COLORING COMPLEXES AND ARRANGEMENTS

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Abstract. Steingrimsson's coloring complex is interpreted in terms of hyperplane arrangements. This viewpoint leads to a short proof that all coloring complexes have convex ear decompositions. These convex ear decompositions impose strong new restrictions on the chromatic polynomials of all finite graphs. Similar results are obtained for characteristic polynomials of submatroids of type $\mathcal{B}_n$ arrangements.

1. Introduction

Since its introduction by Birkhoff almost a century ago [Bi], the chromatic polynomial has been the object of intense study. Nonetheless, a satisfactory answer to Wilf's question, "What polynomials are chromatic?" [Wi] remains elusive. In [Jo], Jonsson proved that Steingrimsson's coloring complex is Cohen-Macaulay, and thereby established new restrictions on such polynomials. Our main result is that the coloring complex has a convex ear decomposition, which implies that the chromatic polynomials of all finite graphs satisfy much stronger inequalities than those provided by [Jo, Theorem 1.4].

We also apply our methods to submatroids of type $\mathcal{B}_n$ arrangements. On the other hand, we give examples indicating that these results cannot be extended to the characteristic polynomials of all matroids or even to large classes that seem to be particularly natural candidates.

The coloring complex Δ_G of a graph was introduced in [Ste] and was proven to be constructible, hence Cohen-Macaulay, in [Jo]. The $(r-1)$-dimensional faces of the coloring complex are ordered lists $T_1|T_2|T_3|\cdots|T_r$ of nonempty disjoint sets of vertices with the property that at least one T_i includes a pair of vertices that comprise an edge of G and $\cup_{1\leq i\leq r} T_i \neq V(G)$. Steingrimsson showed that the h-polynomial of the double cone of the coloring complex is related to the chromatic

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polynomial by the following formula.

$$(1) \qquad (1-t)^n \sum_{j=0}^{\infty} [(j+1)^n - P_G(j+1)]t^j = h_0 + h_1 t + \cdots + h_n t^n.$$

This expression allows any new constraints on the h-vector of the coloring complex to be translated into new constraints on chromatic polynomials of all finite graphs. Steingrimsson proved this formula by a Hilbert series calculation, so next we describe the rings involved.

Following [Ste], let G be a graph with vertex set $V = [n]$. Set $A = k[x_S | S \subseteq [n]]$, $I = \langle x_S x_T | S \not\subseteq T \text{ and } T \not\subseteq S \rangle$, and let $R = A/I$. By definition, $R = k[\Delta(B_n)]$, the Stanley-Reisner ring of the order complex of the Boolean algebra B_n. Let K_G be the ideal in R generated by monomials $x_{S_1} x_{S_2} \cdots x_{S_r}$ such that for each $i \geq 1$ we have that $S_i \setminus S_{i-1}$ does not include any pairs $\{i_1, i_2\}$ in $E(G)$, the edge set of G. By convention, $S_0 = \emptyset$ so that $S_1 \setminus S_0 = S_1$ must be a disconnected set of vertices. K_G is often called the coloring ideal of G. It turns out that R/K_G is the Stanley-Reisner ring of the double cone of Δ_G.

In [Br], Brenti asked whether there exists, for an arbitrary graph G, a standard graded algebra whose Hilbert polynomial is the chromatic polynomial of G. In general it is not possible for the Hilbert function of a standard graded algebra to agree identically with the values of the chromatic polynomial of a graph since the latter is zero below the graph's chromatic number. However, Steingrimsson showed that K_G is an ideal whose Hilbert function agrees (up to a shift of one) with the values of the chromatic polynomial [Ste], and thereby obtained the above formula as a corollary. In [Ste], he also attributes to G. Almkvist an earlier, nonconstructive affirmative answer to Brenti's question.

Steingrimsson's idea was to give a correspondence between the monomials in K_G of degree r and the proper $r+1$ colorings of G as follows: the monomial $(x_{S_1})^{d_1} \cdots (x_{S_l})^{d_l}$ corresponds to the coloring in which the vertices in S_1 are colored 1, the vertices in $S_2 \setminus S_1$ are colored $d_1 + 1$, the vertices in $S_3 \setminus S_2$ are colored $d_1 + d_2 + 1$, etc. Note that $S_1 = \emptyset$ if no vertices are colored 1. We then have $r = \sum d_i$, in other words, the degree of the monomial.

