From the Weak Bruhat Order to Crystal Posets

Patricia Hersh

North Carolina State University

(joint work with Cristian Lenart, SUNY Albany)

Perspective & Main Goal:

- Study crystal graphs regarded as posets via poset map to weak Bruhat order, namely via the (right) key map.

Poset Structure for Many Crystals

$$u <\cdot_{crystal} v \iff u \xrightarrow[f_i]{} v \text{ for some } i$$

- Transfer properties of weak Bruhat order to crystals.

Weak Bruhat order:

$$u <\cdot_{weak} s_i u \quad \text{if } \ell(s_i u) > \ell(u)$$

- Discover surprising new reln's amongst crystal operators

Motivations for Crystals

- Study representation theory of Kac-Moody algebras (e.g. affine Lie algebras)
- Take universal enveloping algebra, & its quantum algebra w/ parameter q
- $q \rightsquigarrow 1$ yields $U(A)$ for Kac-Moody algebra A
- $q \rightsquigarrow 0$ yields algebra with same dimensions of weight spaces encoded by combinatorics of "crystal graphs" (which are often posets)
- poset elts $\leftrightsquigarrow$ basis vectors for weight spaces (with remarkable properties)
- cover rel'ns $\leftrightsquigarrow$ Kashiwara raising operators

(Type A) Crystals of Highest Weight Representations & their Kashiwara Lowering Operators

e.g. $\lambda = \begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$

$$
\begin{tikzcd}
 & \begin{smallmatrix}2&3\\3&\end{smallmatrix} & \\
\begin{smallmatrix}2&2\\3&\end{smallmatrix} \arrow[ur, "f_2"] & & \begin{smallmatrix}1&3\\3&\end{smallmatrix} \arrow[ul, "f_1"'] \\
\begin{smallmatrix}1&2\\3&\end{smallmatrix} \arrow[u, "f_1"'] & & \begin{smallmatrix}1&3\\2&\end{smallmatrix} \arrow[u, "f_2"] \\
\begin{smallmatrix}1&1\\3&\end{smallmatrix} \arrow[u, "f_1"'] & & \begin{smallmatrix}1&2\\2&\end{smallmatrix} \arrow[u, "f_2"] \\
 & \begin{smallmatrix}1&1\\2&\end{smallmatrix} \arrow[ul, "f_2"] \arrow[ur, "f_1"'] &
\end{tikzcd}
$$

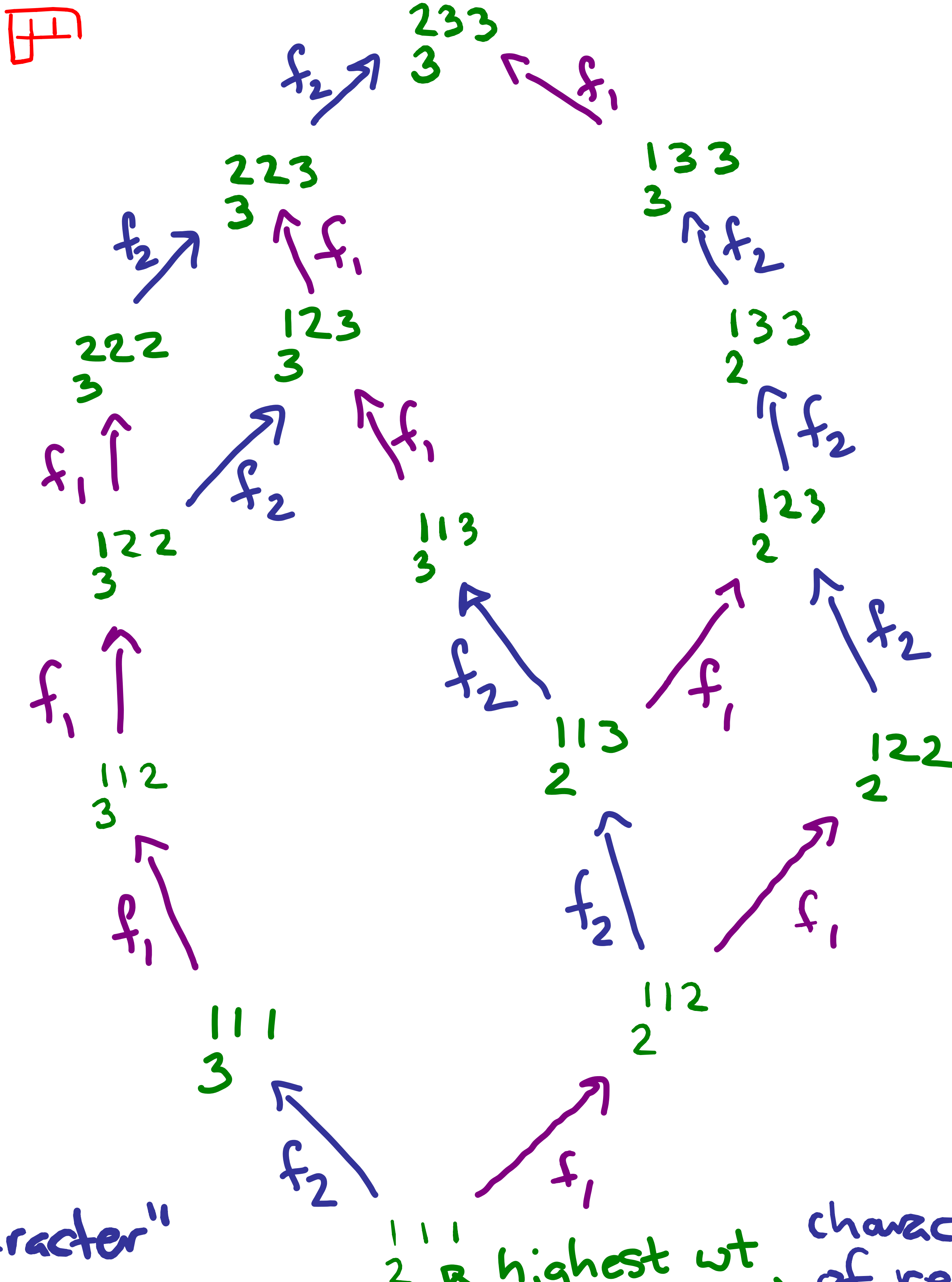
changes weight by $(-1,1,0)$

weight $(2,0,1)$ $x_1^2x_3$

weight $(1,2,0)$ $x_1x_2^2$

weight $(2,1,0)$ $x_1^2x_2$

$\lambda =$



↖ highest wt vector (3,1,0)

"character" of crystal $= x_1^3 x_2 + x_1^2 x_2^2 + \ldots = \text{weight}\begin{pmatrix} 1\,1\,1 \\ 2 \end{pmatrix} + \text{weight}\begin{pmatrix} 1\,1\,2 \\ 2 \end{pmatrix} + \ldots$

character of rep'n ||

Type A crystal for highest weight rep'n of type λ

1. $\hat{O} = \begin{matrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 & \\ 3 & 3 & \cdots & & \\ \vdots & & & & \end{matrix}$ of shape λ = "highest weight vector"

2. $u \xrightarrow[i]{} v$ for v obtained from u by incrementing to an $i+1$ the rightmost i that is not in a "parenthesization pair" w/ an "$i+1$"

Parenthesization Pairs: Read leftmost column bottom to top, then subsequent columns L to R, ignoring all but i & $i+1$; pair up consec. $i+1, i$; delete; repeat...

