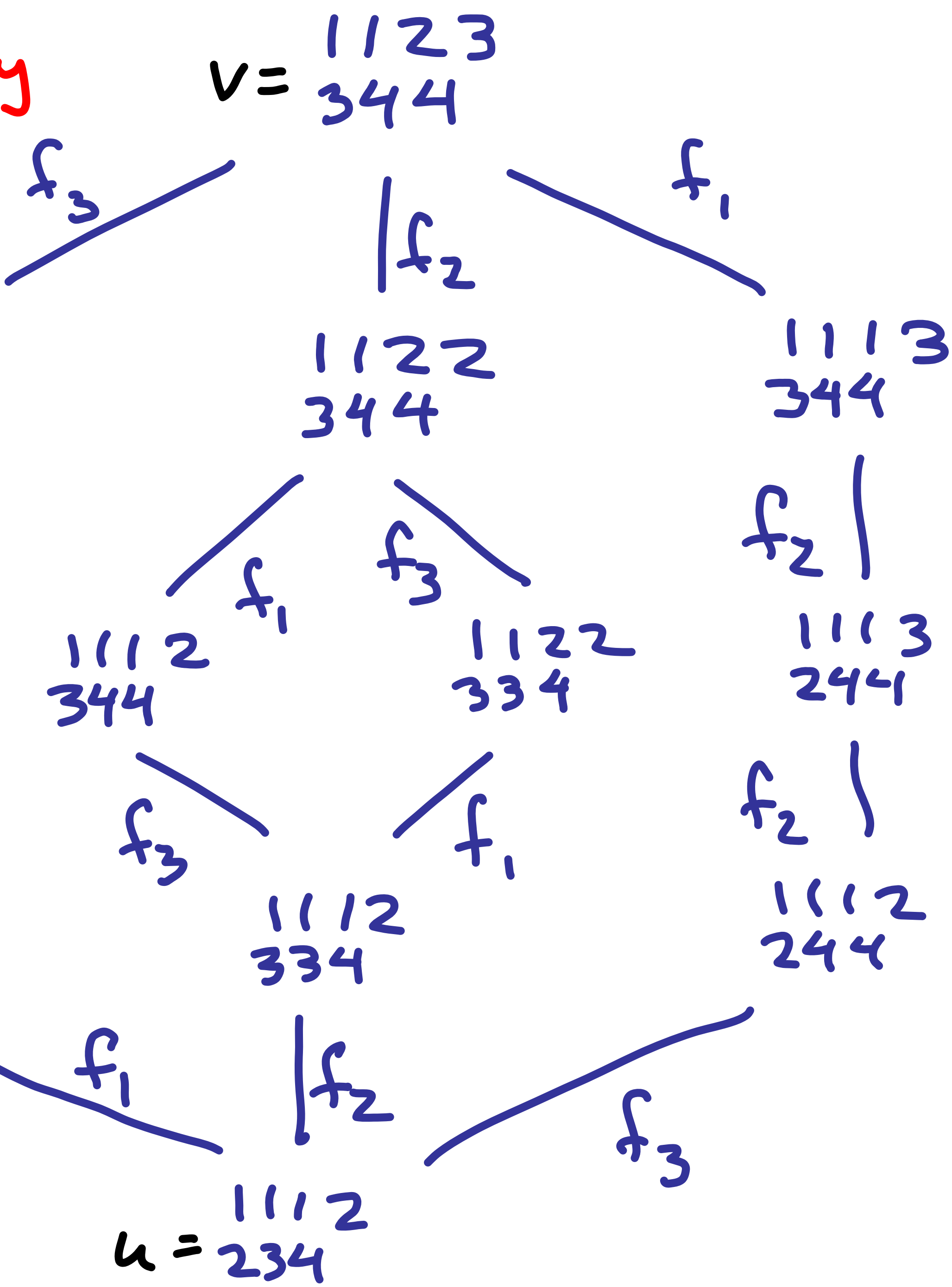
Applic: $M(u,v) \neq 0, \pm 1 \Rightarrow$ rel'n within $[u,v]$ not generated by Stembridge local rel'ns (w/ Lenart)

VI. More Examples, etc. (time permitting)

# Examples with Unexpected(?) Structure

"Base Case":

(discovered by computer)

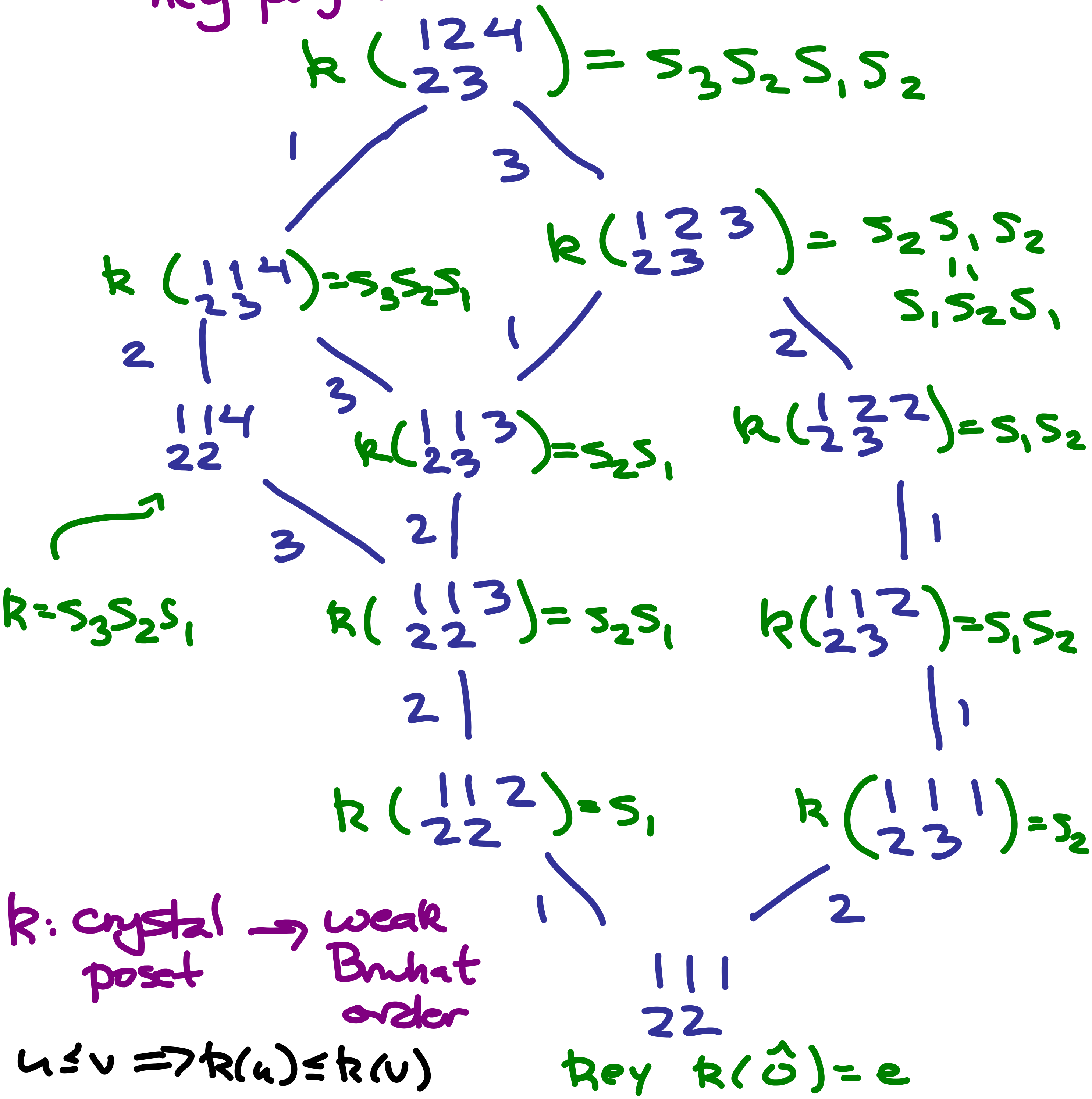



$\mu_P(u,v) = 2$

& not connected by "Stembridge moves"

