Symmetric Chain Decomposition for Cyclic Quotients of Boolean Algebras and Relation to Cyclic Crystals

Patricia Hersh
North Carolina State University

Anne Schilling
University of California, Davis

Background

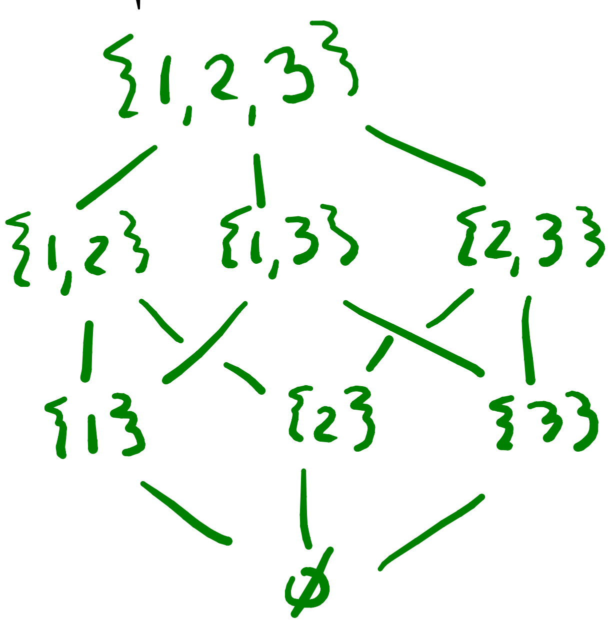
- The rank generating function $a_0 + a_1 t + a_2 t^2 + \dots + a_r t^r$ of a graded poset P is **unimodal** if $a_0 \le a_1 \le \dots \le a_p \ge \dots \ge a_r$ for some p.

It is **symmetric** if $a_i = a_{r-i}$ for all i.

Example: $P =$ Boolean algebra B_n

i.e. poset of subsets of $\{1, \dots, n\}$

$B_3 =$



$1 + 3t + 3t^2 + t^3$

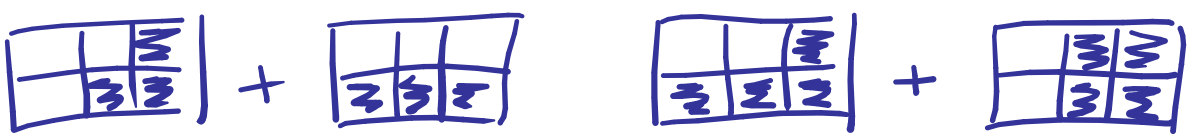
$\parallel$

rank generating function

Thm (Stanley, Proctor, & others): $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is symmetric & unimodal, i.e. the polynomial counting partitions in a $k \times (n-k)$ rectangle by # boxes is symmetric & unimodal.

e.g.