e.g. $u = \begin{matrix} 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & & & & \\ \boxed{3} & 4 & 4 & & & & & \end{matrix}$ $\underset{i=3}{\rightsquigarrow}$ $\boxed{3}\,4\,4\,\underbrace{3\,3}\,4\,4\,4$ $\xrightarrow{f_3}$ $4\,4\,4\,3\,3\,4\,4\,4$

Stembridge Crystals: "$\mathfrak{g}$-crystals"

(Crystals of highest weight rep'n's in Simply laced case)

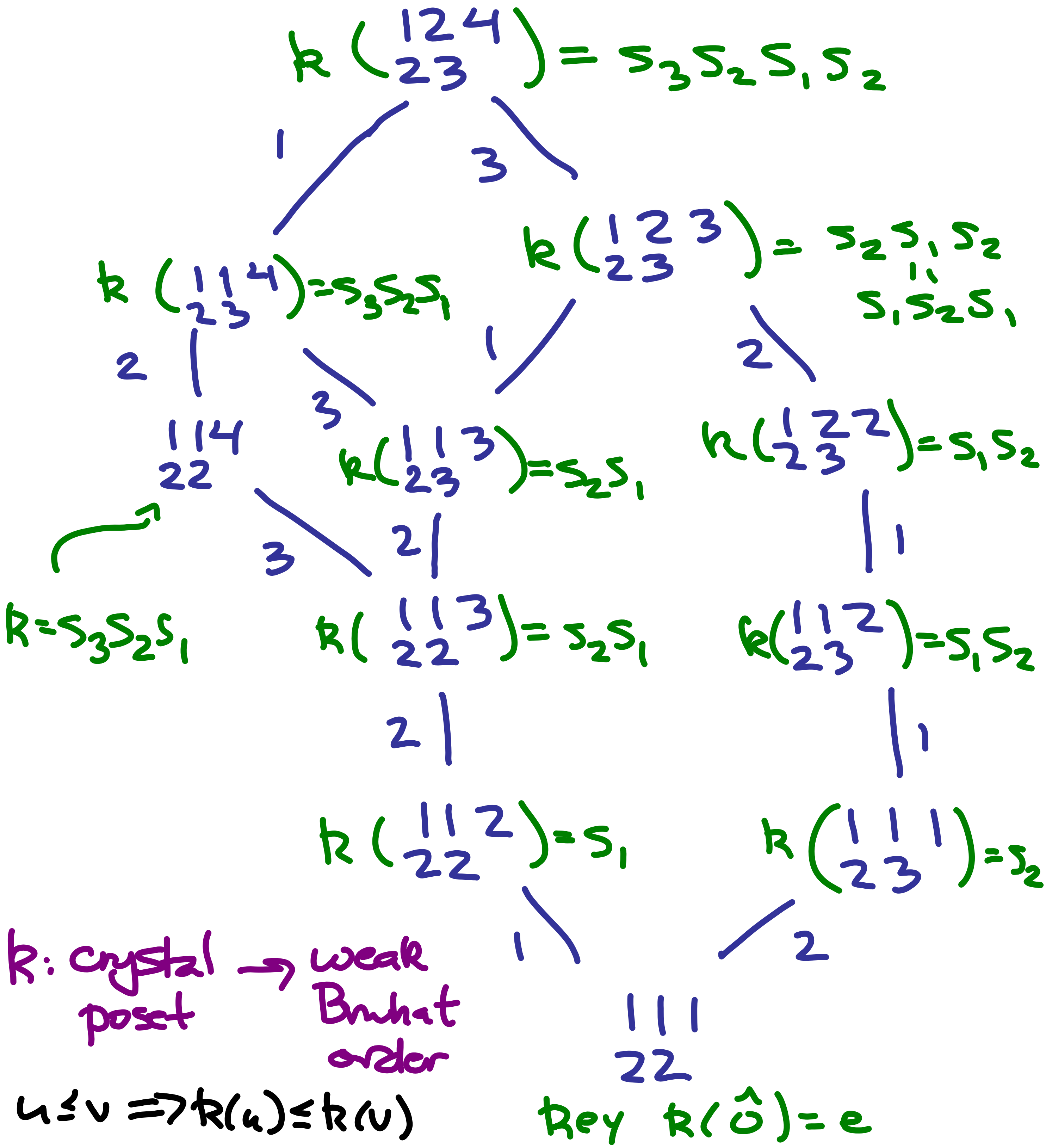
a) $$\chi_{B(\lambda)}(t) = \sum_{b \in B(\lambda)} t^{wt(b)}$$ = character of irrep $B(\lambda)$

b)
$$\begin{tikzcd}[column sep=small]
f_i(x) & & f_j(x) \\
& x \arrow[ul, dash, "i"] \arrow[ur, dash, "j"'] &
\end{tikzcd}
\Rightarrow
\begin{tikzcd}[column sep=small]
& \substack{f_j f_i(x) \\ \shortparallel \\ f_i f_j(x)} & \\
f_i(x) \arrow[ur, dash, "j"] & & f_j(x) \arrow[ul, dash, "i"'] \\
& x \arrow[ul, dash, "i"] \arrow[ur, dash, "j"'] &
\end{tikzcd}
\quad \text{OR} \quad
\begin{tikzcd}[column sep=small]
& \bullet & \\
\bullet \arrow[ur, dash, "i"] & & \bullet \arrow[ul, dash, "j"'] \\
\bullet \arrow[u, dash, "j"] & & \bullet \arrow[u, dash, "i"'] \\
\bullet \arrow[u, dash, "j"] & & \bullet \arrow[u, dash, "i"'] \\
& x \arrow[ul, dash, "i"] \arrow[ur, dash, "j"'] &
\end{tikzcd}$$

c) likewise for e_i, e_j operators

d) axioms yield this & characterize crystals of highest weight rep'n's in simply laced case

Right key "k" of a KM-crystal



k: crystal poset $\to$ weak Bruhat order

$u \le v \Rightarrow k(u) \le k(v)$

New Algorithm to Calculate Right Key of a KM-Crystal

(1) $\mathrm{key}(\hat{0}) = e$

(2) if $\hat{0} \xrightarrow[i]{} a$, then $\mathrm{key}(a) = s_i$

(i.e. $\hat{0} \lessdot a$)

(3) if v covers 2 or more elements

then $\mathrm{key}(v) = \bigvee_{\{u \mid u \to v\}} \mathrm{key}(u)$

(for join taken in weak order)

(4) if $u \xrightarrow[i]{} v$ and v does not cover any other elements, then:

(a) $\mathrm{key}(v) = \mathrm{key}(u)$ if $\exists\, u' \xrightarrow[i]{} u$

(b) $\mathrm{key}(v) = s_i \cdot \mathrm{key}(u)$ otherwise

Key Polynomials & right/left key

(see Lascoux-Schutzenberger & e.g. Reiner-Shimozono)

Motivations: (1) Schubert poly. $\mathfrak{S}_w$ is positive sum of "key polynomials"