# II. Right key "$\kappa$" of a "KM"-crystal

(related to Lascoux-Schutzenberger key polynomials)




$\kappa$: crystal poset $\to$ weak Bruhat order

$u \le v \Rightarrow \kappa(u) \le \kappa(v)$

# New Algorithm to Calculate Right Key of a KM-Crystal

(1) $\text{key}(\hat{0}) = e$

(2) if $\hat{0} \xrightarrow[i]{} a$, then $\text{key}(a) = s_i$

(i.e. $\hat{0} \lessdot a$)

(3) if $v$ covers 2 or more elements

then $\text{key}(v) = \bigvee_{\{u \mid u \to v\}} \text{key}(u)$

(for join taken in weak order)

(4) if $u \xrightarrow[i]{} v$ and $v$ does not cover any other elements, then:

(a) $\text{key}(v) = \text{key}(u)$ if $\exists\, u' \xrightarrow[i]{} u$

(b) $\text{key}(v) = s_i \cdot \text{key}(u)$ otherwise

# Key Polynomials & right/left key

(see Lascoux-Schutzenberger & e.g. Reiner-Shimozono)

Motivations: (1) Schubert poly. $\mathfrak{S}_w$ is positive sum of "key polynomials"

(2) Key polynomial records character for Demazure module

(3) The (closely related) right & left key maps determine smallest Demazure modules containing a given crystal element

(4) Key maps give poset maps from KM-crystal to weak order enabling transfer of properties.

# III. Analogy with Weak Order for Lower Intervals in Crystals

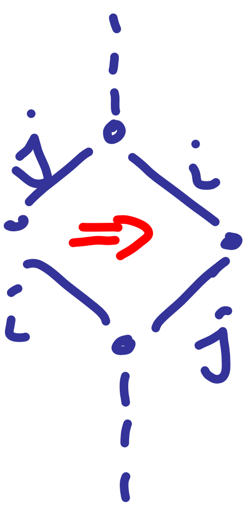
Thm (H.-Lenart) Given $u$ in a symmetrizable Kac-Moody type crystal "KM-crystal", then $\mu(\hat{0},u)=0,\pm 1$ with $\mu(\hat{0},u)=0$ unless $\mathrm{key}(u)=w_0(J)$ for some parabolic subgroup $W_J$ with $u=\min\{z \mid z\in \mathrm{key}^{-1}(w_0(J))\}$ in which case $\mu(\hat{0},u)=(-1)^{|J|}$. Moreover, $\Delta(\hat{0},u)\simeq S^{|J|-2}$ or ball.

Proof: Quillen fibre lemma via

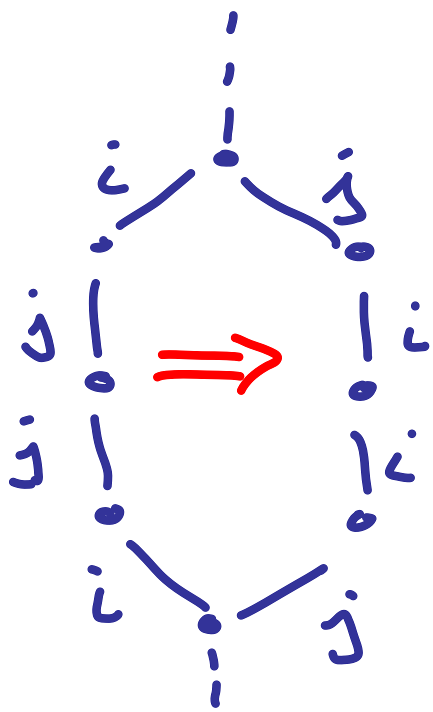
$$f:\text{crystal}\longrightarrow \text{Boolean algebra}$$

$$x\longmapsto \max\{S \mid w_0(J_S)\le_{\text{weak}} \mathrm{key}(x)\}$$

**Thm (H.-Lenart):** Given any lower interval $(\hat{0}, u)$ in a $\mathfrak{g}$-crystal, then set of saturated chains from $\hat{0}$ to $u$ is connected by "Stembridge moves", namely moves of the form



