From the Weak Bruhat Order to Crystal Posets

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Perspective & Main Goal:

- Study crystal graphs regarded as posets via poset map to weak Bruhat order, namely via the (right) key map.

Poset Structure for Many Crystals

\[ u \prec_{\text{crystal}} v \iff u \xrightarrow{f_i} v \text{ for some } i \]

- Transfer properties of weak Bruhat order to crystals.

Weak Bruhat order:

\[ u \prec_{\text{weak}} s_i u \text{ if } l(s_i u) > l(u) \]
Motivations for Crystals

- Study representation theory of Kac–Moody algebras (e.g. affine Lie algebras)

- Take universal enveloping algebra, & its quantum algebra w/ parameter $\gamma$

- $\gamma \rightsquigarrow 1$ yields $\mathcal{U}(\mathfrak{A})$ for Kac–Moody algebra $\mathfrak{A}$

- $\gamma \rightsquigarrow 0$ yields algebra with same dimensions of weight spaces, described by combinatorics of "crystal graphs"
(Type A) Crystals of Highest Weight
Representations & their Kashiwara
Lowering Operators

E.g. \( \lambda = \square \)

\[
\begin{array}{c}
\frac{1^2}{2} \quad \frac{1}{2} \\
\downarrow f_1 \\
\frac{1^2}{2} \quad \frac{1}{2} \\
\downarrow f_2 \\
\frac{1^3}{2} \quad \frac{1}{2} \\
\downarrow f_2 \\
\frac{1^3}{3} \\
\downarrow f_1 \\
\frac{2^2}{3} \\
\downarrow f_2 \\
\frac{2^2}{3} \\
\end{array}
\]
(We study dual poset)

\[ \lambda = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline \end{array} \]

Type A highest weight vector

\[ f_1 \]
\[ f_2 \]

\[ \begin{array}{|c|c|c|c|} \hline 111 & 2 & 111 & 3 \\ \hline 2 & f_1 & 2 & f_2 \\ \hline 3 & f_1 & 3 & f_2 \\ \hline \end{array} \]

\[ \begin{array}{|c|c|c|} \hline 122 & 113 & 122 \\ \hline 2 & f_1 & 3 \\ \hline 3 & f_1 & 3 \\ \hline \end{array} \]

\[ \begin{array}{|c|c|c|} \hline 112 & 2 & 112 & 3 \\ \hline 2 & f_1 & 2 & f_2 \\ \hline 3 & f_1 & 3 & f_2 \\ \hline \end{array} \]

\[ \begin{array}{|c|c|c|} \hline 123 & 212 & 222 \\ \hline 2 & f_1 & 3 \\ \hline 3 & f_1 & 3 \\ \hline \end{array} \]

\[ \begin{array}{|c|c|c|} \hline 133 & 313 & 323 \\ \hline 3 & f_1 & 3 \\ \hline 3 & f_1 & 3 \\ \hline \end{array} \]

\[ \begin{array}{|c|c|c|} \hline 233 & 323 \\ \hline 3 & f_1 & 3 \\ \hline \end{array} \]

\[ f_1 \text{ ignores letters other than } i \leftrightarrow i+1, \]

pairs \( i+1 \) followed by \( i \), then \( f_i: i^{r(i+1)} \mapsto i^{r-1}(i+1)^{s+1} \)
Talk Outline:

I. Background Review

II. New Algorithm to Calculate Right Key of Crystal
   - does not depend on choice of model

III. Positive Results for Lower Intervals \([0,u]\)
   - Möbius function & homotopy type
   - Connectedness of saturated chains under "Stembridge moves" & "Sternberg moves"

IV. Negative Results for Arbitrary Type A Intervals \([u,v]\)
   - Arbitrarily large Möbius functions
   - Arbitrarily high degree non-redundant "relations" amongst crystal operators
Base Case for Negative Examples

\[ v = \begin{bmatrix}
1 & 1 & 2 & 3 \\
3 & 4 & 4
\end{bmatrix} \]

\[ \begin{align*}
M_p(u,v) &= 2 \\
u &= \begin{bmatrix}
1 & 1 & 1 \\
2 & 3 & 4
\end{bmatrix}
\end{align*} \]