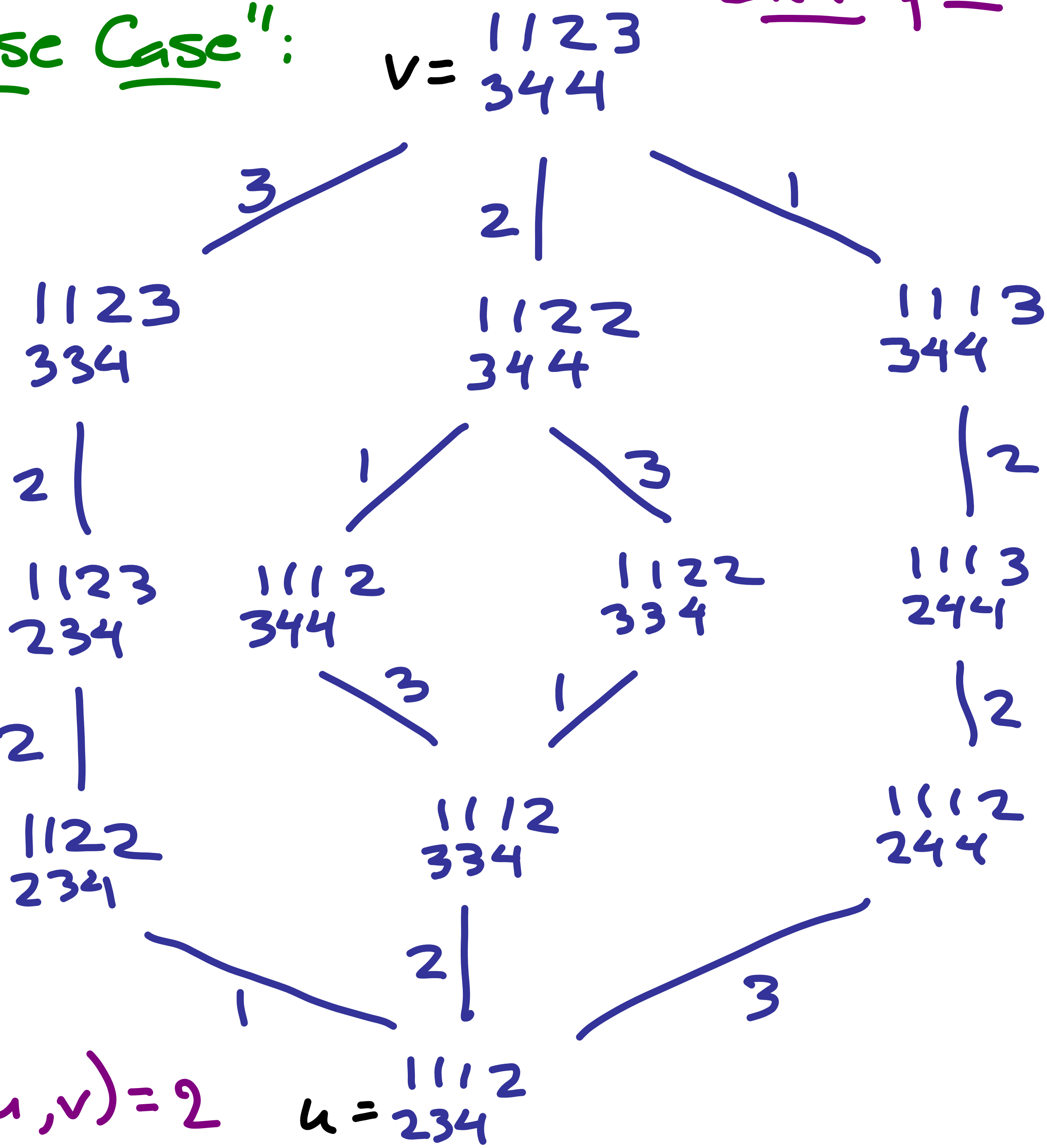
(2) Key polynomial records character for Demazure module

(3) The (closely related) right & left key maps determine smallest Demazure modules containing a given crystal element

(4) These will give us poset map from $\mathfrak{gl}$-crystal to weak Bruhat order, transferring properties

Infinite Families of (Negative) Examples

"Base Case":

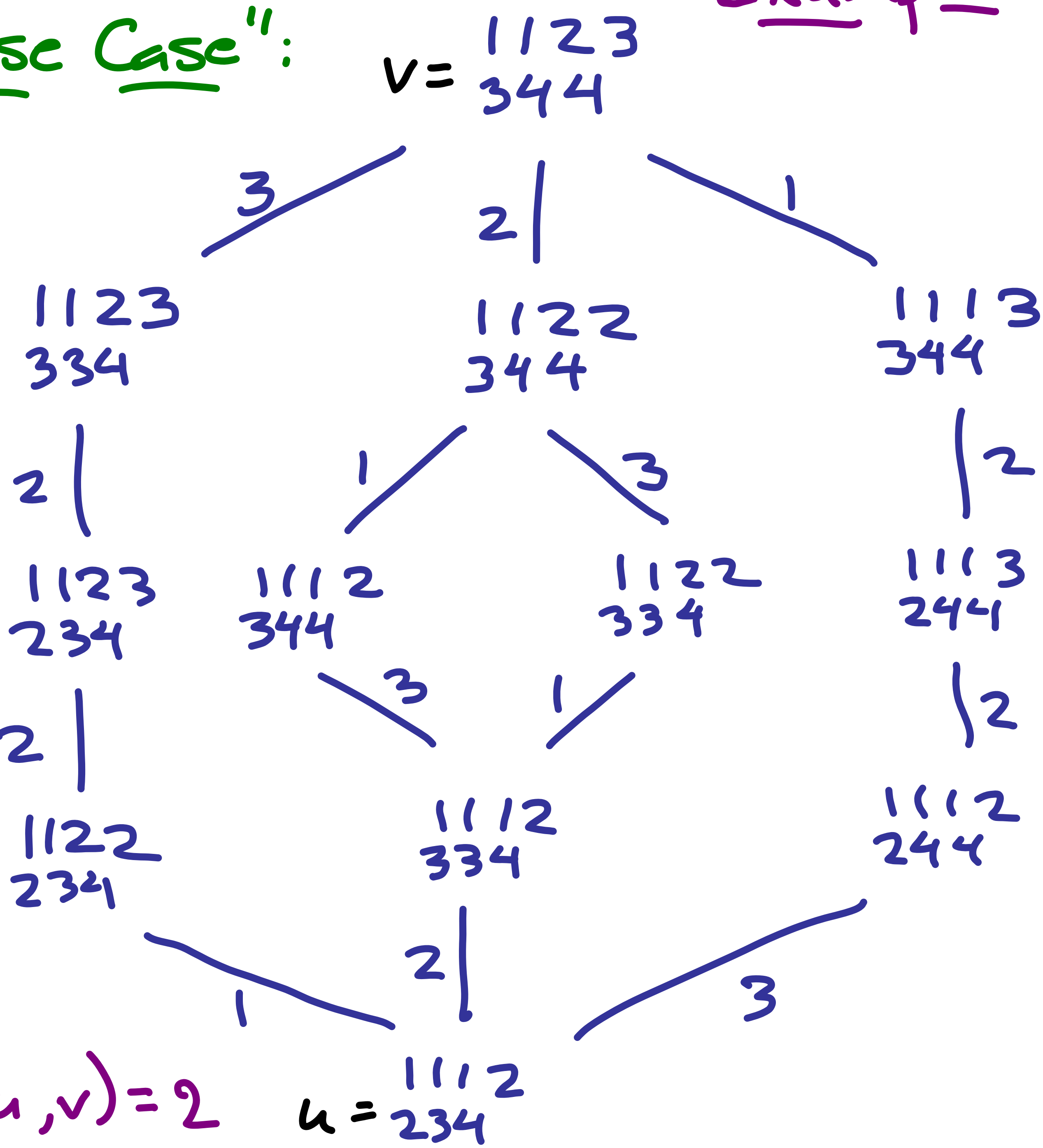


$M_P(u,v) = 2$

& not connected by "Stembridge moves"

Infinite Families of (Negative) Examples

"Base Case":



$M_P(u,v)=2$

& not connected by "Stembridge moves"