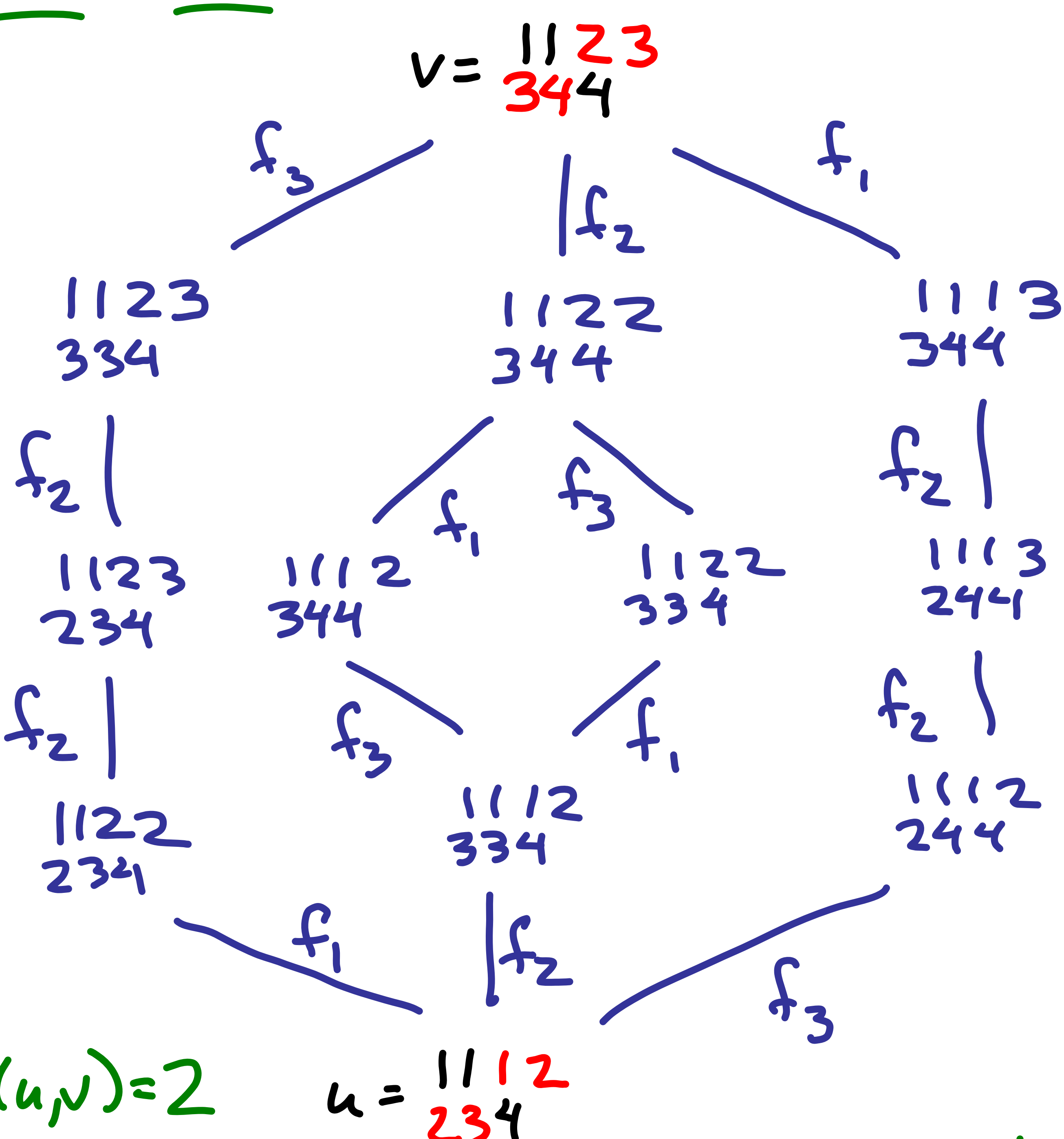

and




__Note:__ Likewise in doubly-laced case via "Sternberg moves".

# IV. Non-analogy to Weak Order for Arbitrary Intervals

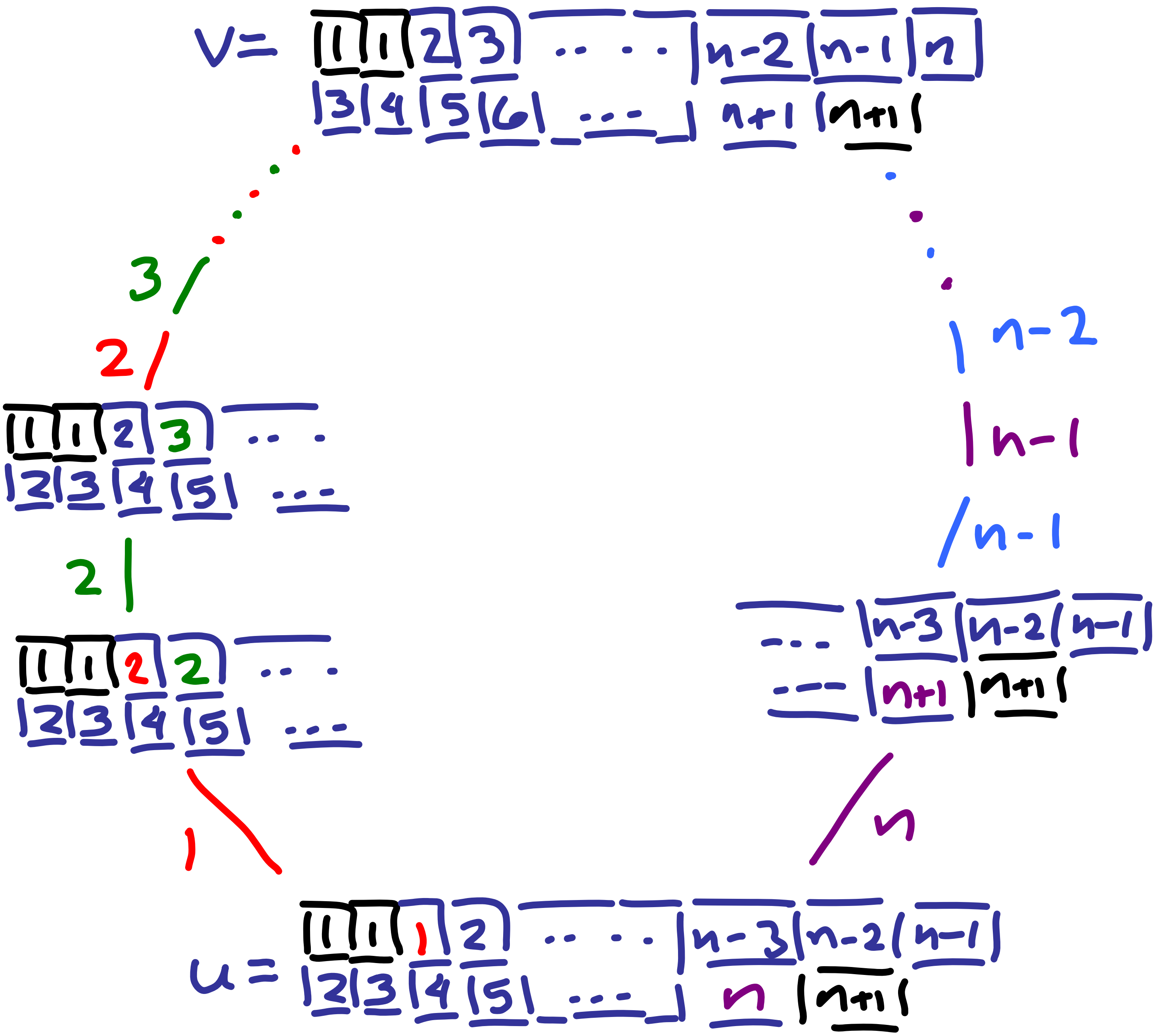
"Base Case":




$\mu_P(u,v)=2$

& not connected by "Stembridge moves"

# Arbitrarily High Rank Disconnected Open Intervals




label sequences: 1, 2, 2, 3, 3, 4, 4, ..., n-1, n-1, n

& n, n-1, n-1, ..., 2, 2, 1 in distinct components

**Consequence:** Arbitrarily high degree relations $f_{i_1}\cdots f_{i_d}(u)=f_{j_1}\cdots f_{j_d}(u)$ amongst crystal operators applied to $u$ not implied by any lower degree relations.

**Systematic Method to Discover Such Unexpected Relations?**

- Möbius functions, using the theory of SB-labelings.

**Thm (H.–Lenart):** There exist $u<v$ in type A $\mathfrak{g}$-crystals with $M(u,v)=2^j$ for every positive integer $j$ (hence arbitrarily large).

