From the Weak Bruhat Order to Crystal Posets

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**Perspective & Main Goal:**

- Study crystal graphs regarded as posets via poset map to weak Bruhat order, namely via the (right) key map.

**Poset Structure for Many Crystals**

\[ u \prec_{\text{crystal}} v \iff u \xrightarrow{f_i} v \text{ for some } i \]

- Transfer properties of weak Bruhat order to crystals.

**Weak Bruhat order:**

\[ u \prec_{\text{weak }} s_i u \text{ if } l(s; u) > l(u) \]

- Discover surprising new reln's amongst crystal operators
Motivations for Crystals

- Study representation theory of Kac-Moody algebras (e.g. affine Lie algebras)
- Take universal enveloping algebra, its quantum algebra w/ parameter \( q \)
- \( q \rightarrow 1 \) yields \( U(A) \) for Kac-Moody algebra \( A \)
- \( q \rightarrow 0 \) yields algebra with same dimensions of weight spaces encoded by combinatorics of "crystal graphs" (which are often posets)
  - basis vectors for poset elts \( \rightarrow \) weight spaces (with remarkable properties)
- cover relns \( \rightarrow \) Kashiwara raising operators
(Type A) Crystals of Highest Weight
Representations & their Kashiwara
Lowering Operators

e.g. \( \lambda = \begin{array}{c} \\
\end{array} \)

weight \( (2,0,1) \)
\( x_1^2 x_3 \)

changes weight by \( (1,1,0) \)

weight \( (1,2,0) \)
\( x_1 x_2^2 \)

weight \( (2,1,0) \)
\( x_2^2 x_1 \)
\[ \lambda = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \]

"character" of crystal

\[ = x_1^3 x_2^2 + x_1^2 x_2^3 + \ldots = \text{weight}(z_{11}) + \text{weight}(z_{112}) + \ldots \]
Type A crystal for highest weight rep'n of type A

1. $\hat{\Theta} = \begin{array}{c}
\frac{111 - 1}{22 - 2} \\
\frac{33 - 3}{\cdots}
\end{array}$

2. $u \mapsto \nu$ for $\nu$ obtained from $u$

by incrementing to an $i+1$ the rightmost $i$ that is not in a "parenthesization pair" w/ an "$i+1"

 Parenthoodization Pairs: Read leftmost column bottom to top, then subsequent columns L to R, ignoring all but $i, i+1$; pair up consec. $i, i+1$; delete; repeat ...

c.g., $\mathbf{1} \, \mathbf{1} \, \mathbf{1} \, \mathbf{1} \, \mathbf{1} \, 4 \, 4 \, 4 \, 3 \, 3 \, 4 \, 4 \, 4$

$u = \begin{array}{c}
\frac{2233}{344}
\end{array}$

$\nu = \begin{array}{c}
\frac{34433444}{\nu_3}
\end{array}$
Stambridge Crystals: "\(g\)-crystals"
(Crystals of highest weight repn's in Simply laced case)

a) \(X^{(t)}_{B\lambda} = \sum_{b \in B(\lambda)} t^{\text{wt}(b)} = \text{character of irrep } B(\lambda)\)

b) \(f_i(x) f_j(x) \Rightarrow f_i f_j(x) \text{ or } f_j f_i(x)\)

c) likewise for \(e_i, e_j\) operators

d) axioms yield this & characterize crystals of highest weight repn's in Simply laced case
Right key "k" of a KM-crystal

\[ k(\frac{124}{23}) = s_3 s_2 s_1 s_2 \]

\[ k(\frac{114}{23}) = s_3 s_2 s_1 \]

\[ k(\frac{123}{23}) = s_2 s_1 s_2 s_1 \]

\[ k(\frac{113}{23}) = s_2 s_1 \]

\[ k(\frac{112}{22}) = s_1 s_2 \]

\[ k(\frac{111}{22}) = s_1 \]

\[ k(\frac{122}{23}) = s_1 s_2 \]

\[ k(\frac{1111}{22}) = s_2 \]

\[ k : \text{crystal \to weak post} \]

\[ k(\frac{111}{22}) = e \]

\[ \forall u \leq v \Rightarrow k(u) \leq k(v) \]
New Algorithm to Calculate Right Key of a KM-Crystal

(1) $\text{key}(\hat{\sigma}) = e$

(2) If $\hat{\sigma} \rightarrow a$, then $\text{key}(a) = s_i$
   (i.e. $\hat{\sigma} < a$)

(3) If $v$ covers 2 or more elements then $\text{key}(v) = v \text{ key}(u) \bigcap \{u \rightarrow v\}$
   (for join taken in weak order)