Jonsson proved that coloring complexes are constructible in [Jo]. By examining these complexes from the viewpoint of hyperplane arrangements we will prove that the coloring complex has a convex ear decomposition. From this, we obtain new restrictions on the chromatic polynomials of all finite graphs in Section 4. See Section 3 for the definition of convex ear decomposition. Applying this idea to subarrangements of type $\mathcal{B}_n$ arrangements leads to restrictions on their characteristic polynomials.

We assume the reader is familiar with Stanley-Reisner rings and h-vectors of finite simplicial complexes as presented in [Sta]. In Section 5 we assume the reader is familiar with the characteristic polynomial of a matroid and its connection to the chromatic polynomial of a graph. See, for instance, [BO, Section 6.3]

2. An arrangements interpretation for the coloring complex

Given a graph G with n vertices, let A_G be the real hyperplane arrangement generated by the hyperplanes of the form $x_i = x_j$ for each edge $\{i, j\}$ present in $E(G)$. When G is K_n, the complete graph on n vertices, A_{K_n} is usually called the type A braid arrangement. In this case the intersection of all the hyperplanes is the line $x_1 = x_2 = \cdots = x_n$. Let H be the hyperplane $\{(a_1, \ldots, a_n) \in \mathbb{R}^n : \sum a_i = 0\}$. Then $A_{K^n} \cap H$ induces a simplicial cell decomposition on S^{n-2}, the unit sphere of H. The faces of the complex correspond to ordered partitions $S_1|S_2|\cdots|S_{r-1}|S_r,\ r \geq 2$, of $[n]$. A point $(a_1, a_2, \ldots, a_n)$ is in the cell in which S_1 consists of those coordinates which are all equal to each other and are smaller than all other coordinates, and where S_i is defined inductively to consist of all coordinates that are all to equal each other and are smaller than all other elements of $\{a_1, \ldots, a_n\} \setminus (S_1 \cup \cdots \cup S_{i-1})$. The top dimensional faces have dimension $n-2$ and correspond to partitions with $|S_i| = 1$ for all i. Identifying ordered partitions $S_1|S_2|\cdots|S_{r-1}|S_r$ of $[n]$ with ordered partitions $S_1|S_2|\cdots|S_{r-1}$ of proper subsets of $[n]$, the above discussion makes it clear that Δ_{K^n} is simplicially isomorphic to the codimension one skeleton of $S^{n-2} \cap A_{K^n}$. In addition, we can see from its definition, that Δ_G is isomorphic as a simplicial complex to the restriction of A_{K^n} to $(S^{n-2} \cap A_G)$. The above discussion is essentially a special case of an idea appearing in [HRW]. We sum up the above with the following theorem.

Theorem 1. *The coloring complex of G is isomorphic as a simplicial complex to the restriction of $A_{K_n} \cap S^{n-2}$ to the arrangement A_G.*

One consequence is a new, short proof of the following result (also see Theorem 4.2 of [HRW] for a generalization of this result).

Theorem 2 (Jonsson)**.** *The coloring complex of G is homotopy equivalent to a wedge of spheres, where the number of spheres is the number of acyclic orientations of G, and each sphere has dimension $n-3$.*

Proof. First notice that the number of regions into which A_G subdivides the sphere is the number of acyclic orientations of G, since points in the same region are all linear extensions of the associated acyclic orientation. Therefore, Δ_G is the codimension one skeleton of a regular cell decomposition of an $(n-2)$-ball obtained by removing any single $(n-2)$-cell of S^{n-2}. Since the ball has $A_G - 1$ cells of dimension $n-2$, its $(n-3)$-skeleton, and hence Δ_G, is homotopy equivalent to a wedge of $A_G - 1$ spheres, all of dimension $n-3$. □

Jonsson also proved that Δ_G is constructible, and hence Cohen-Macaulay. As we will see below, Δ_G has a convex ear decomposition which implies, by [Sw, Theorem 4.1], that it is in fact doubly Cohen-Macaulay. Specifically, if we remove any vertex from A_G it remains an $(n-2)$-dimensional Cohen-Macaulay complex.