\# not connected by "Stembridge moves"
I. **Background**

**Defn:** The (left)weak Bruhat order on Coxeter system \((W, S)\) is the partial order with cover relations 
\[ u < v \iff v = s_i u \text{ for } u, v \in W \text{ with } l(v) = l(u) + 1, \text{ for } l(v) = \min \{ r \mid v = s_{i_1} \cdots s_{i_r} \} \]

*Note:* \(W = S_m\)

\[ S = \{ s_1, s_2, \ldots, s_{n-1} \} \text{ for } s_i = (i, i+1) \]

with relations:
\[ s_i^2 = e, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \quad (\text{for } |j-i| > 1) \]

"braid moves"
e.g. **Left weak order for** $S_3$

\[
s_1s_2s_1 = s_2s_1s_2
\]

**Key Fact:** Saturated chains from $e$ to $w$ naturally labeled w/ the "reduced expressions" $s_{i_1} \ldots s_{i_\ell}(w)$ for $w$. Likewise, saturated chains from $u$ to $v$ (\(\sim\) reduced expressions for $vu$)!
Properties of Reduced Expressions

\textbf{Weak Bruhat Order}

(see Björner-Brenti book for details)

**Thm 3.2.1:** Weak order on \( W \) is a meet-semilattice.

**Corollary:** Each \( [u,v] \) is a lattice.

**Lemma 3.2.3:** For \( J \subseteq S \), the join of atoms \( v_{\cup j} \) exists \( \iff W_J = \langle j | j \in J \rangle \) \( j \in J \) finite, in which case \( v_{\cup j} = \omega_0 (W_J) \) \( j \in J \) longest element

**Useful Related Fact:** \( v_j \leq u \iff \exists \text{ reduct. exp. for } u \text{ s.t. } S_j \text{ rightmost parabolic subgroup generated by } J \)
**Connectedness under Braid Moves**

Thm 3.3.1 (Björner-Brenti) Let \((W,S)\) be a Coxeter group w/ \(w \in W\). Then every two reduced expressions for \(w\) are connected via braid moves.

c.g. \(s_1s_2s_3s_2 \rightarrow s_1s_3s_2s_3 \rightarrow s_3s_1s_2s_3\)

**Right weak order:**

\[ w = s_1s_2s_3s_2 = s_1s_3s_2s_3 = s_3s_1s_2s_3 \]

**Note:** Proof via lattice property for \([4,4]\)
**Defn**: The order complex (or nerve) of a poset $P$ is the simplicial complex $\Delta(P)$ whose $i$-dimensional faces are the $(i+1)$-chains $v_0 \prec \ldots \prec v_i$ in $P$.

**Recall**: $M_P(u,v) = \tilde{X}(\Delta(u,v))$  

\[\{z \in P \mid u \prec z \prec v\}\]  

($M_P(u,v) = 0,\pm 1$ suggests ball or sphere)
Crystals + q-Crystals

A crystal $B$ of type $\Phi$ is a nonempty set $B$ with raising $e_i$, lowering operators $f_i$, $\xi$, $\varphi$: $\xi, \varphi : B \rightarrow \mathbb{Z} \cup \{\pm \infty\}$, $\text{wt} : B \rightarrow \Lambda = \text{weight lattice of type } \Phi$ s.t.

(A1) $x, y \in B$, then $e_i(x) = y \Rightarrow x = f_i(y)$ both implying $\text{wt}(y) = \text{wt}(x) + \alpha_i$

(A2) $\varphi_i(x) - \xi_i(x) = \langle \text{wt}(x), \alpha_i \rangle$
Stambridge Crystals: "q-crystals"

(Crystals of highest weight repn's in Simply laced case)

• \( X(t) = \sum_{b \in B(\lambda)} t^{\text{wt}(b)} \) - character of inner \( B(\lambda) \)

• if \( y = e_i(x) \neq 0 \) and \( z = e_j(x) \neq 0 \) for \( i \neq j \) then either:
  • \( e_i e_j(x) = e_j e_i(x) \neq 0 \)
  • \( e_i e_j^2 e_i(x) = e_j e_i^2 e_j(x) \neq 0 \)

• likewise for \( f_i \) operators

• axioms yield this to characterize crystals of highest weight repn's in simply laced case
Additional **Important facts**

- Stembridge crystals are posets with cover reln’s
  \[ u \prec v \iff v = f_i(u) \]