Talk Outline (for 2nd Half)

I. Background Review (cont.)

II. Positive Results for Lower Intervals $[\hat{0}, u]$

- Möbius function & homotopy type
- Connectedness of saturated chains under "Stembridge moves"

III. Negative Results for Arbitrary Type A Intervals $[u,v]$

- Arbitrarily large Möbius functions
- Arbitrarily high degree non-redundant "relations" amongst crystal operators
- $\mu(u,v) \neq 0, \pm 1 \Rightarrow$ rel'n within $[u,v]$ not generated by Stembridge local rel'ns (based on "SB-labelings")

I. Background

Defn: The (left) weak Bruhat order on Coxeter system (W, S) is the partial order with cover relations

$u \lessdot v \iff v = s_i u$ for $u, v \in W$ with $\ell(v) > \ell(u)$, for $\ell(v) = \min\{r \mid v = s_{i_1} \cdots s_{i_r}$ for $s_{i_1}, \ldots, s_{i_r} \in S\}$

e.g. $W = S_n$
$S = \{s_1, s_2, \ldots, s_{n-1}\}$ for $s_i = (i, i+1)$

with relations:

$$s_i^2 = e \;\&\; \underbrace{s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \;\&\; s_i s_j = s_j s_i}_{\text{"braid reln's"}} \text{ (for } |j-i| > 1)$$

e.g. Left weak order for S_3

$$
\begin{tikzcd}
& s_1s_2s_1 = s_2s_1s_2 & \\
s_2s_1 \arrow[ur, dash, "s_1\cdot\_"] & & s_1s_2 \arrow[ul, dash, "s_2\cdot\_"'] \\
s_1 \arrow[u, dash, "s_2\cdot\_"] & & s_2 \arrow[u, dash, "s_1\cdot\_"'] \\
& e \arrow[ul, dash, "s_1\cdot\_"] \arrow[ur, dash, "s_2\cdot\_"'] &
\end{tikzcd}
$$

Key Fact: Saturated chains from e to w naturally labeled w/ the "reduced expressions" $s_{i_1}\cdots s_{i_{\ell(w)}}$ for w. Likewise, saturated chains from u to v $\leftrightsquigarrow$ reduced expressions for vu^{-1}

Connectedness under Braid Moves

Thm 3.3.1 (in Björner-Brenti) Let (W,S) be a Coxeter group w/ $w \in W$. Then every two reduced expressions for w are connected via braid moves.

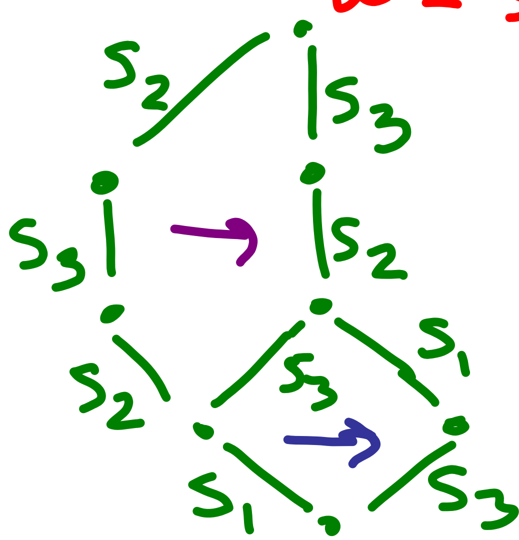
e.g. $\underbrace{s_2 s_3 s_2}\, s_1 \rightarrow \underbrace{s_3 s_2 \overbrace{s_3}}\overbrace{s_1}$

$\rightarrow s_3 s_2 \overbrace{s_1 s_3}$

$w = s_2 s_3 s_2 s_1 = s_3 s_2 s_3 s_1$

$= s_3 s_2 s_1 s_3$

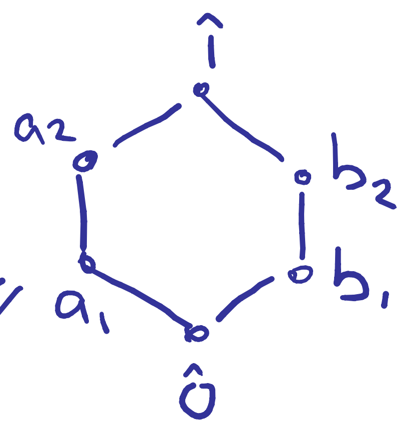
Right weak order:



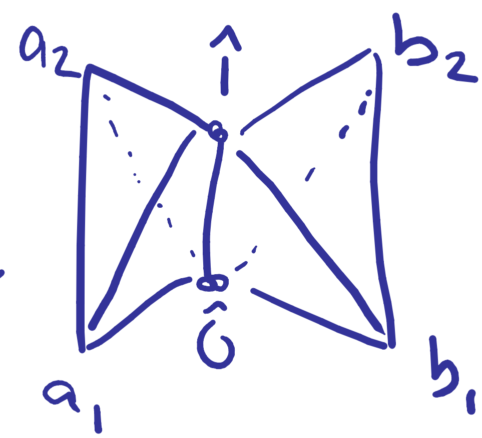
Note: Proof via lattice property for $[u,v]$

Def'n: The order complex (or nerve) of a poset P is the simplicial complex $\Delta(P)$ whose i-dimensional faces are the $(i+1)$-chains $v_0 < \dots < v_i$ in P

e.g. $P \cong$



$\Delta(P) \cong$



Recall: $\mu_P(u,v) = \tilde{\chi}(\Delta(\underbrace{(u,v)}_{= \{z \in P \mid u < z < v\}}))$

($\mu_P(u,v) = 0, \pm 1$ suggests ball or sphere)

II. Positive Results for Lower Intervals $[\hat{0}, u]$

Recall: $\mu_P(u,u) = 1$

$$\mu_P(u,v) = -\sum_{u \le z < v} \mu_P(u,z)$$

Thm 1 (H.-Lenart): Given u in a symmetrizable Kac-Moody type crystal "KM-crystal", then $\mu(\hat{0}, u) = 0, \pm 1$. More specifically, $\mu(\hat{0}, u) = 0$ unless $\mathrm{key}(u) = w_0(J)$ for some parabolic subgroup W_J with u the unique smallest element in $\mathrm{key}^{-1}(w_0(J))$, in which case $\mu(\hat{0}, u) = (-1)^{|J|}$.

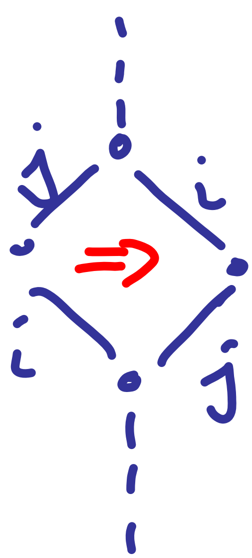
Thm 2 (H.-Lenart): Given a symmet. KM-crystal & given any parabolic W_J, then $\mathrm{key}^{-1}(w_0(W_J))$ has a unique minimal element and a unique maximal element.