The arrangements viewpoint on the coloring complex follows easily from a connection between bar resolutions and arrangements as developed in [HRW] and further exploited in [HW] and [PRW]. In particular, [HRW] deals with rings in which one mods out by ideals in exactly the way the coloring complex arises, and [HRW] makes the connection in its more general setting to arrangements.

3. Convex ear decomposition for the coloring complex

The following notion was introduced by Chari in [Ch].

Definition 3. *Let Δ be a $(d-1)$-dimensional simplicial complex. A convex ear decomposition of Δ is an ordered sequence $\Delta_1, \ldots, \Delta_m$ of pure $(d-1)$-dimensional subcomplexes of Δ such that*

(1) *Δ_1 is the boundary complex of a d-polytope. For each $j \geq 2$, Δ_j is a $(d-1)$-ball which is a proper subcomplex of the boundary of a simplicial d-polytope.*
(2) *For $j \geq 2$, $\Delta_j \cap (\cup_{i<j} \Delta_i) = \partial \Delta_j$.*
(3) *$\bigcup_j \Delta_j = \Delta$.*

The subcomplexes $\Delta_1, \ldots, \Delta_m$ are the *ears* of the decomposition. The key ingredient in proving our main result is the lemma stated next, after requisite terminology is introduced. An arrangement $A = \{H_1, \ldots, H_s\}$ is *central* if each H_i includes the origin, and A is *essential* if $\cap_{i=1}^s H_i$ consists of exactly one point. For A any essential central arrangement in $\mathbb{R}^n$, a *polytopal realization* of $A \cap S^{n-1}$ is any n-polytope containing the origin whose face fan is the fan of the arrangement. Polytopal realizations of A can be constructed by taking the polar dual of Minkowski sums of line segments through the origin perpendicular to the hyperplanes (see, for instance, [Zi]).

Lemma 4 (Sw, Lemma 4.6)**.** *Let $A = \{H_1, \ldots, H_s\}$ be an essential arrangement of hyperplanes in $\mathbb{R}^n$. Let P be any n-polytope whose face fan is the fan of A. Let $H_{i_1}^+, \ldots, H_{i_t}^+$ be closed half-spaces of distinct hyperplanes in A. If $B = \partial P \cap H_{i_1}^+ \cap \cdots \cap H_{i_t}^+$ is nonempty, then ∂B is combinatorially equivalent to the boundary of an $(n-1)$-polytope.*

Theorem 5. *The coloring complex of a graph has a convex ear decomposition. Moreover, any simplicial complex obtained by replacing A_{K_n} in Theorem 1 by an essential, central, simplicial arrangement and A_G by any subarrangement will have a convex ear decomposition.*

Proof. Suppose that G is connected. Then $A_G \cap H$ is an essential arrangement. Let P be a polytopal realization of $S^{n-2} \cap A_G$, and let $F_1, F_2, \ldots, F_t$ be a line shelling of the facets of P (as in e.g. [Zi]). Identify each facet with the corresponding region of $A_G \cap S^{n-2}$ and, after further subdivision, a subcomplex of $A_{K^n} \cap S^{n-2}$. By the lemma (applied in $A_{K^n} \cap S^{n-2}$), the boundary of each such region is combinatorially equivalent to the boundary of a simplicial polytope. Theorem 1 and the properties of line shellings imply that setting $\Delta_1 = \partial F_1$, and for $2 \leq i \leq t-1, \Delta_i$ equal to the closure of $\partial F_i \setminus (\partial F_1 \cup \cdots \cup \partial F_{i-1})$, produces a convex ear decomposition of Δ_G.

For general finite graphs G, the intersection of all of the hyperplanes in A_G is a k-dimensional subspace of $\mathbb{R}^n$, where k is the number of components of G. The lemma still implies that as a subcomplex of $A_{K^n} \cap S^{n-2}$ the boundary of each region of $A_G \cap S^{n-2}$ is combinatorially equivalent to the boundary of a simplicial polytope. Let H' be the subspace of $\mathbb{R}^n$ orthogonal to the intersection of all of the hyperplanes in A_G. Then the collection $A' = \{H_1 \cap H', H_2, \cap H', \ldots, H_s \cap H'\}$, where the H_i are the hyperplanes in A_G, is an essential arrangement in H'. The facets of a polytopal realization of A' correspond to the regions of $A_G \cap S^{n-2}$. Order the regions of $A_G \cap S^{n-2}$ in a way which corresponds to a line shelling of a polytopal representation of A'. Proceeding as before gives a convex ear decomposition of Δ_G. Indeed, the ears (and their intersections) are $(k-1)$-fold suspensions of a convex ear decomposition of the codimension one skeleton of a polytopal representation of A'.