$$1 + q + 2q^2$$



$$+ 2q^3 + 2q^4$$

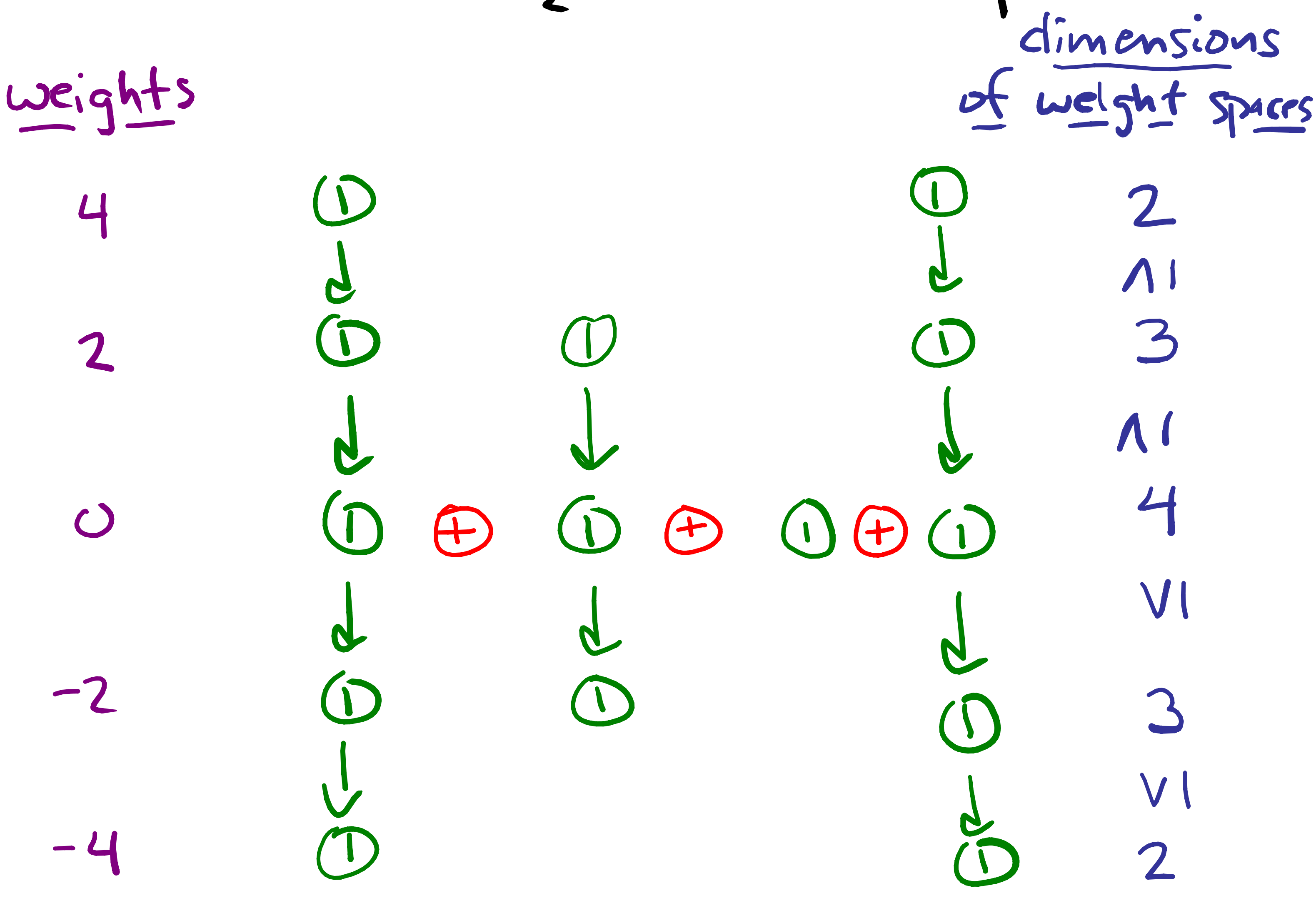
$$+ q^5 + q^6$$

$$= 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$$

$$1 \le 1 \le 2 \le 2 \ge 2 \ge 1 \ge 1$$

Idea: Construct vector spaces $V_1, V_2, \dots, V_{k(n-k)}$ which are the weight spaces of an sl_2-representation with $\dim(V_i) =$ # partitions of "area" i within a rectangle

Deduce unimodality from decomposition into irreducible repn's + nature of sl_2 irreducible representations



<u>Calculation</u>: Let $V_i = \left\langle e_S \,\middle|\, \begin{array}{l} S \subseteq \{1,2,\ldots k(n-k)\} \\ |S| = i \end{array} \right\rangle$

i.e. vector spaces from ranks of Boolean alg.

$(UD - DU)(e_S) = $ (# elements covered by S − # elements covering S) e_S

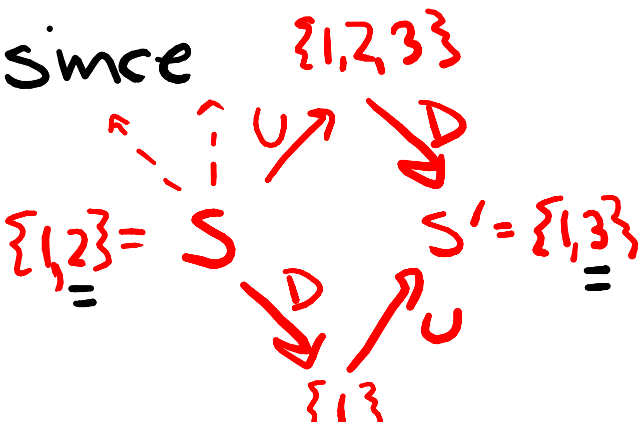
$= (|S| - (n - |S|))e_S$

$= (2|S| - n)e_S$

U = "up operator" sends e_S to formal sum of poset elements covering S

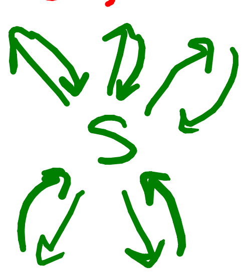
D = "down operator"