# V. SB-labelings (w/ Mészáros)

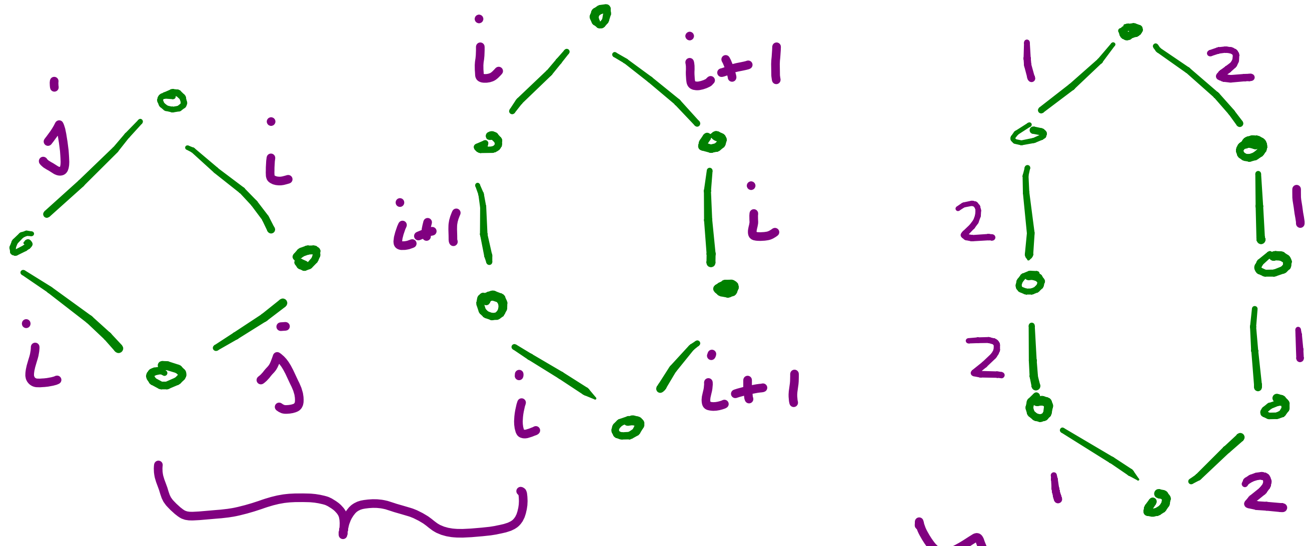
**Def'n:** Let $\lambda$ be an edge labeling of a finite lattice $L$ s.t. all $u \lessdot v$ & $u \lessdot w$ in $L$ with $v \neq w$ meet the conditions:

(1) $\lambda(u,v) \neq \lambda(u,w)$ for $v \neq w$

(2) Every saturated chain from $u$ to $v \vee w$ uses each of the labels $\lambda(u,v)$ & $\lambda(u,w)$ at least once, but does not use any other label.

Then $\lambda$ is an SB-labeling (index 2 formulation).

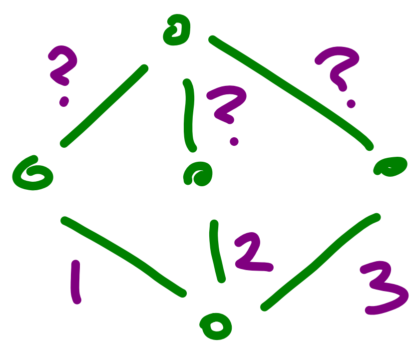
e.g.




weak order

crystal graph

Non-Example:




**Thm (H.-Meszáros):** If finite lattice $L$ has labeling $\lambda$ that is SB-labeling, then $M(u,v) = 0, \pm 1$ for $u, v \in L$. $\Delta(u,v) \simeq$ ball or sphere.

**Plan for crystals:** Find $M(u,v) \neq 0, \pm 1$ & use that edge coloring is nearly SB-labeling to deduce properties of such intervals $[u,v]$

# Examples (of Lattices with SB-labelings)

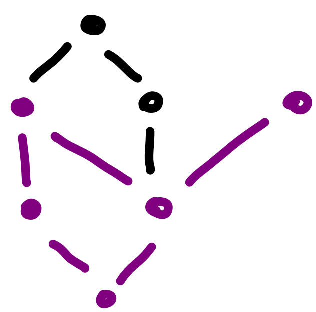
## 1. Finite distributive lattices

• Let $\lambda(S \lessdot S \cup \{i\}) = i$ (for $L = J(P)$

"order ideals"

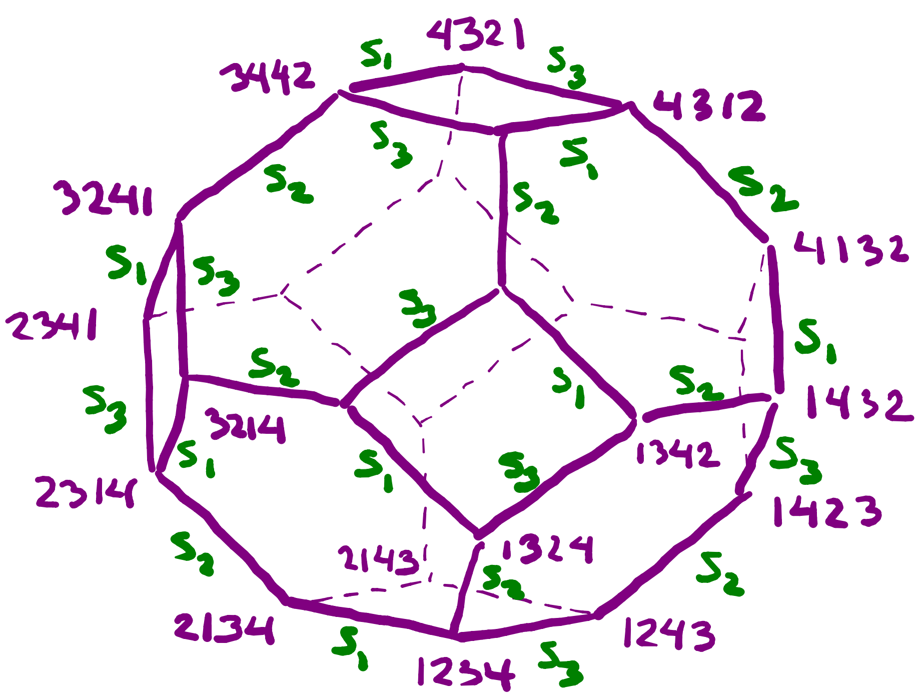
= poset of order ideals in $P$

(including Young's lattice intervals)