(4) If $u \rightarrow v$ and $v$ does not cover any other elements, then:
   (a) $\text{key}(v) = \text{key}(u)$ if $\exists u' \rightarrow u$
   (b) $\text{key}(v) = s_i \cdot \text{key}(u)$ otherwise
Key Polynomials & right/left key
(see Lascoux-Schützenberger & e.g. Reiner-Shimozono)

Motivations:
1. Schubert poly. $G_w$ is positive sum of "key polynomials"
2. Key polynomial records character for Demazure module
3. The (closely related) right/left key maps determine smallest Demazure modules containing a given crystal element
4. These will give us poset map from $q$-crystal to weak Bruhat order, transferring properties
Infinite Families of (Negative) Examples

"Base Case": \( v = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 4 \end{pmatrix} \)

\[
M_p(u, v) = 2 \quad u = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix}
\]

\( \) not connected by "Stembridge moves"
Infinite Families of (Negative) Examples

"Base Case": \( v = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \end{pmatrix} \)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3 \\
2 \\
1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1123 \\
334 \\
234
\end{array}
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\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1122 \\
344 \\
2341
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\begin{array}{c}
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\begin{array}{c}
1112 \\
334 \\
244
\end{array}
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\end{array}
\]

\[
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\begin{array}{c}
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\]

\[ M_p(u,v) = 2 \quad u = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix} \]

\[ \text{not connected by "Stembridge moves"} \]
Talk Outline (for 2nd Half)

I. Background Review (cont.)

II. Positive Results for Lower Intervals $[0,u]$
   - Möbius function & homotopy type
   - Connectedness of saturated chams under "Stembridge moves"

III. Negative Results for Arbitrary Type A Intervals $[u,v]$
   - Arbitrarily large Möbius functions
   - Arbitrarily high degree non-redundant "relations" amongst crystal operators
   - $M(u,v) \neq 0 \Rightarrow$ rel in within $[u,v]$ not generated by Stembridge local rels'ns (based on "SB-labelings")
I. Background

Defn: The (left)weak Bruhat order on Coxeter system \((W,S)\) is the partial order with cover relations
\[ u < v \equiv v = s_i u \text{ for } u, v \in W \text{ with } l(v) - l(u) = 1 \text{ for } s_i \in S \]

e.g. \( W = S_n \)
\[ S = \{s_1, s_2, \ldots, s_{n-1}\} \text{ for } s_i = (i, i+1) \]

with relations:
\[ s_i^2 = e \quad s_i s_k = s_k s_i \quad (s_i s_j = s_j s_i) \text{ for } |i - j| > 1 \]

"braid reln's"
**e.g.** Left weak order for $S_3$

\[
S_1 S_2 S_1 = S_2 S_1 S_2
\]

**Key Fact:** Saturated chains from $e$ to $w$ naturally labeled with the "reduced expressions" $s_{i_1} \ldots s_{i_k} \text{ for } w$. Likewise, saturated chains from $u$ to $v$ correspond to reduced expressions for $vu'$. 
Connectedness under Braid Moves

Thm 3.3.1 (in Björner-Brenti) Let \((W,S)\) be a Coxeter group \(w/w \in W\). Then every two reduced expressions for \(w\) are connected via braid moves.

c.g. \(s_2 s_3 s_2 s_1 \rightarrow s_3 s_2 s_3 s_1 \rightarrow s_3 s_2 s_1 s_3 = s_2 s_3 s_2 s_1 = s_3 s_2 s_3 s_1 = s_3 s_2 s_1 s_3\)

Right weak order:

\[
\begin{array}{c}
\ s_2 \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ s_3 \\
\ s_3 \ \rightarrow \ \ s_2 \\
\ s_2 \ \rightarrow \ \ s_3 \\
\ s_3 \rightarrow \ s_1 \\
\ s_1 \rightarrow \ s_3
\end{array}
\]

Note: Proof via lattice property for \([u,v]\)
**Defn:** The order complex (or nerve) of a poset $P$ is the simplicial complex $\Delta(P)$ whose $i$-dimensional faces are the $(i+1)$-chains $v_0 < \ldots < v_i$ in $P$.