since



$\Rightarrow (UD - DU)e_S = 0 \cdot e_{S'} + \ldots\ldots$

and



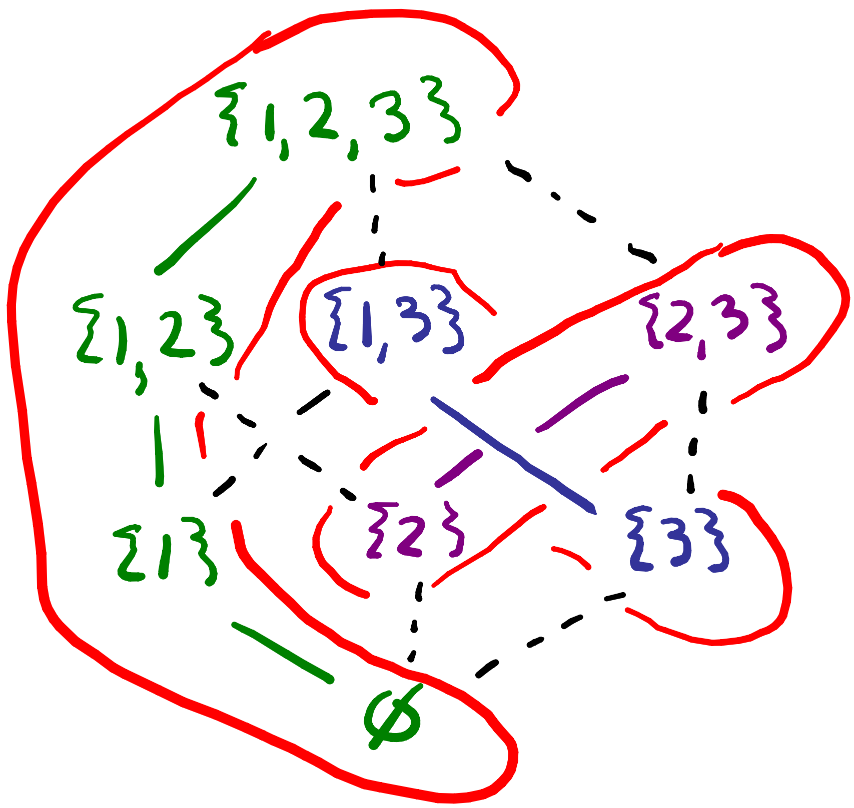
← n − |S| ways for D∘U

← |S| ways for U∘D

• Consider subspaces $V_i^{S_k \wr S_{n-k}}$ whose dimensions count partitions in $k \times (n-k)$ box by area. Use that D & U commute with group action to get sl_2-rep'n

Def'n: A poset P has a symmetric chain decomposition (SCD) if it decomposes into nonoverlapping, saturated chains symmetric about the middle rank(s)

Example:



Easy Fact: Symmetric chain decomposition $\Rightarrow$ rank generating function is symmetric & unimodal; largest antichain middle rank

Thm (Greene-Kleitman) Any Boolean algebra (& more generally any product of chains) has a symmetric chain decomposition.