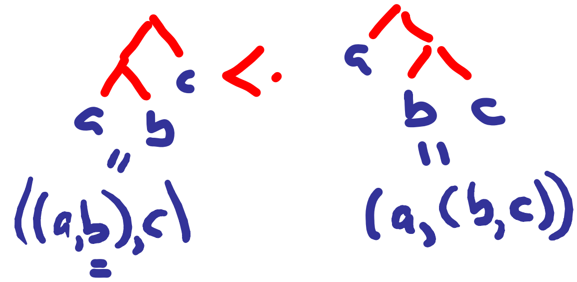



## 2. Weak Bruhat Order

Idea: Use $\lambda(u \lessdot s_i u) = s_i$



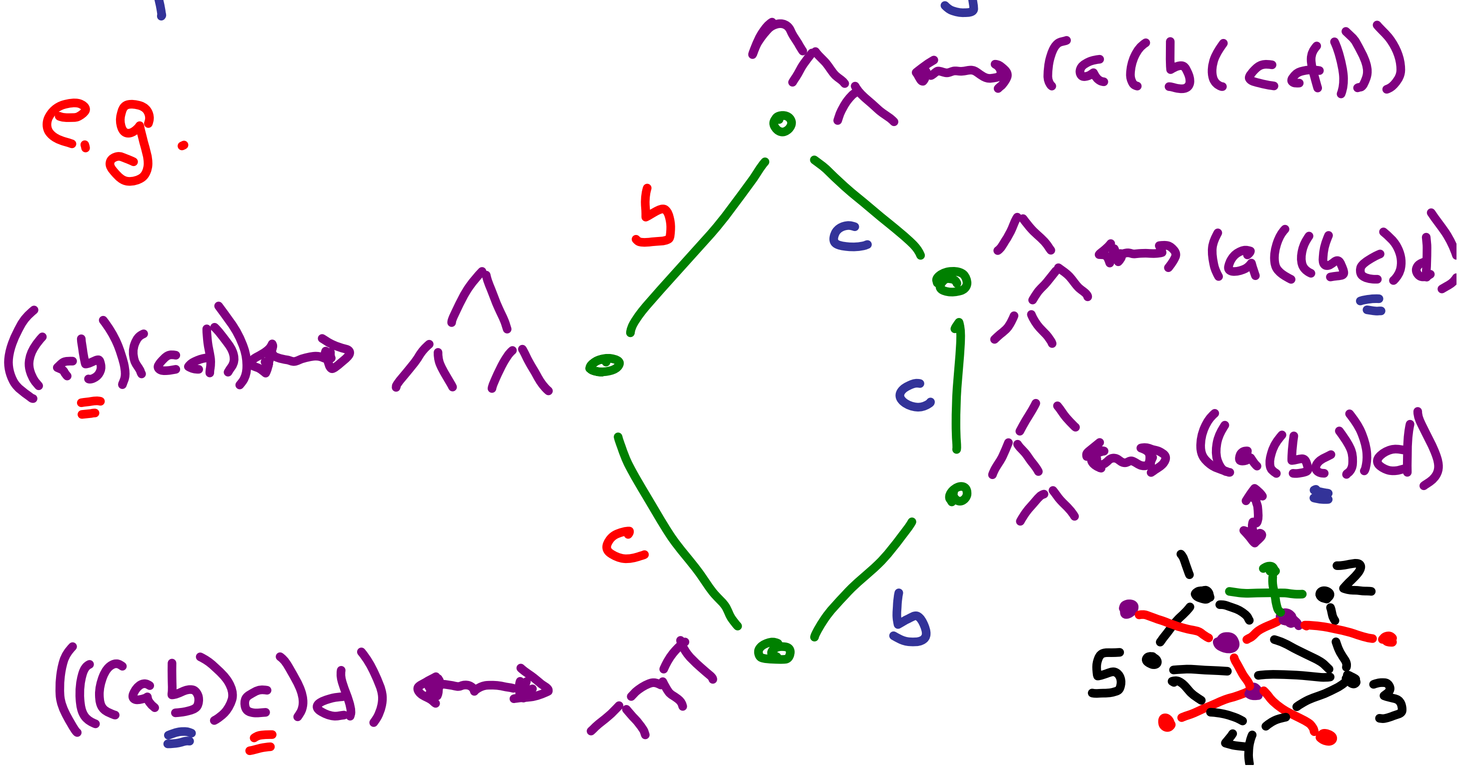
## 3. Tamari lattice (1-skeleton of Stasheff polytope/associahedron)



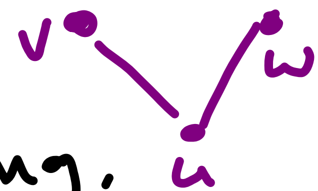

- poset of binary trees w/ n leaves (or parenthesizations or triangulations)
- $\lambda(u \lessdot v)$ := letter to immediate left of right parenthesis being moved (e.g. $\lambda($  $) = b$)

(nonpure lex. shellable – Björner & Wachs)

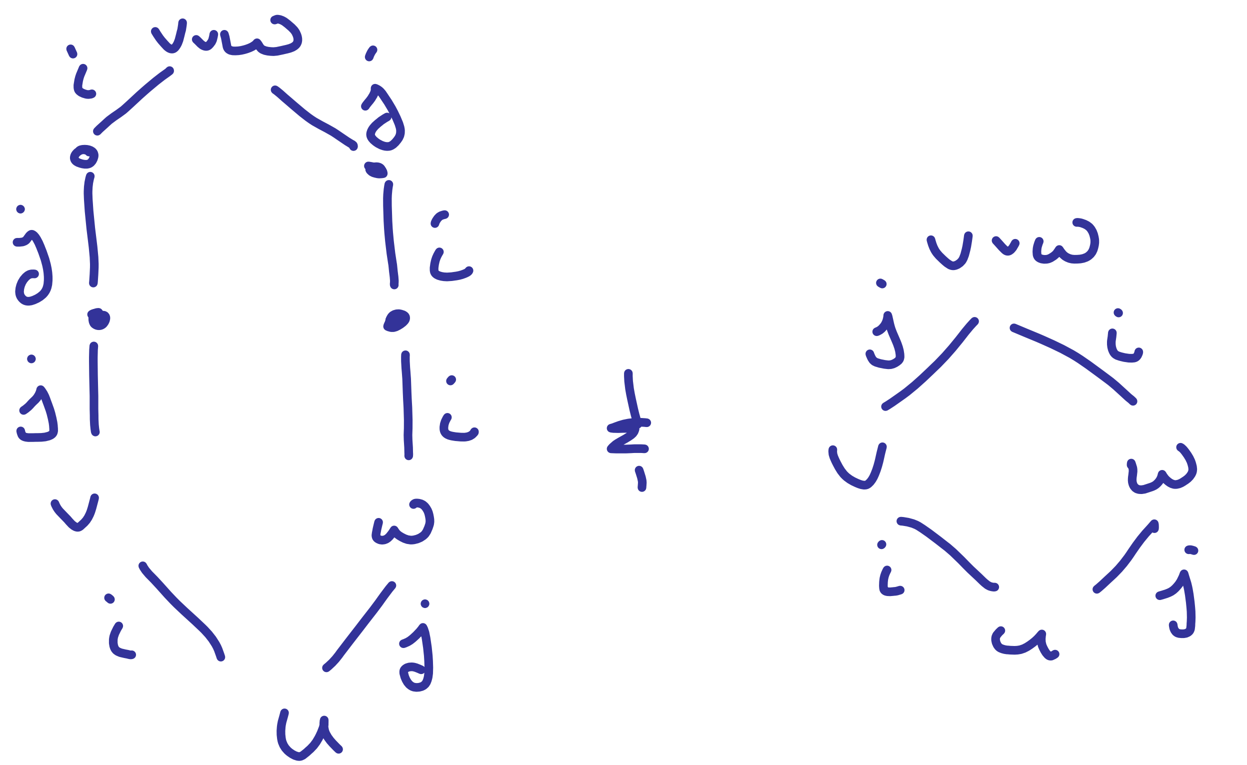
e.g.



**Observation (H.–Lenart):** If $\mathfrak{g}$-crystal were a lattice s.t. Stembridge local upper bounds were the least upper bounds for every $v \neq w$ s.t. $u \lessdot v$ & $u \lessdot w$, then edge coloring would be SB-labeling.




e.g.