- Sternberg gave analogous “relations” for doubly laced crystals (not characterization)

**A Crystal that is not a Poset**

```
\[ A_3^{(1)} \]
```

```plaintext
\[
\begin{align*}
\frac{1}{2} & \rightarrow \frac{1}{3} & \rightarrow \frac{2}{3} & \rightarrow \frac{1}{4} & \rightarrow \frac{2}{4} & \rightarrow \frac{3}{4} \\
\frac{1}{3} & \rightarrow \frac{2}{3} & \rightarrow \frac{1}{4} & \rightarrow \frac{2}{4} & \rightarrow \frac{3}{4} & \rightarrow \frac{1}{4}
\end{align*}
\]```
Type A SSYT Model for Crystals

A semistandard Young tableau (SSYT) of shape \( \lambda \) for \( \lambda = n \) is a filling of a "Ferrers diagram" having \( \lambda_i \) boxes in row \( i \) for \( \lambda_1 \geq \lambda_2 \geq \ldots \) with positive integers \( \{ a_{i,j} \mid 1 \leq i \leq \lambda_i, 1 \leq j \leq \lambda_i \} \) such that:

\[
\begin{align*}
    a_{11} &\leq a_{12} \leq a_{13} \leq \ldots \leq a_{1\lambda_1} \\
    & \vdots \\
    a_{21} &\leq a_{22} \leq a_{23} \leq \ldots \leq a_{2\lambda_2} \\
    & \vdots \\
    a_{\lambda_1 1} &\leq a_{\lambda_1 2} \leq a_{\lambda_1 3} \leq \ldots \leq a_{\lambda_1 \lambda_1}
\end{align*}
\]

E.g. \( \lambda = (4,2,1,1) \):

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 \\
| & | & | & | & | & | & |
\end{array}
\]
Type A highest weight rep's of type $\lambda$

1. $\hat{\Delta} = \begin{array}{c} 1 \ 2 \ 3 \vdots \\
   \ 2 \ 3 \ \vdots \\
   \ \ 3 \ \vdots \\
\end{array}$ of shape $\lambda$

2. $u \rightarrow v$ for $v$ obtained from $u$ by increment rightmost $i$ not in "parenthesization pair" $w_1 \cdot i + 1$ to an $i + 1$

**Parenthesization Pairs:** Read leftmost column bottom to top, then subsequent columns L to R, ignoring all but $i \cdot i + 1$; omit consec $i + 1, i$ and repeat.

E.g. $11111444 \quad 44433444 \rightarrow e_i(u)$

$u = \begin{array}{c} 2 \ 2 \ 3 \ 3 \\
   3 \ 4 \ 4 \ \vdots \\
\end{array}$

$z = 3 \quad \begin{array}{c} 3 \ 4 \ 4 \ 3 \ 3 \ 4 \ 4 \ 4 \\
\end{array}$
II. Right key "k" of a KM-crystal

\[ k(124) = s_3s_2s_1s_2 \]
\[ k(123) = s_2s_1s_2s_1 \]
\[ k(114) = s_3s_2s_1 \]
\[ k(123) = s_2s_1 \]
\[ k(113) = s_2s_1 \]
\[ k(112) = s_1s_2 \]
\[ k(122) = s_1s_2 \]
\[ k(111) = s_2 \]

\[ k: \text{crystal} \rightarrow \text{weak poset} \]
\[ \text{Brauer order} \]
\[ u \leq v \Rightarrow k(u) \leq k(v) \]
\[ \text{key } k(\emptyset) = e \]
New Algorithm to Calculate Right Key of a KM-Crystal

1. \( \text{key}(\hat{\delta}) = \epsilon \)

2. If \( \hat{\delta} \rightarrow a \), then \( \text{key}(a) = s_i \)
   (i.e. \( \hat{\delta} \geq a \))

3. If \( v \) covers 2 or more elements then \( \text{key}(v) = v \cdot \text{key}(u) \)
   \( \exists u \) such that \( u \rightarrow v \)

   (for join taken in weak order)

4. If \( u \rightarrow v \) and \( v \) does not cover any other elements, then:
   a) \( \text{key}(v) = \text{key}(u) \) if \( \exists u' \rightarrow u \)
   b) \( \text{key}(v) = s_i \cdot \text{key}(u) \) otherwise