(Proof via alcove path model)

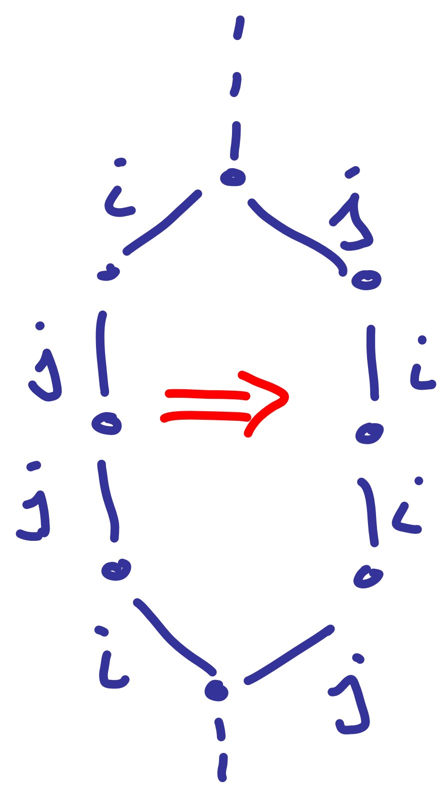
Thm 3 (H-Lenart): Each lower interval $(\hat{0}, u)$ in a symmet. KM-crystal has $\Delta(\hat{0}, u) \simeq$ ball or sphere, getting $S^{|J-2|}$ for $u = \min(k^{-1}(w_0(J)))$. Likewise for upper intervals in finite KM-crystals.

(Proof via Quillen fibre lemma)

Thm 4 (H.-Lenart): Given any lower interval $(\hat{0}, u)$ in a $\mathfrak{g}$-crystal, then set of saturated chains from $\hat{0}$ to u is connected by "Stembridge moves", namely moves of the form



and

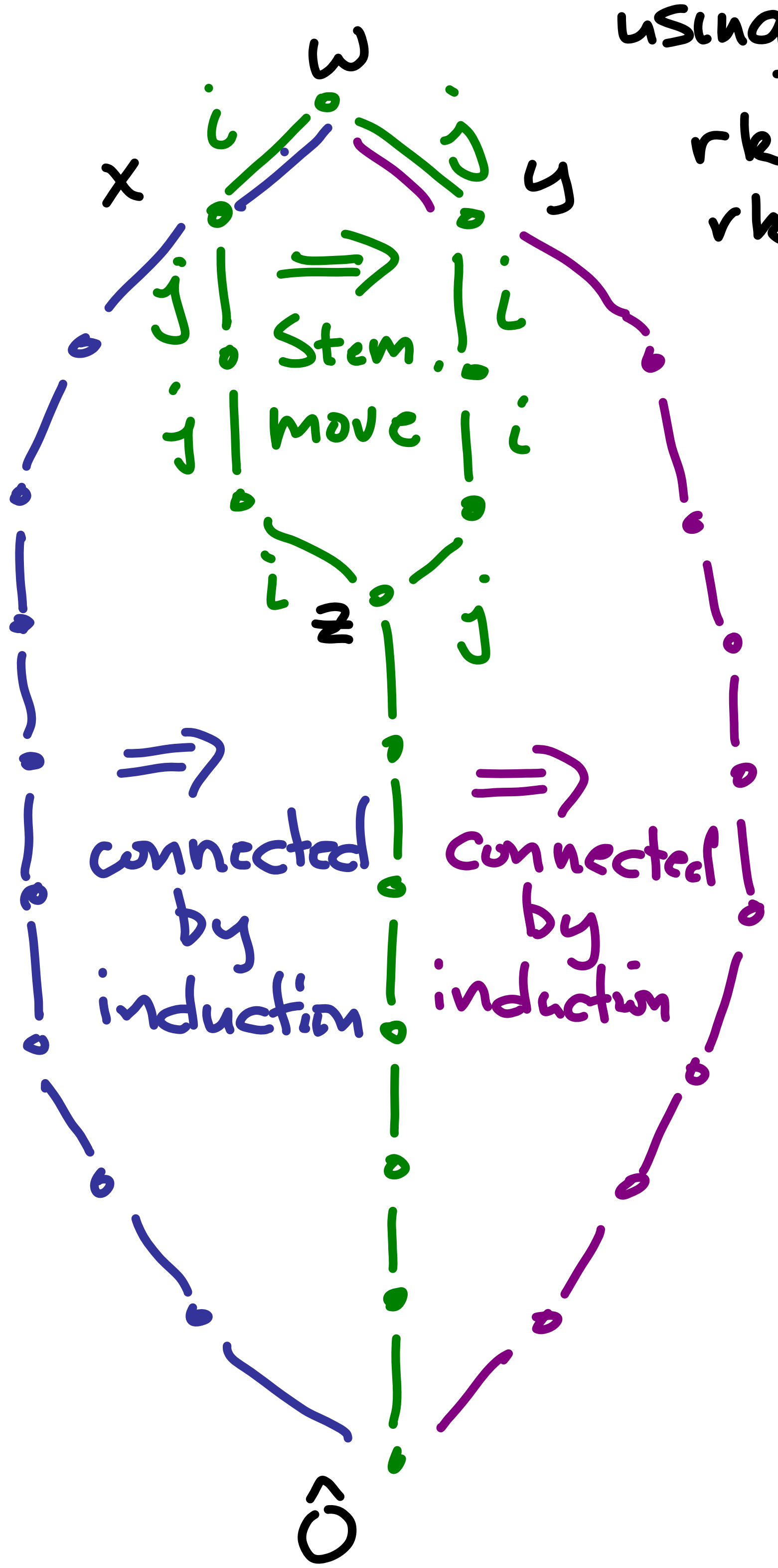


Note: Likewise in doubly-laced case via "Sternberg moves".

Proof Idea: Induction on rank using $z \geq \hat{0}$ &

$rk(x) < rk(w)$

$rk(y) < rk(w)$



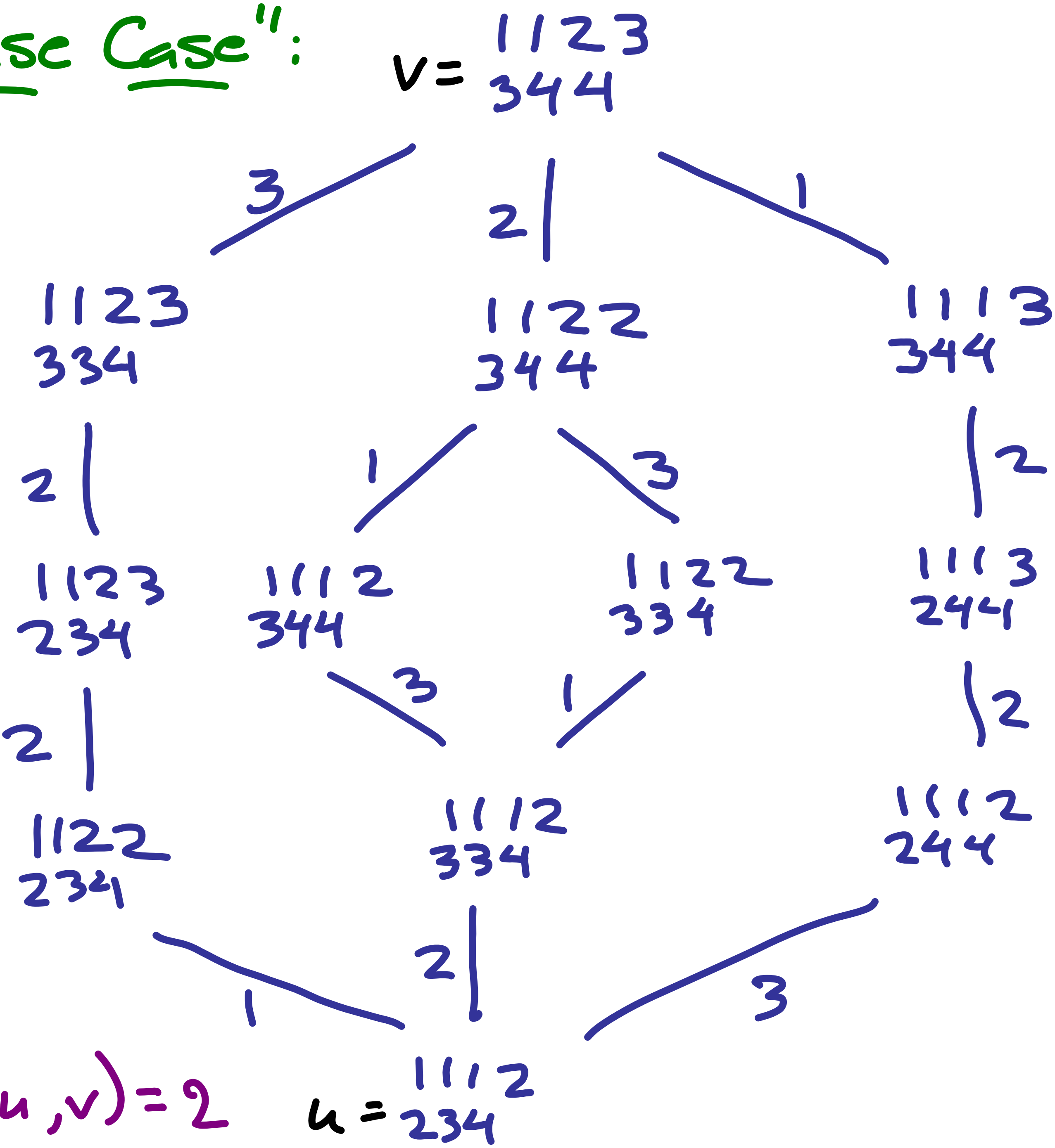
III. Negative Results for Arbitrary (not necessarily lower) Crystal Poset Intervals (type A)