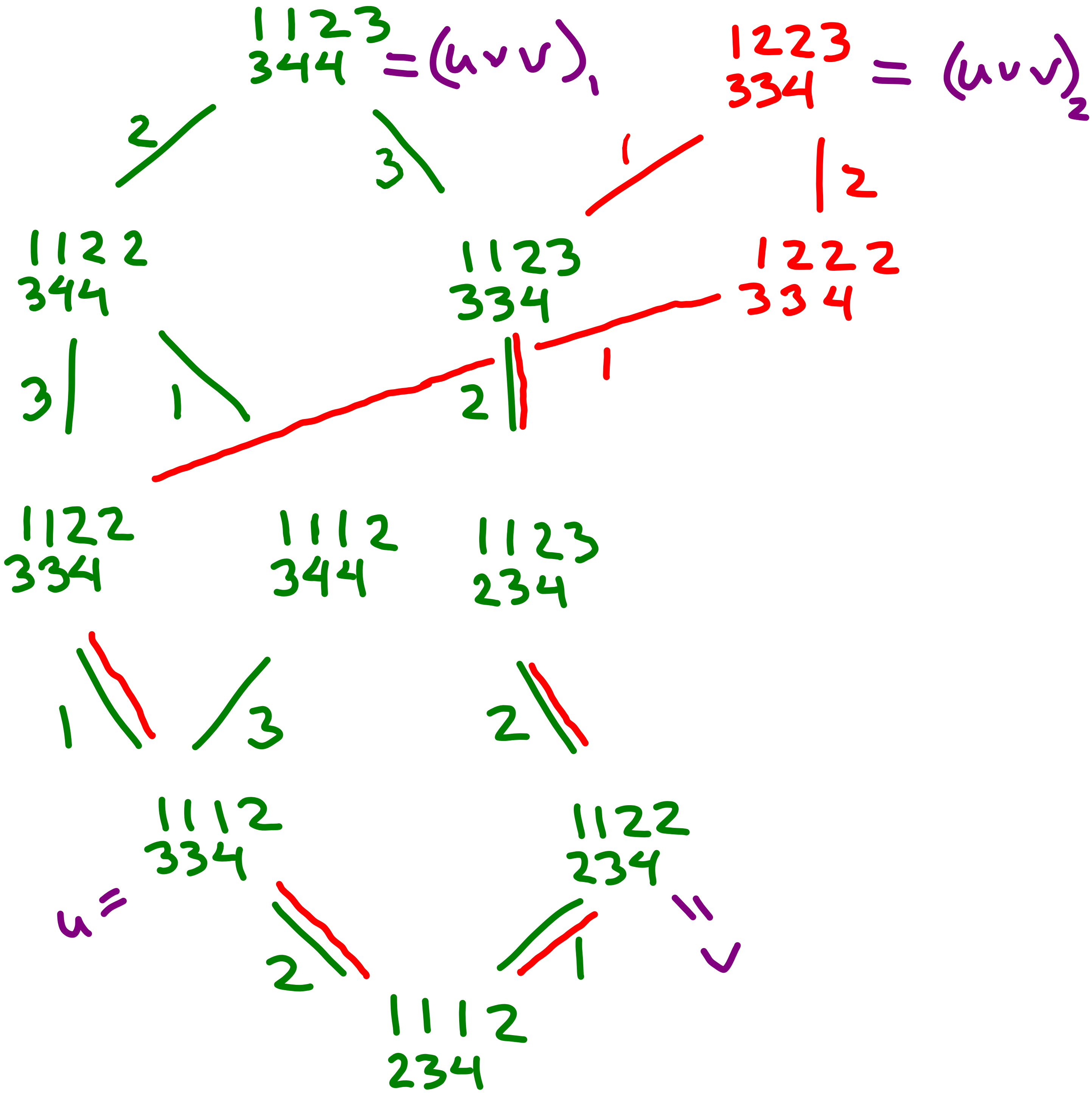



But $M(u,v) \neq 0, \pm 1$ precludes this!

**Thm** (H–Lenart): If $\mathfrak{g}$-crystal has $\mu(x,y) \neq 0, \pm 1$, then there exists rel'n amongst crystal operators within $[x,y]$ not implied by Stembridge local rel'ns.

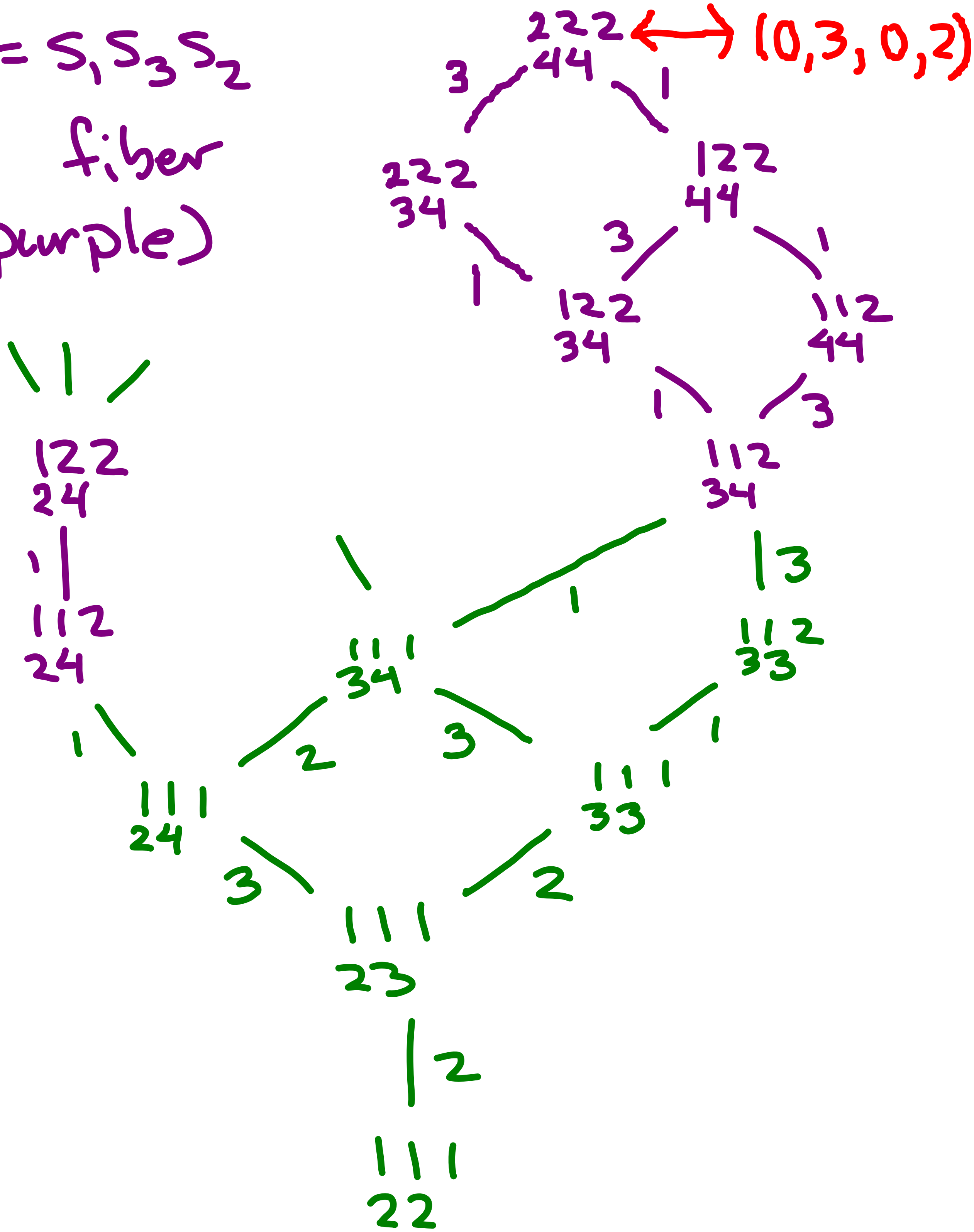
# VI. Further Examples, etc.