**Example:**

![Diagram](image)

$P = a_1 \leq a_2 \leq b_1 \leq b_2$,

$\Delta(P) = \Delta(P)$

**Recall:** $M_{P}(u,v) = \tilde{X}(\Delta(u,v))$

$\{z \in P | u \leq z \leq v \}$

$(M_{P}(u,v) = 0, \pm 1$ suggests ball or sphere$)$
II. Positive Results for Lower Intervals $[\delta, u]$ 

Recall: 

\[ M_p(\delta, u) = 1 \]

\[ M_p(u, v) = - \sum_{u \leq z < v} M_p(u, z) \]

Thm 1 (H.-Lenart): Given $u$ in a symmetrizable Kac-Moody type crystal "$K$-crystal", then $M(\delta, u) = 0, \pm 1$. More specifically, $M(\delta, u) = 0$ unless $\text{key}(u) = \omega_0(v)J$ for some parabolic subgroup $W_J$ with $u$ the unique smallest element in $\text{key}^{-1}(\omega_0(v)J)$, in which case $M(\delta, u) = (1)^{|J|}$. 
Thm 2 (H.-Lenart): Given a symmet. KM-crystal & given any parabolic $\omega_J$, then $\text{key}^-(\omega_0(\omega_J))$ has a unique minimal element and a unique maximal element. (Proof via alcove path model)

Thm 3 (H.-Lenart): Each lower interval $(\hat{0}, u)$ in a symmet. KM-crystal has $\Delta(\hat{0}, u) = \text{ball or sphere}$, getting $S^{1|J-2}/u = \min(k^n(\omega_0(J)))$. Likewise for upper intervals in finite KM-crystals. (Proof via Quillen fibre lemma)
Thm 4 (H.-Lenart): Given any lower interval \((\hat{0}, u)\) in a \(\gamma\)-crystal, then set of saturated chains from \(\hat{0}\) to \(u\) is connected by "Stembridge moves", namely moves of the form and

Note: Likewise in doubly-laced case via "Stemberg moves".
Proof Idea: Induction on rank using $z^\circ \downarrow$

$\Rightarrow \quad \Rightarrow$

$\Rightarrow$ connected by induction

$\Rightarrow$ connected by induction

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III. Negative Results for Arbitrary (not necessarily lower) Crystal Poset Intervals (type A)

Thm 5 (H.-Lenart): There exist elements $u, v$ in type A $q$-crystals with $M(u, v) = 2j$ for every positive integer $j$.

Thm 6 (H. Lenart): There exist type A intervals $[u, v]$ with $rk(u) - rk(v)$ arbitrarily large s.t. $(u, v)$ is disconnected
Infinite Family of Examples

"Base Case": \[ v = \begin{array}{c} 1123 \\ \hline 344 \end{array} \]

\begin{align*}
M_p(u,v) &= 2 \\
\text{not connected by "Stembridge moves"}
\end{align*}
Arbitrarily High Rank
Disconnected Open Intervals

\[
V = \begin{bmatrix}
1 & 1 & 2 & 3 \\
2 & 3 & 4 & 5 \\
\vdots & \vdots & \ddots & \ddots \\
3 & 4 & 5 & 6 \\
n-2 & n-1 & n \\
n+1 & n+1 & \ddots & \ddots \\
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
1 & 1 & 2 & 3 \\
2 & 3 & 4 & 5 \\
\vdots & \vdots & \ddots & \ddots \\
3 & 4 & 5 & 6 \\
n-3 & n-2 & n-1 \\
n & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

Label sequences: 1, 2, 2, 3, 3, 4, 4, \ldots, n-1, n-1, n

\[n, n, n, \ldots, 2, 2, 1\] in distinct components
Consequence: Arbitrarily high degree relations $e_{ij} = e_{jd}(u) = e_{ji} = e_{jd}(u)$ amongst crystal operators applied to $u$ not implied by any lower degree relations.

Systematic Method to Find Such Unexpected Relations?

- Möbius functions, due to theory of "SB-labelings" of H.-Meszáros
**Defn (H.-Meszáros):** A finite lattice $L$ has SB-labeling if it has edge labeling $\lambda$ s.t. for $u < v$ if $u < w$

1. $\lambda(u, v) \neq \lambda(u, w)$ for $v \neq w$

2. Each saturated chain from $u$ to $vuw$ uses both these labels and no others. (We call this "index 2 formulation")

**e.g.**

![Diagram of lattice elements labeled with indices](attachment:diagram.png)

meet the requirements!
Thm (H.-Meszáros): \( \lambda \) satisfies index 2 formulation for SB-labeling
\( \iff \lambda \) satisfies global formulation for SB-labeling.

Thm (H.-Meszáros): If a finite lattice \( L \) has SB-labeling, then
\( M_L(x, y) = 0, \pm 1 \) for all \( x, y \in L \).
Observation (H.-Lenart): If a crystal with unique minimal element were a lattice s.t. Stembridge local structure gives least upper bounds for \( v \neq w \) s.t. \( u \lessdot v \) \& \( u \lessdot w \), then edge coloring would be SB-labeling.