Thm 5 (H.-Lenart): There exist elements u, v in type A $\mathfrak{g}$-crystals with $\mu(u,v) = 2^j$ for every positive integer j.

Thm 6 (H. Lenart): There exist type A intervals $[u,v]$ with $rk(v) - rk(u)$ arbitrarily large s.t. (u,v) is disconnected

Infinite Family of Examples

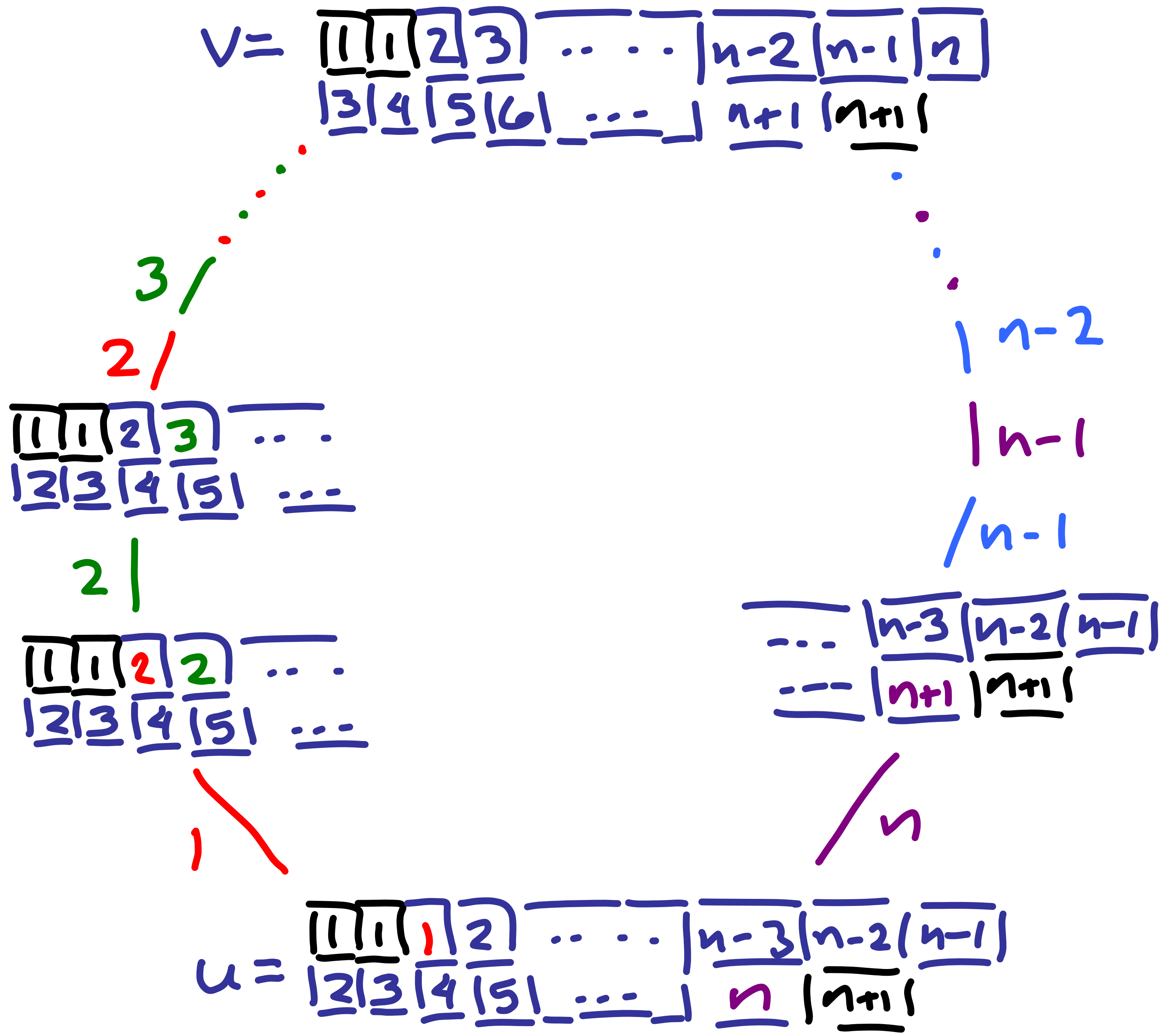
"Base Case":



$M_P(u,v) = 2$

& not connected by "Stembridge moves"

Arbitrarily High Rank Disconnected Open Intervals



label sequences: $1, 2, 2, 3, 3, 4, 4, \ldots, n-1, n-1, n$

& $n, n-1, n-1, \ldots, 2, 2, 1$ in distinct components

Consequence: Arbitrarily high degree relations $e_{i_1}\cdots e_{i_d}(u) = e_{j_1}\cdots e_{j_d}(u)$ amongst crystal operators applied to u not implied by any lower degree relations.

Systematic Method to find Such Unexpected Relations?