## Non-lattice crystal



# Fiber with Multiple Minimal Elements

Key $= s_1 s_3 s_2$ fiber (in purple)



# Key Polynomials (Viewpoint from e.g. Reiner-Shimozono)

- $\partial_i = \dfrac{1-s_i}{x_i - x_{i+1}}$ & $\pi_i = \partial_i x_i$

- $K_\alpha = \pi_{i_1} \cdots \pi_{i_r} x^{\lambda(\alpha)}$ for $\alpha$ composition of $n$ & $s_{i_1} \cdots s_{i_r}$ sorting $\alpha$ to $\lambda(\alpha)$

e.g. $K_{(1,0,2,1)} = \pi_2 \pi_1 \pi_3 \underbrace{x^{(2,1,1,0)}}_{= x_1^2 x_2 x_3}$

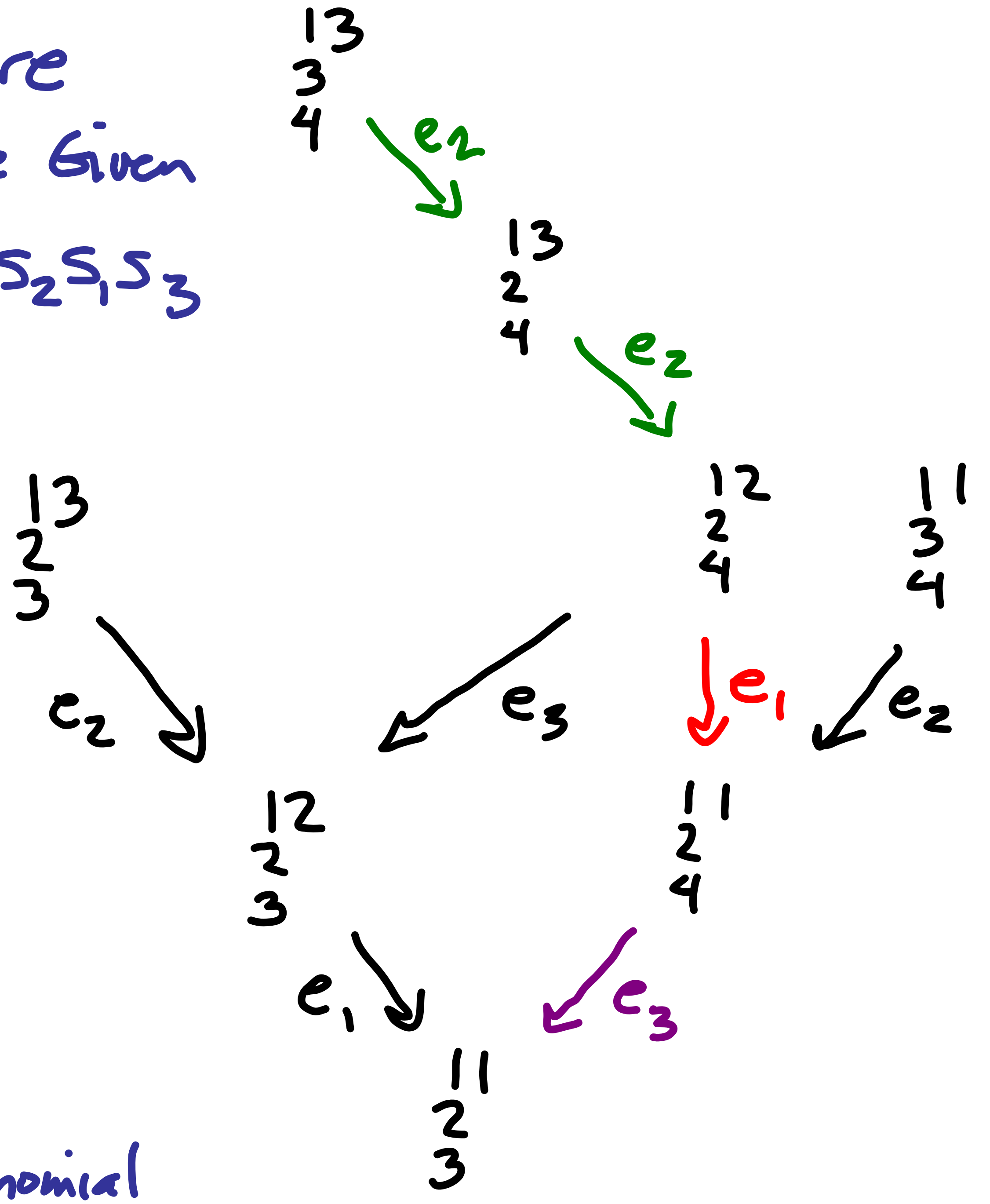
$$= \pi_2 \pi_1 \left( x_1^2 x_2 (x_3 + x_4) \right)$$

$$= \pi_2 \big( x_1 x_2 x_3 (x_1 + x_2) + x_1 x_2 x_4 (x_1 + x_2) \big)$$

$$= x_1^2 x_2 x_3 + x_1 x_2 x_3 (x_2 + x_3) + x_1^2 x_4 (x_2 + x_3) + x_1 x_4 (x_2^2 + x_2 x_3 + x_3^2)$$

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

# Demazure Module Given by $w = s_2 s_1 s_3$




Key Polynomial

$\parallel$

$$K_{(1,0,2,1)} = \sum_{\substack{T' \leq T \\ \text{componentwise}}} x^{T'}$$

$\Updownarrow$

$K(T') \leq_{\text{Bruhat}} K(T)$ w/ no higher $e_i$ exponents

# Examples with $M(u,v) = 2^j$

$j=1$: $u = \begin{matrix}1112\\234\end{matrix}$ $\qquad$ $v = \begin{matrix}1123\\344\end{matrix}$

$j=2$: $u^{(2)} = \begin{matrix} & 1111 & \boxed{\begin{matrix}1112\\234\end{matrix}} \\ & 2222 & \\ \boxed{\begin{matrix}6667\\789\end{matrix}} & & \end{matrix}$ $\qquad$ $v^{(2)} = \begin{matrix} & 1111 & \boxed{\begin{matrix}1123\\344\end{matrix}} \\ & 2222 & \\ \boxed{\begin{matrix}6678\\899\end{matrix}} & & \end{matrix}$