The only property of A_{K_n} used above was the fact that it was a simplicial arrangement, so the above proof carries over immediately to the more general setting. □

Remark 6. *When G is connected, the above reasoning also leads to an obvious shelling of Δ_G. However, the question of shellability is more* *subtle for G having $k > 1$ components since not all the facets of the coloring complex actually intersect with the perpendicular space H' to the k-dimensional space U shared by all the hyperplanes in A_G. See [Hu] for a shelling of the coloring complex for any G.*

4. Enumerative consequences

The following connection between the coloring complex Δ_G and the chromatic polynomial $P_G(t)$ was first given in [Ste].

Theorem 7. [Ste] *Let Δ_G be the coloring complex of G and let $(h_0, \ldots, h_n)$ be the h-vector of the double cone of Δ_G. Then*

$$(1-t)^n \sum_{j=0}^{\infty} [(j+1)^n - P_G(j+1)]t^j = h_0 + h_1 t + \cdots + h_n t^n. \tag{2}$$

Since the h-vector of a cone equals the h-vector of the original complex, $h_{n-1} = h_n = 0$. In order to state the enumerative consequences of Theorem 5, we first recall the definition of an M-vector.

Definition 8. *A sequence of nonnegative integers $(h_0, h_1, \ldots, h_d)$ is an* **M-vector** *if it is the Hilbert function of a homogeneous quotient of a polynomial ring. Equivalently, the terms form a degree sequence of an order ideal of monomials.*

Another definition given by arithmetic conditions is due to Macaulay. Given positive integers h and i there is a unique way of writing

$$h = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

so that $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. Define

$$h^{<i>} = \binom{a_i + 1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j + 1}{j+1}.$$

Theorem 9. [Sta, Theorem 2.2] *A sequence of nonnegative integers $(h_0, \ldots, h_d)$ is an M-vector if and only if $h_0 = 1$ and $h_{i+1} \leq h_i^{<i>}$ for all $1 \leq i \leq d-1$.*

Theorem 10. *Suppose Δ is a $(d-1)$-dimensional complex with a convex ear decomposition. Then,*

1. $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$.
2. *For $i \leq d/2$, $h_i \leq h_{d-i}$.*
3. $(h_0, h_1 - h_0, \ldots, h_{\lceil d/2 \rceil} - h_{\lceil d/2 \rceil - 1})$ *is an M-vector.*

Proof. The first two inequalities are due to Chari [Ch]. The last statement is in [Sw]. □

Theorem 11. *Let G be a graph with n vertices. Define $h_0, \ldots, h_n$ by the generating function equation*

$$h_0 + h_1 t + \cdots + h_n t^n = (1-t)^n \sum_{j=0}^{\infty} [(j+1)^n - P_G(j+1)] t^j.$$

Then

(1) $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor (n-2)/2 \rfloor}$.
(2) *For* $i \leq (n-2)/2$, $h_i \leq h_{n-2-i}$.
(3) $(h_0, h_1 - h_0, \ldots, h_{\lceil (n-2)/2 \rceil} - h_{\lceil (n-2)/2 \rceil - 1})$ *is an M-vector.*

Proof. Theorems 5, 7 and 10. □

Let A be a subarrangement of the $\mathcal{B}_n$ arrangement. The $\mathcal{B}_n$ arrangement consists of all the hyperplanes in A_{K_n} and all coordinate hypeplanes $x_i = 0$. In [Hu] Hultman proved the following relationship between $\chi_A(t)$, the characteristic polynomial of A viewed as a matroid, and $(h''_0, \ldots, h''_{n-1})$, the h-vector of $\mathcal{B}_n \cap S^{n-1}$ restricted to A.

Theorem 12. [Hu] *Let A be a subarrangement of $\mathcal{B}_n$ and let r be the rank of A as a matroid. Then*

$$(3) \quad h''_0 + \cdots + h''_{n-1} t^{n-1} = (1-t)^n \sum_{j=0}^{\infty} [(2j+1)^n - \chi_A(2j+1)(2j+1)^{n-r}] t^j.$$

Combining Theorem 5, Theorem 10 and (3) we obtain the following.