Idea:
- Elements of B_n can be represented as sequences in $\{0,1\}^n$

e.g. $\{1,3,4,7,8\} \in B_9 \longleftrightarrow 101100110$

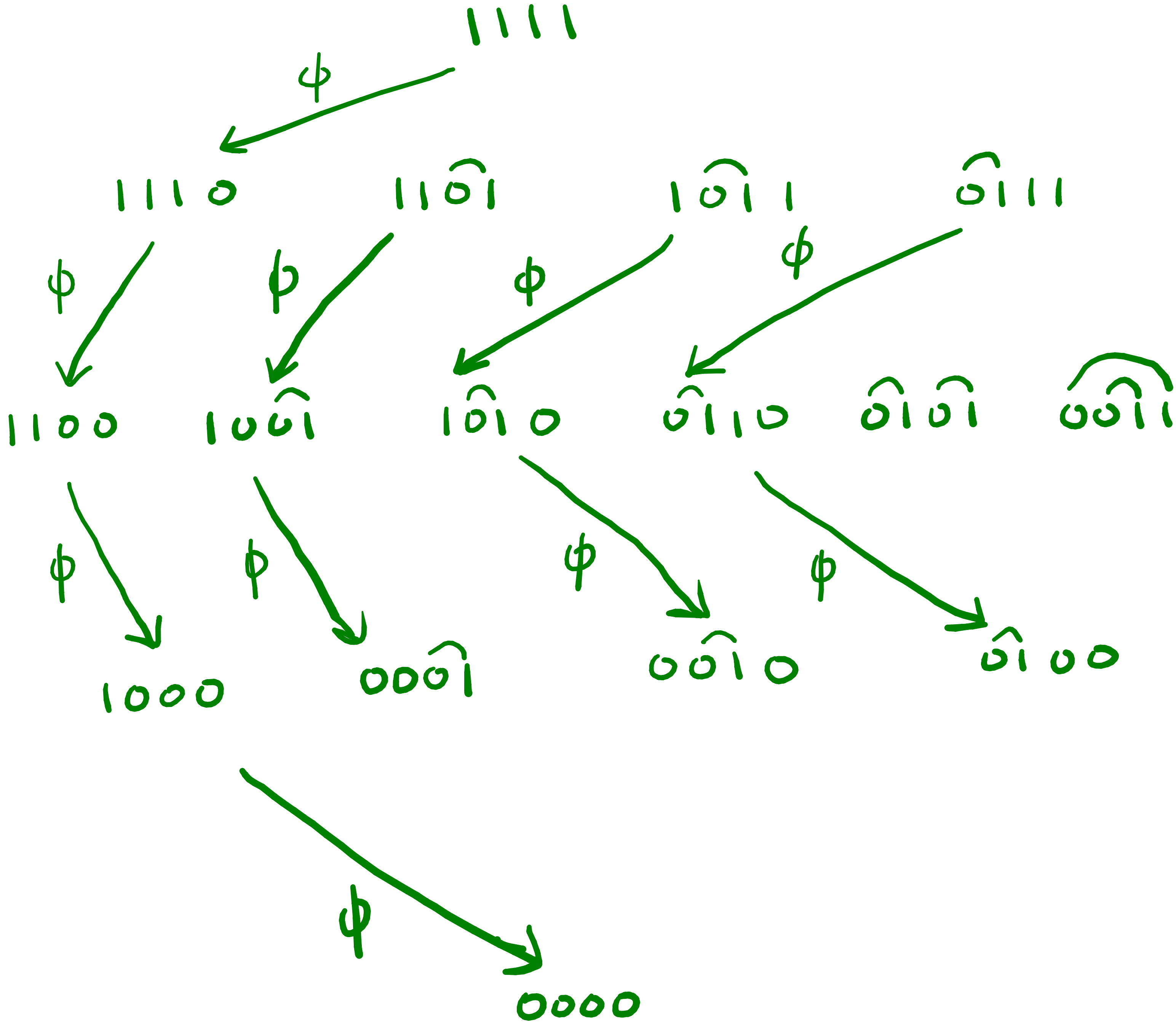
positions 1,3,4,7,8

- Parenthesize consecutive 01 pairs, removing pairs from further consideration & continuing

e.g. $1\,(01)\,1\,(0\,(01)\,1)\,0$, i.e., $\underline{1}\,01\,\underline{1}\,0011\,\underline{0}$

- Unmatched part of form $1^r 0^s$
- Symmetric chain obtained by letting r range from 0 to $r+s$

e.g. $\underline{1}\,01\,\underline{1}\,0011\,\underline{1} \to \underline{1}\,01\,\underline{1}\,0011\,\underline{0} \to \underline{1}\,01\,\underline{0}\,0011\,\underline{0} \to \ldots$



Remark: Boolean algebras also have symmetric & unimodal rank generating function since

$$\binom{n}{0}=\binom{n}{n}\le\binom{n}{1}=\binom{n}{n-1}\le\cdots\le\binom{n}{\lfloor \frac{n}{2}\rfloor}=\binom{n}{\lceil \frac{n}{2}\rceil}$$

Qn: What about quotient posets of B_n?