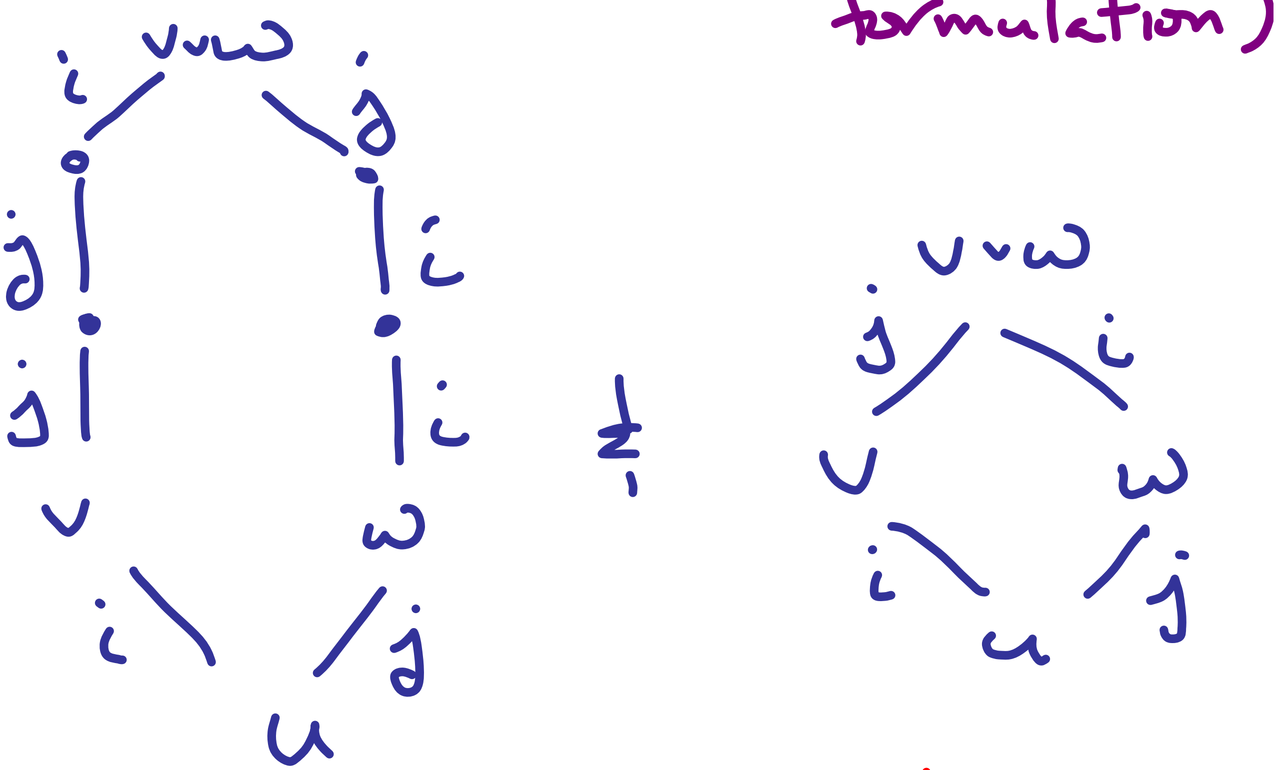
- Möbius functions, due to theory of "SB-labelings" of H.-Mészáros

Defn (H.-Mészáros): A finite lattice L has SB-labeling if it has edge labeling λ s.t. for $u \lessdot v$ & $u \lessdot w$

(1) $\lambda(u,v) \neq \lambda(u,w)$ for $v \neq w$

(2) Each saturated chain from u to $v \vee w$ uses both these labels & no others. (We call this "index 2 formulation")

e.g.



meet the requirements!

Thm (H.-Mészáros): λ satisfies index 2 formulation for SB-labeling $\Longleftrightarrow$ λ satisfies global formulation for SB-labeling.

Thm (H.-Mészáros): If a finite lattice L has SB-labeling, then $\mu_L(x,y) = 0, \pm 1$ for all $x, y \in L$.

Observation (H.–Lenart): If a crystal w/ unique minimal element were a lattice s.t. Stembridge local structure gives least upper bounds for $v \neq w$ s.t. $u \lessdot v$ & $u \lessdot w$, then edge coloring would be SB-labeling.

Lemma (H.–Lenart): If crystal has $u \lessdot f_i(u)$ & $u \lessdot f_j(u)$ s.t. $x = f_i f_j(u) = f_j f_i(u)$ (resp. $x = f_i f_j^2 f_i(u) = f_j f_i^2 f_j(u)$), then there does not exist x' s.t. $u < x' < x$.

Thm (H–Lenart): If crystal of rep'n has $\mu(x,y) \neq 0, \pm 1$, then have rel'n of crystal operators within $[x,y]$ that is not implied by Stembridge local rel'ns.

Appendix: a few slides with extra details...

Crystals

A **crystal** B of type Φ is a nonempty set B with raising & lowering operators e_i, f_i & fns

$$\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \cup \{-\infty\}$$

$$wt : B \longrightarrow \Lambda = \text{weight lattice of type } \Phi$$

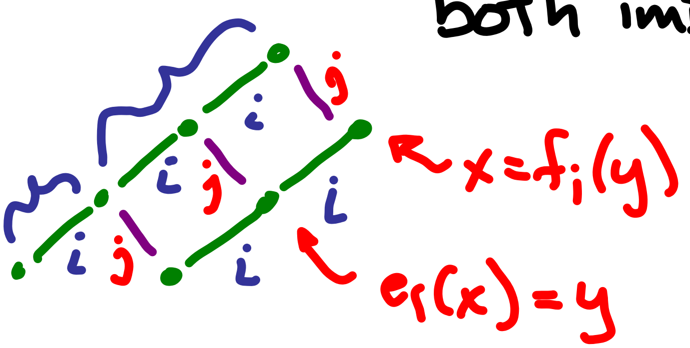
s.t.

(A1) $x, y \in B$, then $e_i(x) = y \iff x = f_i(y)$

both implying $wt(y) = wt(x) + \alpha_i$

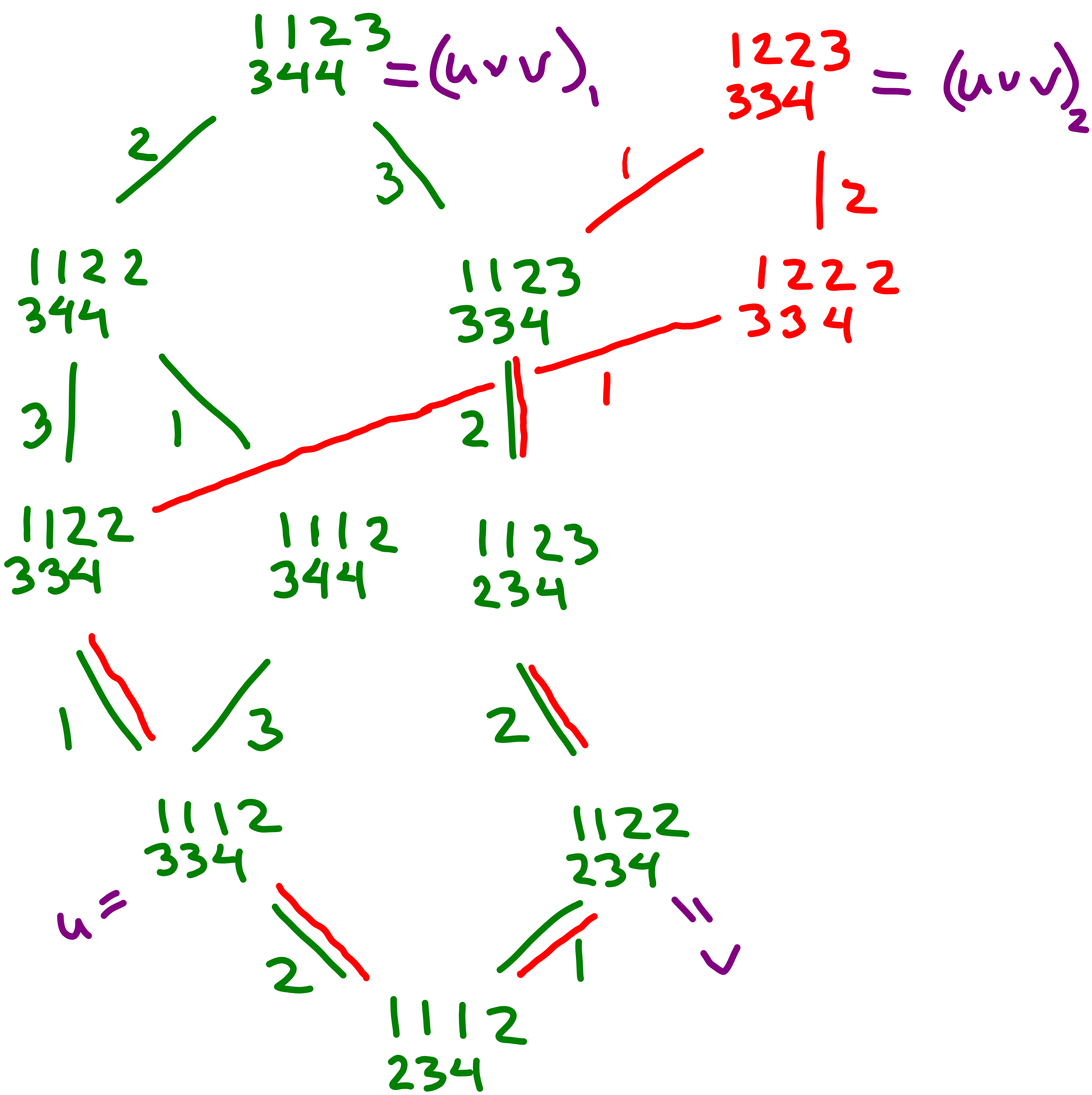
& $\varepsilon_i(y) = \varepsilon_i(x) - 1$