Crucial Properties of Key

Thm (Littlemann): Given any symmetrizable Kac-Moody algebra $A$, the key of any crystal of type $A$ satisfies:

$$K(f_{p+1}(F)) = \begin{cases} K(F) \text{ if } e_p(F) \neq 0 \\ s_p K(F) \text{ or } K(F') \text{ if } e_p(F) = 0 \end{cases}$$

Also, if $e_p(F) = 0$ then $s_p K(F) > K(F)$

Corollary: If $K(F) = s_{i_1} \cdots s_{i_r}$ then there exists saturated chain from $F$ to $0$ given by applying $e_{i_r} \cdots e_{i_1}$ to $F$ for some $d_1, \ldots, d_r > 0$. 
Fiber with Multiple Minimal Elements

Key = $S_1S_3S_2$

$\text{fiber}$

$(0,3,0,2)$
Key Polynomials & right/ left key

(see Lascoux-Schützenberger & e.g. Reiner-Shimozono)

Motivations: (1) Schubert poly. $G_w$ is positive sum of "key polynomials"

(2) Key polynomial records character for Demazure module

(3) The (closely related) right & left key maps determine smallest Demazure modules containing a given crystal element

(4) These will give us poset map from $q$-crystal to weak Bruhat order, transferring properties
Relation to Reiner-Shimozono Viewpoint on Key Polynomials

- $\delta_i = 1 - s_i \quad \text{and} \quad \Pi_i = \delta_i x_i$

- $K_\alpha = \Pi_{i_1} \cdots \Pi_{i_r} x^{\lambda(\alpha)}$ for a composition of $n \uparrow s_i \cdots s_i$ sorting $\alpha$ to $\lambda(\alpha)$

E.g., $K_{(1,0,2,1)} = \Pi_2 \Pi_1 \Pi_3 x^{(2,1,1,0)}$

\[
= \Pi_2 \Pi_1 (x_1^2 x_2 (x_3 + x_4))
\]

\[
= \Pi_2 (x_1 x_2 x_3 (x_1 + x_2) + x_1 x_2 x_4 (x_1 + x_2))
\]

\[
= x_1^2 x_2 x_3 + x_1 x_2 x_3 (x_2 + x_3) + x_1^2 x_4 (x_2 + x_3) + x_1 x_4 (x_2 + x_3 + x_4^2)
\]
Key Polynomial

\[ K(1921) = \sum_{T} x^T \cdot T^{1921} \]

Componentwise

By \( \omega = 52153 \)

Mazure Module Given

- \( e_2 \)
- \( e_3 \)
- \( e_4 \)
- \( e_5 \)
- \( e_6 \)
- \( e_7 \)
- \( e_8 \)
- \( e_9 \)
- \( e_{10} \)

No higher e.
III. Positive Results for Lower Intervals $[\hat{0}, u]$

Recall: 
\[
M_D(u,u) = 1 \\
M_D(u,v) = \sum_{u \leq z < v} M_D(u,z)
\]

Thm 1 (H.-Lenart): Given $u$ in a symmetrized $\mathbf{M}$-crystal, then $M(\hat{0}, u) = 0, \pm 1$. More specifically, $M(\hat{0}, u) = 0$ unless $\text{key}(u) = \omega_0(\mathcal{J})$ for some parabolic subgroup $\mathcal{W}_\mathcal{J}$ with $u$ the unique smallest element in $\text{key}^{-1}(\omega_0(\mathcal{J}))$, in which case $M(\hat{0}, u) = (-1)^{|\mathcal{J}|}$. 
**Proof Ingredients:**

**Thm 2 (H.-Lenart):** Given a symmetrized Kac-Moody crystal $\mathfrak{g}$, given any parabolic $W_J$, then $\text{key}^{-1}(w_0(w_J))$ has a unique minimal element and a unique maximal element. (Proof via alcove path model)

**Prop'n:** Each $w \in W$ has unique maximal element $u \leq w$ in weak Bruhat order s.t.

- $u \leq_{\text{weak}} w$
- there exists parabolic subgroup $W_J$ s.t. $u = w_0(J) = \text{longest elt}$
