A BIJECTIVE PROOF OF THE DIMENSION OF THE FUSS-CATALAN ALGEBRAS

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ABSTRACT. D. Bisch and V.F.R. Jones defined the Fuss-Catalan algebras as a generalization of the Temperley-Lieb algebras and obtained a dimension formula in [BJ] using generating functions. Landau proved further results about Fuss-Catalan algebras in [La]. We provide a combinatorial proof of the dimension formula by giving a bijection between a class of planar trees which naturally corresponds to Fuss-Catalan generators and another class of trees which clearly has cardinality $\frac{1}{kn+1}\binom{(k+1)n}{n}$.

We shall give a bijective proof that the dimension of the Fuss-Catalan algebra is $\frac{1}{kn+1}\binom{(k+1)n}{n}$. This will restrict to a proof of the dimension of the Temperley-Lieb algebra, so let us begin by reviewing the Temperley-Lieb bijection. Each Temperley-Lieb algebra generator may be represented by a wire diagram as in Figure 1.

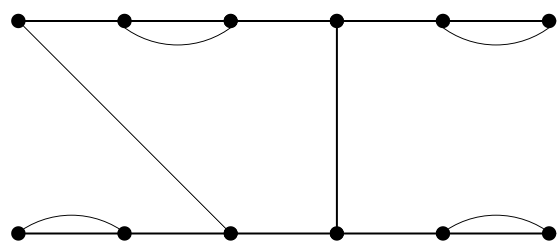

FIGURE 1. A Temperley-Lieb generator

Place a tree node in each region of the diagram and an edge between any two adjacent regions. In this manner, each wire diagram gives rise to a rooted unlabelled planar tree, and one may easily check that this gives a bijection between trees on $n+1$ nodes and wire diagrams with n wires. Dashed lines represent the edges of such a tree in Figure 2.

The Fuss-Catalan algebra $FC(n, a_1, \ldots, a_k)$ is generated by those Temperley-Lieb generators with kn wires which have a property we refer to as k-consistency. An example is given in Figure 3. When the the endpoints of the wires in a wire diagram are labelled left to right by a sequence of the form $a_1, a_2, \ldots, a_{k-1}, a_k, a_k, a_{k-1}, \ldots, a_2, a_1, a_1, a_2, \ldots, a_2, a_1$ which has length a multiple of $2k$, then the labels of the two endpoints for each wire are required to agree for a wire diagram to be **k-consistent**.

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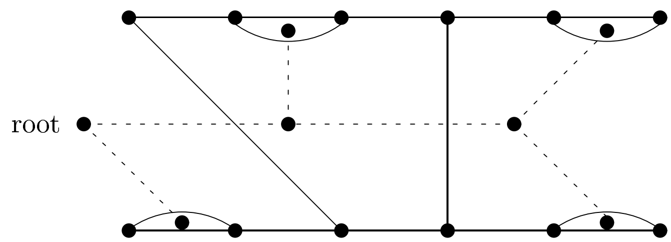

FIGURE 2. An example of the Temperley-Lieb bijection

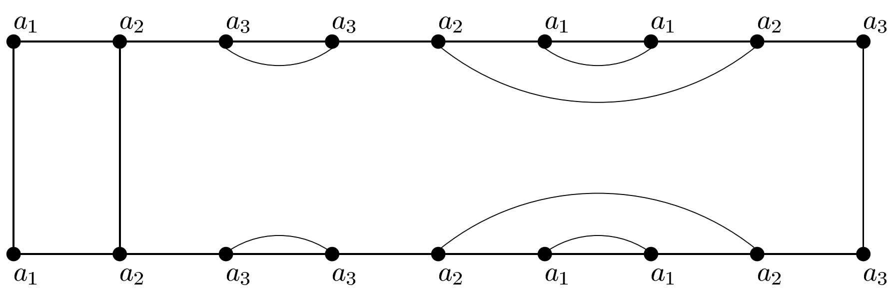

FIGURE 3. A Fuss-Catalan generator

This leads to a class of planar trees which we call k-consistent trees by applying the Temperley-Lieb bijection to the wire-diagrams associated to Fuss-Catalan generators. We shall provide a bijection between k-consistent trees and another class of trees which is easily seen to have the desired cardinality. We will refer to this latter class of trees as k-step trees. When $k = 1$, all unlabelled rooted planar trees are both k-step and k-consistent.

Definition 0.1. *An unlabelled rooted planar tree is* k**-consistent** *if the edges may be labelled with a sequence* $1, 2, \ldots, k-1, k, k, k-1, \ldots, 2, 1, 1, 2, \ldots, k, k-1, \ldots, 2, 1$ *in the following way: edges are labelled by the sequence in the order they are encountered in a depth-first search; each edge is visited twice, but for k-consistency we require that the label assigned to each edge on its downward pass agrees with the label assigned on the upward pass.*

This condition on trees corresponds via the Temperley-Lieb bijection to the requirement that wires in Fuss-Catalan generators must have endpoints which agree; a depth-first search of a tree crosses the wires sequentially, with the labels of the downward and upward passes through each tree edge corresponding to the labels of the two endpoints of a wire.