Thm (J. Griggs): Let P be a ranked poset of rank n with N_k elements of rank k such that:

(1) $N_0 = N_n \leq N_1 = N_{n-1} \leq \dots \leq N_{\lfloor \frac{n}{2} \rfloor} = N_{\lceil \frac{n}{2} \rceil}$

(2) P has the LYM property

Then P has a symmetric chain decomposition.

What is the LYM property?

Ans: For every antichain F (i.e. every collection of incomparable elements of P), $\displaystyle\sum_{x \in F} \frac{1}{N_{\text{rank}(x)}} \leq 1$

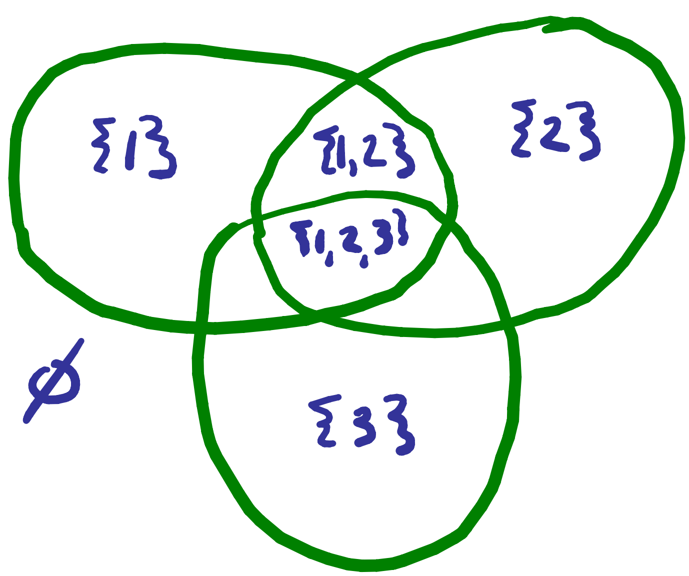
Note: $F = \{\text{elements of rk } k\} \Rightarrow \displaystyle\sum_{x \in F} \frac{1}{N_k} = \frac{N_k}{N_k} = 1$

Symmetric Venn Diagrams & the Quotient of B_n by Cyclic Group

(work of Griggs-Killian-Savage)

A **Venn diagram** is a collection of simple closed curves s.t. each subset of $\{1,2,\ldots,n\}$ is represented by distinct region

e.g



Qn: For which n is there a Venn diagram of subsets of $\{1,\ldots,n\}$ with C_n symmetry?

Thm (Griggs-Killian-Savage): For n prime, these do exist. These are constructed from a symmetric chain decomposition for B_n / C_n.

Idea of GKS Sym. Chain Decomp.

Associate cyclic composition to each element of B_n / C_n

e.g. $\underbrace{110}_{3}\,\underbrace{1000000}_{7}\,\underbrace{110}_{3}\,\underbrace{1100}_{4} \mapsto (3,7,3,4)$

Rotate to get lexicographically smallest composition

e.g. $\underbrace{110}_{3}\,\underbrace{1100}_{4}\,\underbrace{110}_{3}\,\underbrace{1000000}_{7} \mapsto (3,4,3,7)$