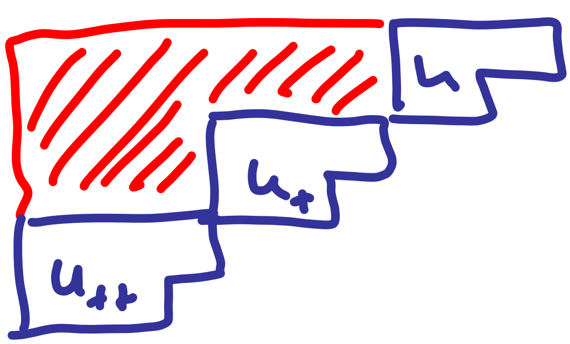
$\begin{matrix}1112\\234\end{matrix} = u$ $\qquad$ $\begin{matrix}1123\\344\end{matrix} = v$

$u_+ := u+5 = \begin{matrix}6667\\789\end{matrix}$ $\qquad$ $v_+ := v+5 = \begin{matrix}6678\\899\end{matrix}$

$$[u^{(2)}, v^{(2)}] \cong [u,v] \times [u,v]$$

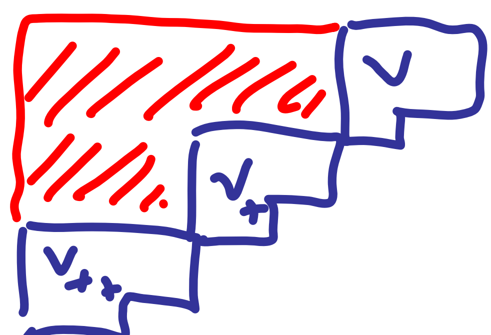
so $M(u^{(2)}, v^{(2)}) = 2^2$

$u^{(3)} =$





$v^{(3)} =$





$$[u^{(k)}, v^{(k)}] \cong \underbrace{[u,v] \times \cdots \times [u,v]}_{k\text{-fold}} \qquad M = 2^k$$

# A Crystal that is not a Poset

$A_3^{(1)}$

$$
\begin{tikzcd}[column sep=large]
\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \arrow[r, "2"] &
\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \arrow[r, "1"] \arrow[rr, bend left=60, "3"] &
\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \arrow[rr, bend left=60, "3"] &
\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} \arrow[r, "1"] &
\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} \arrow[r, "2"] \arrow[llll, bend left=40, "0"] &
\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \arrow[llll, bend left=40, "0"]
\end{tikzcd}
$$

# Another Motivation for Crystals: Proving Schur-Positivity

(cf. e.g. Morse-Schilling)

- Express symmetric fn as positive sum of monomials
- Associate combinatorial objects (e.g. SSYT) w/ monomials as weights
- Arrange into directed graph w/ colored edges recording wt change
- Check digraph axioms that guarantee it is crystal graph of a $GL_n$ polynomial rep'n denoted $\rho$
- Conclude sym fn is character of $\rho$, hence Schur-positive.

# Crystals

A **crystal** $B$ of type $\Phi$ is a nonempty set $B$ with raising & lowering operators $e_i, f_i$ & fns

$$\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \cup \{-\infty\}$$

$$wt : B \longrightarrow \Lambda = \text{weight lattice of type } \Phi$$

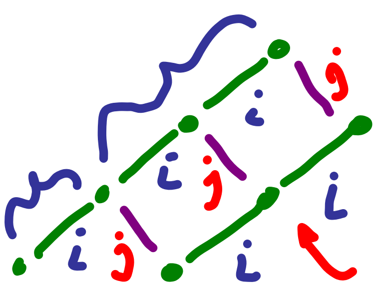
s.t.

(A1) $x, y \in B$, then $e_i(x) = y \iff x = f_i(y)$

both implying $wt(y) = wt(x) + \alpha_i$

& $\varepsilon_i(y) = \varepsilon_i(x) - 1$

$\varphi_i(y) = \varphi_i(x) + 1$




$x = f_i(y)$

$e_i(x) = y$

(A2) $\varphi_i(x) - \varepsilon_i(x) = \langle wt(x), \alpha_i^\vee \rangle$

# Crucial Properties of Key

**Thm (Littlemann):** Given any symmetrizable Kac-Moody algebra A, the key of any crystal of type A satisfies:

$$K(f_p(F)) = \begin{cases} K(F) & \text{if } e_p(F) \neq 0 \\ s_p K(F) \text{ or } K(F) & \text{if } e_p(F) = 0 \end{cases}$$

Also, If $e_p(F)=0$ then $s_p K(F) > K(F)$

**Corollary:** If $K(F) = s_{i_1} \cdots s_{i_r}$ then there exists saturated chain from $F$ to $\hat{0}$ given by applying $e_{i_r}^{d_r} \cdots e_{i_1}^{d_1}$ to $F$ for some $d_1, \ldots, d_r > 0$.

# SB-Labeling (General Index Formulation)

- Given a finite lattice $L$ with atoms $A(L)$, an edge-labeling with label set $S$ is a lower SB-labeling if:

  (1) $A(L) \subseteq S$ and $\lambda(\hat{0}, a) = a$ for each $a \in A(L)$

  (2) If $x \in L$ satisfies $x = \underbrace{a_{i_1} \vee \dots \vee a_{i_r}}_{\text{join of atoms}}$ then all saturated chains $M$ on $[\hat{0}, x]$ use exactly the labels $\{a_{i_1}, \dots, a_{i_r}\}$ each with positive multiplicity.

- If these conditions are met for every interval $[u,v]$ then $\lambda$ is an SB-labeling.

"Sphere" or "Ball"

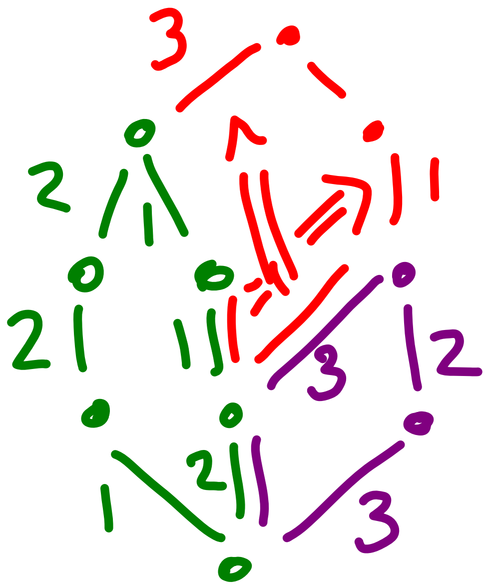
# Main Results on SB-labelings

**Thm 1.:** An edge labeling $\lambda$ on finite lattice $L$ is SB-labeling (index 2 formulation) $\iff$ $\lambda$ is SB-labeling (general index formulation).

- (therefore call either type of labeling "SB-labeling")

**Thm 2:** If finite lattice $L$ has edge labeling $\lambda$ which is SB-labeling, then $\Delta_L(u,v)$ is homotopy equivalent to ball or sphere for each $u < v$.

# Index 2 $\Rightarrow$ General Index



**Idea:** Structure propagates upward

# General Index $\Rightarrow$ Homotopy Type & Möbius function

SB-labeling

**Idea:** poset map to Boolean algebra + Quillen fiber lemma