Definition 0.2. *An unlabelled rooted planar tree is a* k**-step tree** *if a depth first search of the tree may be broken into downward steps consisting of k consecutive edges and upward steps of a single edge.*

A depth first search of a k-step tree may be encoded by a sequence of interspersed d's and u's: there are n downward steps represented by d's and kn upward steps given by u's. Such a sequence is associated to a k-step tree if and only if for each

$1 \le i \le (k+1)n$, the number of u's occuring before position i in the sequence is at most k times the number of d's. There are $\frac{1}{kn+1}\binom{(k+1)n}{n}$ such sequences of length $(k+1)n$ by the standard argument for counting Catalan numbers (cf. [St]). We will refer to these sequences of d's and u's as **k-Catalan sequences** and use them interchangeably with k-step trees.

Theorem 0.1. *There is a bijection between k-step trees on $km+1$ nodes and k-consistent trees on $km+1$ nodes.*

PROOF. We give an invertible map ϕ from k-step trees to k-consistent trees by specifying how to construct a k-consistent tree from a k-Catalan sequence. Figure 3 provides a very simple example of this bijection.

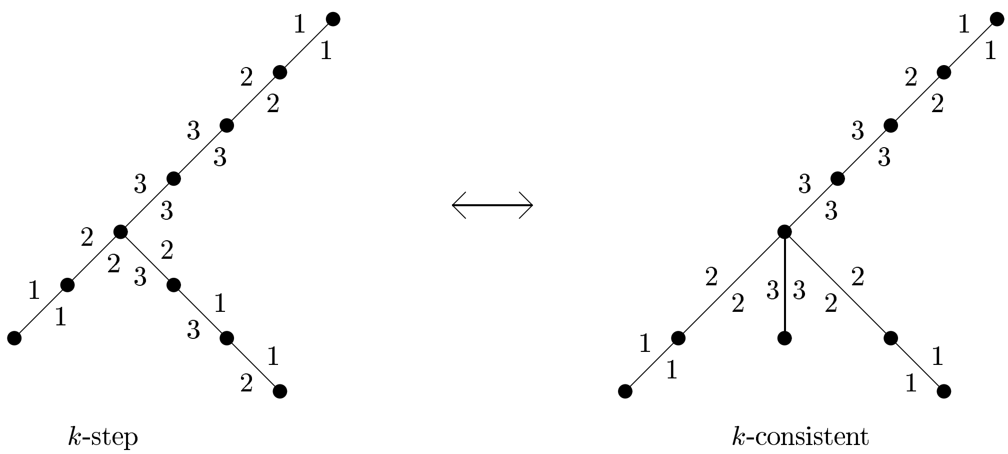

FIGURE 4. An example for $k = 3$

A k-Catalan sequence gives rise to a k-step tree by specifying a depth-first search of the tree; we shall push and pop numbers on a stack to record (in a reversible way) alterations to the depth-first search which lead to the corresponding k-consistent tree. The k-step tree is obtained by taking k consecutive downward steps for each d in the sequence and a single upward step for each u. To turn this into a k-consistent tree, each time a d is encountered immediately after a u, we record the total number of edges traversed so far mod k (keeping track of edge multiplicity, since each edge is visited twice altogether) and push this number i along with the current node N on a stack. Instead of travelling down k steps to account for this d, we travel down $k - i$ steps in the tree; we will later pop the number i off the stack and at that time travel i extra steps downward. Thus we postpone i downward steps in a way ensures k-consistency. We proceed in the tree construction blind to this shift until we revisit the node at which i was placed on the stack, that is, the node N which we encountered immediately before traveling down $k-i$ steps. At this point, (i, N) will be at the top of the stack and is popped off the stack. The shift of downward steps is now completed by inserting i downward steps from the node N before continuing to read off the k-Catalan sequence.

Let us check that each tree thus obtained is k-consistent. First note that the number of edges encountered between any two leaves (with the root treated as a leaf) is a multiple of k, because we travel a multiple of k steps between the root and each leaf after shifting. This implies k-consistency each time we begin to travel upward from a leaf. We refer to a tree node as a branch point if it has at least two children and none of its descendents are branch points. Consider the last branch point encountered on the initial downward path in a depth first search from the root. The number of edges in the subtree below this branch point must be congruent to $k - i$ mod $2k$ where i is the number of edges traversed mod k in travelling from the root to this branch point; this is again a consequence of shifting. This implies that we may inductively reduce the task of showing k-consistency by eliminating the subtree below this branch point and deleting the corresponding subsequence of length congruent to $2(k-i)$ mod k from the k-Catalan sequence which gives rise to this tree; simply note that the upward step from the branch point will be consistent with the downward step through that edge. Induction this gives k-consistency

Finally, ϕ^{-1} exists because we may recover the k-Catalan sequence by a depth first search of a k-consistent tree which reverses the shifting which must have occured. That is, we record a d for each k consecutive downward steps, a u for each upward step, and when we may only travel $0 < i < k$ downward steps before reaching a leaf, we record a d in the sequence and push an i on the stack. When we first revisit the node where we put i on the stack, we ignore the next i downward steps in recording a k-Catalan sequence and pop i off the stack. It is not hard to check that this map is well-defined and that it is the inverse of the map from k-Catalan sequences to k-consistent trees. □

Acknowledgments

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References

[BJ] D. Bisch and V.F.R. Jones, Algebras associated to intermediate subfactors, *Inventiones Mathematicae*, vol 128, No. 1, 1997.

[La] Z. Landau. *Intermediate Subfactors*, Ph.D. thesis, Univ. of Calif., Berkeley, 1998.

[St] R. Stanley. Enumerative Combinatorics, vol. II. Cambridge Univ. Press, Cambridge/New York, 1999.

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