Theorem 13. *Let A be a subarrangement of $\mathcal{B}_n$. Define $(h''_0, \ldots, h''_{n-1})$ by (3). Then*

(1) $h''_0 \leq h''_1 \leq \cdots \leq h''_{\lfloor (n-1)/2 \rfloor}$.
(2) *For* $i \leq (n-1)/2$, $h''_i \leq h''_{n-1-i}$.
(3) $(h''_0, h''_1 - h''_0, \ldots, h''_{\lceil (n-1)/2 \rceil} - h''_{\lceil (n-1)/2 \rceil - 1})$ *is an M-vector.*

Remark 14. *Characteristic polynomials of subarrangements of $\mathcal{B}_n$ correspond to chromatic polynomials of signed colorings introduced by Zaslavsky. See* [Za].

In order to apply these methods to other arrangements it is essential that subarrangements with the same characteristic polynomial (as matroids) have the same h-vector when restricted to the unit sphere. In particular, all the simplicial subdivisions of the codimension one spheres corresponding to the hyperplanes must have the same h-vector.

Question 15. *Are there other (classes) of hyperplane arrangements such that the h-vectors of subcomplexes induced by subarrangements only depend on the characteristic polynomials of the subarrangements?*

5. Matroids

Given the close connection between the chromatic polynomial of a graph and the characteristic polynomial of the associated cycle matroid, it does not seem unreasonable to hope that it is possible to generalize Theorem 11 or Theorem 13 to matroids. However, as the examples below show, it is not clear that there is any large class of matroids for which this is possible, though it is certainly possible that there is.

In these examples we let $\chi_M(t)$ be the characteristic polynomial of the matroid M. When G is connected, $P_G(t) = t\chi_{M_G}(t)$, where M_G is the cycle matroid of the graph. We will therefore use

$$(4)\quad h_0 + h_1 t + \cdots + h_n t^n = (1-t)^n \sum_{j=0}^{\infty} [(j+1)^n - (j+1)\chi_M(j+1)]t^j.$$

as the analog of the h-vector of the coloring complex for a rank $n-1$ matroid M.

Let us now give examples violating various parts of Theorem 11.

Example 16. *Let M be $PG(5,2)$, the matroid whose elements correspond to the nonzero elements of the five-dimensional vector space over the field of cardinality two with their natural independence relations. Then $\chi_M(t) = t^5 - 31t^4 + 310t^3 - 1240t^2 + 1984t - 1024$. Like the matroid associated to the braid arrangements, M is binary and supersolvable. However, (4) gives, $h_3 = -1678$, a negative integer.*

Example 17. *Let M be the matroid associated to the B_3 arrangement, the hyperplanes fixed by the symmetries of the cube. Like the braid arrangements, B_3 is a free arrangement associated to a root system. $\chi_M(t) = t^3 - 9t^2 + 23t - 15$. Using (4) we find that $h_0 = 1, h_1 = 6, h_2 = 47$. The h_i are nonnegative, but do not form an M-vector.*

Example 18. *Let $\chi_M(t) = (t-1)^3(t-2)(t-8)(t-10)$. Then $\chi_M(t)$ is the characteristic polynomial of the direct sum of 2 coloops and the parallel connection of a 3-point line, 9-point line, and an 11-point line* [Br, Cor. 4.7]. *Now we find*

$$(h_0, \dots, h_5) = (1, 121, 472, 4424, 9167, 2375).$$

This is an M-vector and satisfies (1) and (2) of Theorem 11. However, (3) is not satisfied as

$$(1, 120, 351, 3952)$$

is not an M-vector.

Since every A_G is a subarrangement of the $\mathcal{B}_n$ arrangement, characteristic polynomials of graphic matroids must satisfy Theorem 13. Perhaps this possibly weaker condition is satisfied by all matroids.

Example 19. *Let M be the matroid of PG(2,6). Using (3) as a definition with $n = 6$, we obtain $h_1 = -3047$ and $h_3 = -65638$.*

Let us conclude by mentioning one class of matroids closely related to graphic matroids to which Theorem 11 or 13 could perhaps apply.

Question 20. *Let M be a regular matroid, namely a matroid representable over every field. Does M satisfy either Theorem 11 or Theorem 13?*

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