A BIJECTIVE PROOF OF THE DIMENSION OF THE FUSS-CATALAN ALGEBRAS

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ABSTRACT. D. Bisch and V.F.R. Jones defined the Fuss-Catalan algebras as a generalization of the Temperley-Lieb algebras and obtained a dimension formula in [BJ] using generating functions. Landau proved further results about Fuss-Catalan algebras in [La]. We provide a combinatorial proof of the dimension formula by giving a bijection between a class of planar trees which naturally corresponds to Fuss-Catalan generators and another class of trees which clearly has cardinality \( \frac{1}{n+1} (k+1)^n \).

We shall give a bijective proof that the dimension of the Fuss-Catalan algebra is \( \frac{1}{n+1} (k+1)^n \). This will restrict to a proof of the dimension of the Temperley-Lieb algebra, so let us begin by reviewing the Temperley-Lieb bijection. Each Temperley-Lieb algebra generator may be represented by a wire diagram as in Figure 1.

![Figure 1. A Temperley-Lieb generator](image)

Place a tree node in each region of the diagram and an edge between any two adjacent regions. In this manner, each wire diagram gives rise to a rooted unlabelled planar tree, and one may easily check that this gives a bijection between trees on \( n + 1 \) nodes and wire diagrams with \( n \) wires. Dashed lines represent the edges of such a tree in Figure 2.

The Fuss-Catalan algebra \( FC(n, a_1, \ldots, a_k) \) is generated by those Temperley-Lieb generators with \( kn \) wires which have a property we refer to as \( k \)-consistency. An example is given in Figure 3. When the the endpoints of the wires in a wire diagram are labelled left to right by a sequence of the form \( a_1, a_2, \ldots, a_{k-1}, a_k, a_{k-1}, \ldots, a_2, a_1, a_2, \ldots, a_{k-1}, a_k \) which has length a multiple of \( 2k \), then the labels of the two endpoints for each wire are required to agree for a wire diagram to be \( k \)-consistent.

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This leads to a class of planar trees which we call $k$-consistent trees by applying the Temperley-Lieb bijection to the wire diagrams associated to Fuss-Catalan generators. We shall provide a bijection between $k$-consistent trees and another class of trees which is easily seen to have the desired cardinality. We will refer to this latter class of trees as $k$-step trees. When $k = 1$, all unlabelled rooted planar trees are both $k$-step and $k$-consistent.

**Definition 0.1.** An unlabelled rooted planar tree is $k$-consistent if the edges may be labelled with a sequence $1, 2, \ldots, k-1, k, k-1, \ldots, 2, 1, 2, \ldots, k, k-1, \ldots, 2, 1$ in the following way: edges are labelled by the sequence in the order they are encountered in a depth-first search; each edge is visited twice, but for $k$-consistency we require that the label assigned to each edge on its downward pass agrees with the label assigned on the upward pass.

This condition on trees corresponds via the Temperley-Lieb bijection to the requirement that wires in Fuss-Catalan generators must have endpoints which agree; a depth-first search of a tree crosses the wires sequentially, with the labels of the downward and upward passes through each tree edge corresponding to the labels of the two endpoints of a wire.

**Definition 0.2.** An unlabelled rooted planar tree is a $k$-step tree if a depth first search of the tree may be broken into downward steps consisting of $k$ consecutive edges and upward steps of a single edge.

A depth first search of a $k$-step tree may be encoded by a sequence of interspersed $d$'s and $u$'s: there are $n$ downward steps represented by $d$'s and $kn$ upward steps given by $u$'s. Such a sequence is associated to a $k$-step tree if and only if for each
1 \leq i \leq (k+1)n, the number of $u$'s occurring before position $i$ in the sequence is at most $k$ times the number of $d$'s. There are $\frac{1}{k+1} \binom{(k+1)n}{n}$ such sequences of length $(k+1)n$ by the standard argument for counting Catalan numbers (cf. [St]). We will refer to these sequences of $d$'s and $u$'s as $k$-Catalan sequences and use them interchangeably with $k$-step trees.

**Theorem 0.1.** There is a bijection between $k$-step trees on $km + 1$ nodes and $k$-consistent trees on $km + 1$ nodes.

**Proof.** We give an invertible map $\phi$ from $k$-step trees to $k$-consistent trees by specifying how to construct a $k$-consistent tree from a $k$-Catalan sequence. Figure 3 provides a very simple example of this bijection.

![k-step tree](image)

![k-consistent tree](image)

**Figure 4.** An example for $k = 3$

A $k$-Catalan sequence gives rise to a $k$-step tree by specifying a depth-first search of the tree; we shall push and pop numbers on a stack to record (in a reversible way) all the ways to traverse the depth-first search which lead to the corresponding $k$-consistent tree. The $k$-step tree is obtained by taking $k$ consecutive downward steps for each $d$ in the sequence and a single upward step for each $u$. To turn this into a $k$-consistent tree, each time a $d$ is encountered immediately after a $u$, we record the total number of edges traversed so far mod $k$ (keeping track of edge multiplicity, since each edge is visited twice altogether) and push this number $i$ along with the current node $N$ on a stack. Instead of travelling down $k$ steps to account for this $d$, we travel down $k - i$ steps in the tree; we will later pop the number $i$ off the stack and at that time travel $i$ extra steps downward. Thus we postpone $i$ downward steps in a way that ensures $k$-consistency. We proceed in the tree construction blind to this shift until we revisit the node at which $i$ was placed on the stack, that is, the node $N$ which we encountered immediately before travelling down $k - i$ steps. At this point, $(i, N)$ will be at the top of the stack and is popped off the stack. The shift of downward steps is now completed by inserting $i$ downward steps from the node $N$ before continuing to read off the $k$-Catalan sequence.

Let us check that each tree thus obtained is $k$-consistent. First note that the number of edges encountered between any two leaves (with the root treated as a
leaf) is a multiple of \( k \), because we travel a multiple of \( k \) steps between the root and each leaf after shifting. This implies \( k \)-consistency each time we begin to travel upward from a leaf. We refer to a tree node as a branch point if it has at least two children and none of its descendents are branch points. Consider the last branch point encountered on the initial downward path in a depth first search from the root. The number of edges in the subtree below this branch point must be congruent to \( k - i \mod 2k \) where \( i \) is the number of edges traversed \( \mod k \) in travelling from the root to this branch point; this is again a consequence of shifting. This implies that we may inductively reduce the task of showing \( k \)-consistency by eliminating the subtree below this branch point and deleting the corresponding subsequence of length congruent to \( 2(k - i) \mod k \) from the \( k \)-Catalan sequence which gives rise to this tree; simply note that the upward step from the branch point will be consistent with the downward step through that edge. Induction this gives \( k \)-consistency.

Finally, \( \sigma^{-1} \) exists because we may recover the \( k \)-Catalan sequence by a depth first search of a \( k \)-consistent tree which reverses the shifting which must have occurred. That is, we record a \( d \) for each \( k \) consecutive downward steps, an \( u \) for each upward step, and when we may only travel \( 0 < i < k \) downward steps before reaching a leaf, we record a \( d \) in the sequence and push an \( i \) on the stack. When we first revisit the node where we put \( i \) on the stack, we ignore the next \( i \) downward steps in recording a \( k \)-Catalan sequence and pop \( i \) off the stack. It is not hard to check that this map is well-defined and that it is the inverse of the map from \( k \)-Catalan sequences to \( k \)-consistent trees.

\[ \square \]

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References


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