Bracket consecutive 01 pairs leaving $1^r 0^s$ unpaired

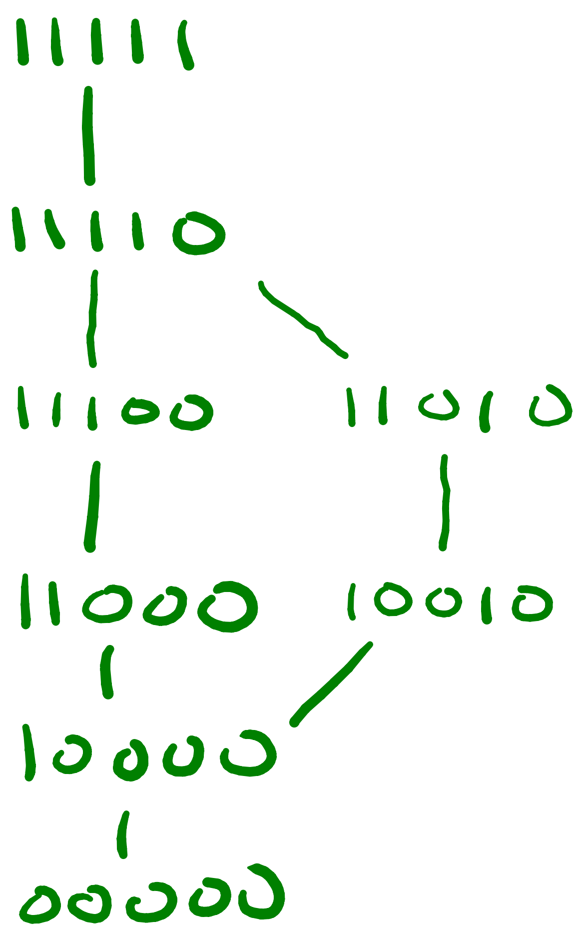
e.g. 11011001101000000

Map to $1^{r-1} 0^{s+1}$ e.g. 11010001101000000

Idea to Obtain Cyclically Symmetric Venn Diagram

- Draw SCD as planar graph, adding edges from top & bottom of symmetric chain to elements covering & covered by them in longer symmetric chains

e.g.



- Cycle diagram around, filling in other orbit elements & take dual graph

Other Related Work:

Thm (Kelly Kross Jordan, 2010): There exists an SCD for B_n/C_n for every n.

Note: One big difference between our approach & other papers - they work on subposet of B_n comprised of orbit representatives, while we work directly on quotient poset. Our approach is also completely explicit.

Theorem (H.-Schilling): There is an explicit symmetric chain decomposition for B_n/C_n for all n via a cyclic analogue of Kashiwara's sl_2-lowering operator from the theory of crystal bases.

Idea: • bracket consecutive 01-pairs cyclically

e.g. 1 1 0 1 1 1 0 1 0 0 0 1 1 1 0 0

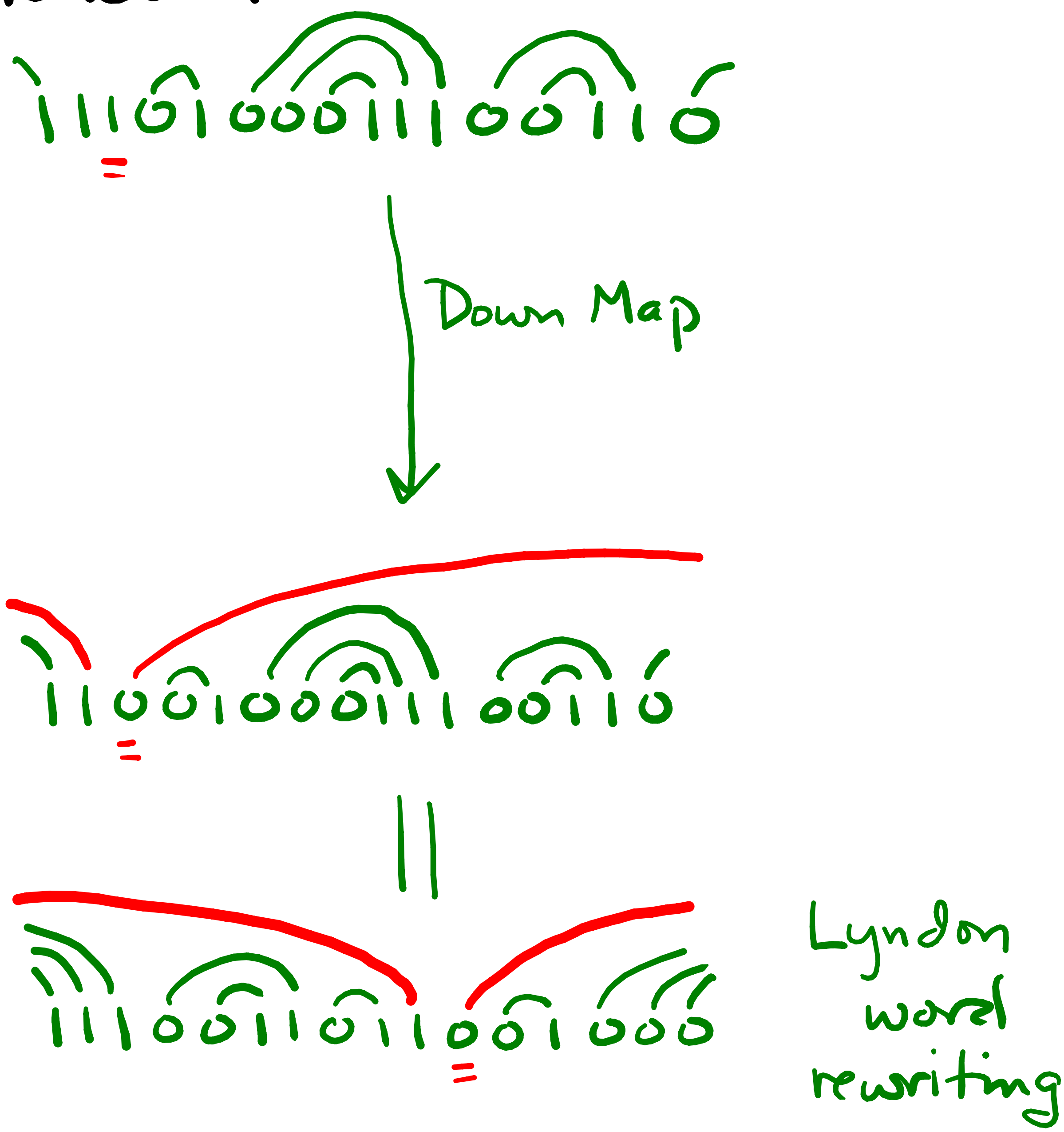
• take the lexicographically earliest cyclic rearrangement of word with alphabet order $1 < 0$, i.e. take the Lyndon word