Lemma (H.-Lenart): If crystal has \( u \lessdot f_i(u) \) \& \( u \lessdot f_j(u) \) s.t. \( x = \uparrow_i f_j(u) = f_i f_j(u) \) (resp. \( x = \downarrow_i f_j(u) = f_j f_i(u) \)), then there does not exist \( x' \) s.t. \( u \lessdot x' \lessdot x \).

Theorem (H.-Lenart): If crystal of rep \( h \) has \( M(x,y) \neq 0, \pm 1 \), then have reln of crystal operators within \([x,y]\) that is not implied by Stembridge local reln's.
Appendix: a few slides with extra details...
Crystals

A crystal $B$ of type $\Phi$ is a nonempty set $B$ with raising $\dagger$ and lowering operators $e_i, f_i : \dagger \subset \Lambda$, such that $e_i, f_i : B \rightarrow \mathbb{Z} \cup \{0\}$. The weight function $w : B \rightarrow \Lambda = \text{weight lattice of type } \Phi$, subject to:

(A1) $x, y \in B$, then $e_i(x) = y \Rightarrow x = f_i(y)$, both implying $w(y) = w(x) + \alpha_i$

(A2) $\varphi_i(x) - \Sigma_i(x) = \langle \omega_t(x), \alpha_i' \rangle$
Non-Lattice Example:

\[
\begin{align*}
\frac{1123}{344} &= (uuv)_1, \\
\frac{1223}{334} &= (uuv)_2
\end{align*}
\]
Examples with $M(u,v) = 2^j$

$j=1$: $u = \frac{1112}{234} \quad v = \frac{1123}{344}$

$j=2$: $u^{(2)} = \begin{bmatrix} 1111 \\ 2222 \\ 6667 \\ 389 \end{bmatrix} \quad v^{(2)} = \begin{bmatrix} 1123 \\ 2222 \\ 6678 \\ 899 \end{bmatrix}$

$u_+^{(2)} = u + 5 = \begin{bmatrix} 6667 \\ 389 \end{bmatrix} \quad v_+^{(2)} = v + 5 = \begin{bmatrix} 6678 \\ 899 \end{bmatrix}$

$[u^{(2)}, v^{(2)}] \cong [u, v] \times [u, v]$

So $M(u^{(2)}, v^{(2)}) = 2^2$

$u^{(3)} = \begin{bmatrix} u_+^{(2)} \\ u_+^{(2)} \end{bmatrix}$ \quad $v^{(3)} = \begin{bmatrix} v_+^{(2)} \\ v_+^{(2)} \end{bmatrix}$

$[u^{(k)}, v^{(k)}] \cong [u, v] \times \ldots \times [u, v] \quad M = 2^k$
Demazure Module Given by \( \omega = s_2 s_1 s_3 \)

Key Polynomial

\[
K_{(1,0,2,1)} = \sum x^{T'} \\
T' \leq T \\
\text{componentwise} \\
K(T') \leq \text{Brm} + K(T) \text{ w/ exponents}
\]
Crucial Properties of Key

Thm (Littlemann): Given any symmetrizable Kac-Moody algebra $A$, the key of any crystal of type $A$ satisfies:

$$K(\ell_p(F)) = \begin{cases} K(F) & \text{if } e_p(F) \neq 0 \\ s_p K(F) \text{ or } K(F') & \text{if } e_p(F) = 0 \end{cases}$$

Also, if $e_p(F) = 0$ then $s_p K(F) > K(F)$

Corollary: If $K(F) = s_{i_1} \ldots s_{i_r}$ then there exists saturated chain from $F$ to 0 given by applying $e_{i_r} \ldots e_{i_1}$ to $F$ for some $d_i > 0$. 

Relation to Reiner-Shimozono Viewpoint on Key Polynomials

- $c_i = 1 - s_i \quad \iff \quad \prod_i = 2_i x_i$
- $K_\alpha = \prod_i \cdots \prod_{i+r} x_\lambda(\alpha)$ for a composition of $n \vdash s_1 \cdots s_r$ sorting $\alpha$ to $\lambda(\alpha)$

E.g., $K_{(1,0,2,1)} = \prod_2 \prod_1 \prod_3 x_{(2,1,1,0)}$

- $= \prod_2 \prod_1 \left( x_1^2 x_2 (x_3 + x_4) \right)$
- $= \prod_2 \left( x_1 x_2 x_3 (x_1 + x_2) + x_1 x_2 x_4 (x_1 + x_2) \right)$
- $= x_1^2 x_2 x_3 + x_1 x_2 x_3 (x_2 + x_3) + x_1 x_4 (x_2 + x_3 + x_4)$