**Thm 3 (H-Lenzart):** Each lower interval \((\hat{0}, u)\) in a symmetric KM-crystal has \(\Delta (\hat{0}, u) \cong \text{ball or sphere}, \text{ getting } S^{1, 1-2} \text{ for } u = \min (k^{2} (w_{0} (J)))\). Likewise for upper intervals in finite KM-crystals.

**Proof Method:** Quillen fiber lemma based upon:

\[
f: \text{Crystal} \rightarrow \text{Boolean Algebra} \quad \text{Poset} \quad \{J \mid J \leq \Xi \}
\]

\[
x \mapsto \max \{J \mid \omega_{0} (J) \leq k(x) \}
\]

**Quillen Fiber Lemma:** Poset map \(f: P \rightarrow Q\) s.t. each \(\Delta (f^{-1} (q))\) is contractible implies \(\Delta (P) \cong \Delta (Q)\).
Thm 4 (H.-Lenart): Given any lower interval $(\hat{0}, u)$ in a $\gamma$-crystal, then set of saturated chains from $\hat{0}$ to $u$ is connected by "Stembridge moves", namely moves of the form and

Note: Likewise in dually-laced case via "Stembridge moves".
**Proof Idea:** Induction on rank using $z \succeq \emptyset$.

- $r(k(x)) < r(k(w))$
- $r(k(y)) < r(k(w))$

Connected by induction.
Non-Lattice Example

\[ \frac{1123}{344} = (uvv), \quad \frac{1223}{334} = (uvv) \]

(Why proof fails for arbitrary intervals)
IV. Negative Results for Arbitrary (not necessarily lower) Crystal Poset Intervals (type A)

Thm 5 (H.-Lenart): There exist elements $u, v$ in type $A$ $\gamma$-crystals with $M(u,v) = 2j$ for every positive integer $j$.

Thm 6 (H. Lenart): There exist type A intervals $[u, v)$ with $rk(u) - rk(v)$ arbitrarily large s.t. $(u,v)$ is disconnected
Infinite Family of Examples

"Base Case": \( v = \begin{pmatrix} 1123 \\ 344 \end{pmatrix} \)

\[ M_p(u,v) = 2 \quad u = \begin{pmatrix} 234 \end{pmatrix} \]

\& not connected by "Stembridge moves"
Examples with $M(u,v) = 2^j$

\[ j = 1: \quad u = \begin{bmatrix} 1112 \\ 234 \end{bmatrix} \quad v = \begin{bmatrix} 1123 \\ 344 \end{bmatrix} \]

\[ j = 2: \quad u^{(2)} = \begin{bmatrix} 1111 \\ 2222 \\ 234 \end{bmatrix} \quad v^{(2)} = \begin{bmatrix} 1111 \\ 2222 \\ 344 \end{bmatrix} \]

\[ u_t = u + 5 = \begin{bmatrix} 6667 \\ 789 \end{bmatrix} \quad v_t = v + 5 = \begin{bmatrix} 6678 \\ 899 \end{bmatrix} \]

\[ [u^{(2)}, v^{(2)}] \cong [u, v] \times [u, v] \]

So $M(u^{(2)}, v^{(2)}) = 2^2$

\[ [u^{(k)}, v^{(k)}] \cong [u, v] \times \ldots \times [u, v] \quad M = 2^k \]
Arbitrarily High Rank
Disconnected Open Intervals

\[ V = \begin{array}{cccc}
1 & 2 & 3 & \ldots \\
4 & 5 & 6 & \ldots \\
7 & 8 & 9 & \ldots \\
\end{array}
\]

\[ u = \begin{array}{cccc}
1 & 2 & 3 & \ldots \\
4 & 5 & 6 & \ldots \\
7 & 8 & 9 & \ldots \\
\end{array}
\]

Label sequences: 1, 2, 2, 3, 3, 4, 4, \ldots, n-1, n-1, n + n, n, n, n, n, \ldots, 2, 2, 1 in distinct components
**Consequence:** Arbitrarily high degree relations $e_i - e_d (n) = e_j - e_d (n)$ amongst crystal operators applied to $n$ not implied by any lower degree relations.

**Some Further Questions:**

1. Positive results in any additional generality (more general intervals)?

2. Interpretation/applications for Möbius function of a crystal?

3. Where/how exactly can the lattice property fail?