$\varphi_i(y) = \varphi_i(x) + 1$



(A2) $\varphi_i(x) - \varepsilon_i(x) = \langle wt(x), \alpha_i^\vee \rangle$

Non-Lattice Example:



Examples with $\mu(u,v) = 2^j$

$j=1$: $u = \begin{matrix} 1112 \\ 234 \end{matrix}$ $\qquad v = \begin{matrix} 1123 \\ 344 \end{matrix}$

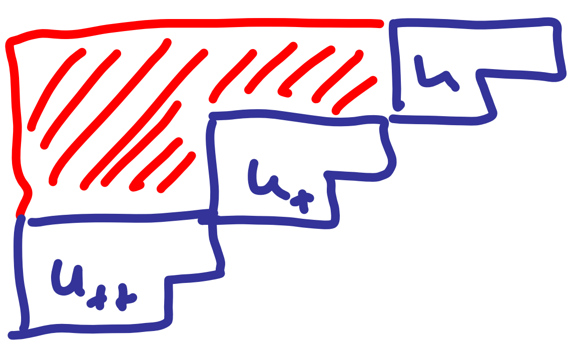
$j=2$: $u^{(2)} = \begin{matrix} 1111 & \boxed{1112} \\ 2222 & 234 \\ \boxed{6667} & \\ 789 & \end{matrix}$ $\qquad v^{(2)} = \begin{matrix} 1111 & \boxed{1123} \\ 2222 & 344 \\ \boxed{6678} & \\ 899 & \end{matrix}$

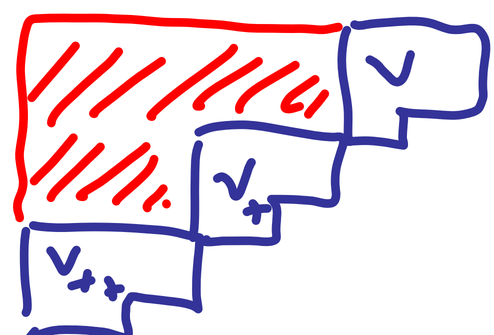
$\boxed{\begin{matrix}1112\\234\end{matrix}} = u$, $\quad \boxed{\begin{matrix}1123\\344\end{matrix}} = v$

$u_+ := u+5 = \begin{matrix} 6667 \\ 789 \end{matrix}$ $\qquad v_+ := v+5 = \begin{matrix} 6678 \\ 899 \end{matrix}$

$$[u^{(2)}, v^{(2)}] \cong [u,v] \times [u,v]$$

so $\mu(u^{(2)}, v^{(2)}) = 2^2$

$u^{(3)} =$ 

$v^{(3)} =$ 

$$[u^{(k)}, v^{(k)}] \cong \underbrace{[u,v] \times \cdots \times [u,v]}_{k\text{-fold}} \qquad \mu = 2^k$$

Demazure Module Given by $w=s_2s_1s_3$

$$
\begin{tikzcd}
 & & \begin{smallmatrix}1&3\\3\\4\end{smallmatrix} \arrow[dr, "e_2"] & & & \\
 & & & \begin{smallmatrix}1&3\\2\\4\end{smallmatrix} \arrow[dr, "e_2"] & & \\
\begin{smallmatrix}1&3\\2\\3\end{smallmatrix} \arrow[drr, "e_2"'] & & & & \begin{smallmatrix}1&2\\2\\4\end{smallmatrix} \arrow[dll, "e_3"] \arrow[d, "e_1"] & \begin{smallmatrix}1&1\\3\\4\end{smallmatrix} \arrow[dl, "e_2"] \\
 & & \begin{smallmatrix}1&2\\2\\3\end{smallmatrix} \arrow[dr, "e_1"'] & & \begin{smallmatrix}1&1\\2\\4\end{smallmatrix} \arrow[dl, "e_3"] & \\
 & & & \begin{smallmatrix}1&1\\2\\3\end{smallmatrix} & &
\end{tikzcd}
$$

Key Polynomial

||

$$K_{(1,0,2,1)} = \sum_{\substack{T' \le T \\ \text{componentwise}}} x^{T'}$$

$\Updownarrow$

$k(T') \le_{\text{Bruhat}} k(T)$ w/ no higher e_i exponents

Crucial Properties of Key

Thm (Littlemann): Given any symmetrizable Kac-Moody algebra A, the key of any crystal of type A satisfies:

$$K(f_p(F)) = \begin{cases} K(F) & \text{if } e_p(F) \neq 0 \\ s_p K(F) \text{ or } K(F) & \text{if } e_p(F) = 0 \end{cases}$$

Also, if $e_p(F) = 0$ then $s_p K(F) > K(F)$

Corollary: If $K(F) = s_{i_1} \cdots s_{i_r}$ then there exists saturated chain from F to $\hat{0}$ given by applying $e_{i_r}^{d_r} \cdots e_{i_1}^{d_1}$ to F for some $d_1, \ldots, d_r > 0$.

Relation to Reiner-Shimozono Viewpoint on Key Polynomials

- $\partial_i = \dfrac{1-s_i}{x_i - x_{i+1}}$ & $\pi_i = \partial_i x_i$
- $K_\alpha = \pi_{i_1} \cdots \pi_{i_r} \, x^{\lambda(\alpha)}$ for α composition of n & $s_{i_1} \cdots s_{i_r}$ sorting α to $\lambda(\alpha)$

e.g. $K_{(1,0,2,1)} = \pi_2 \pi_1 \pi_3 \, x^{(2,1,1,0)}$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

$$\begin{aligned}
&= \pi_2 \pi_1 \left(x_1^2 x_2 (x_3 + x_4)\right) \\
&= \pi_2 \left(x_1 x_2 x_3 (x_1 + x_2) + x_1 x_2 x_4 (x_1 + x_2)\right) \\
&= x_1^2 x_2 x_3 + x_1 x_2 x_3 (x_2 + x_3) \\
&\quad + x_1^2 x_4 (x_2 + x_3) + x_1 x_4 (x_2^2 + x_2 x_3 + x_3^2)
\end{aligned}$$