e.g. 1 1 1 0 1 0 0 0 1 1 1 0 0 1 1 0

"Down map" in top half of quotient poset:

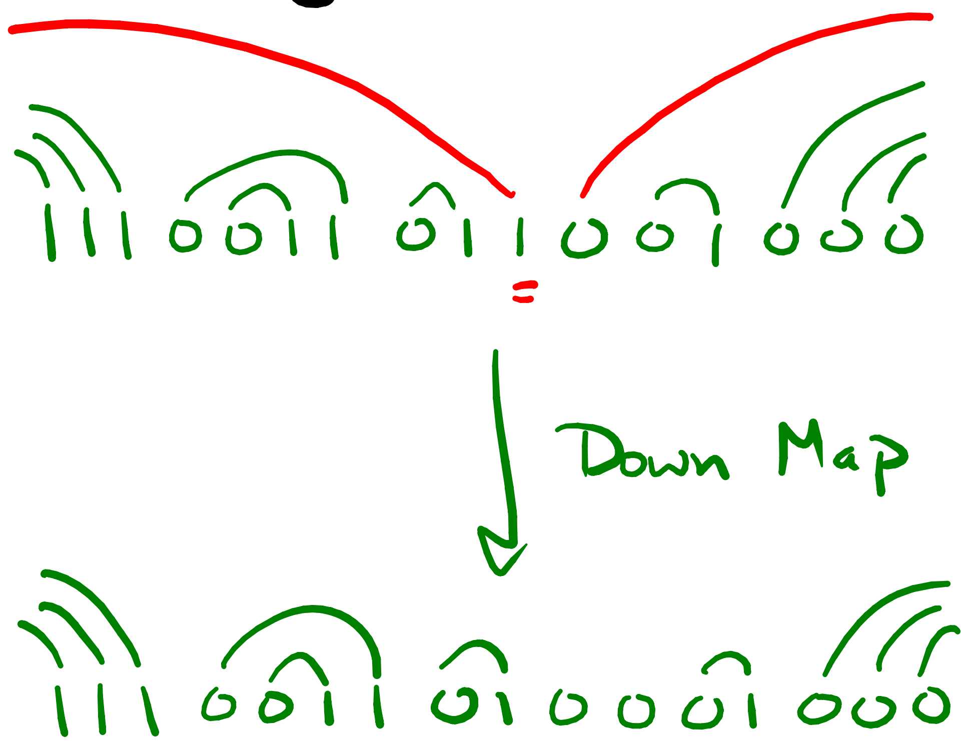
- Turn rightmost unmatched 1 in Lyndon word to 0
  - This new 0 belongs to a new 01 pair unless it was the only unmatched 1

e.g.



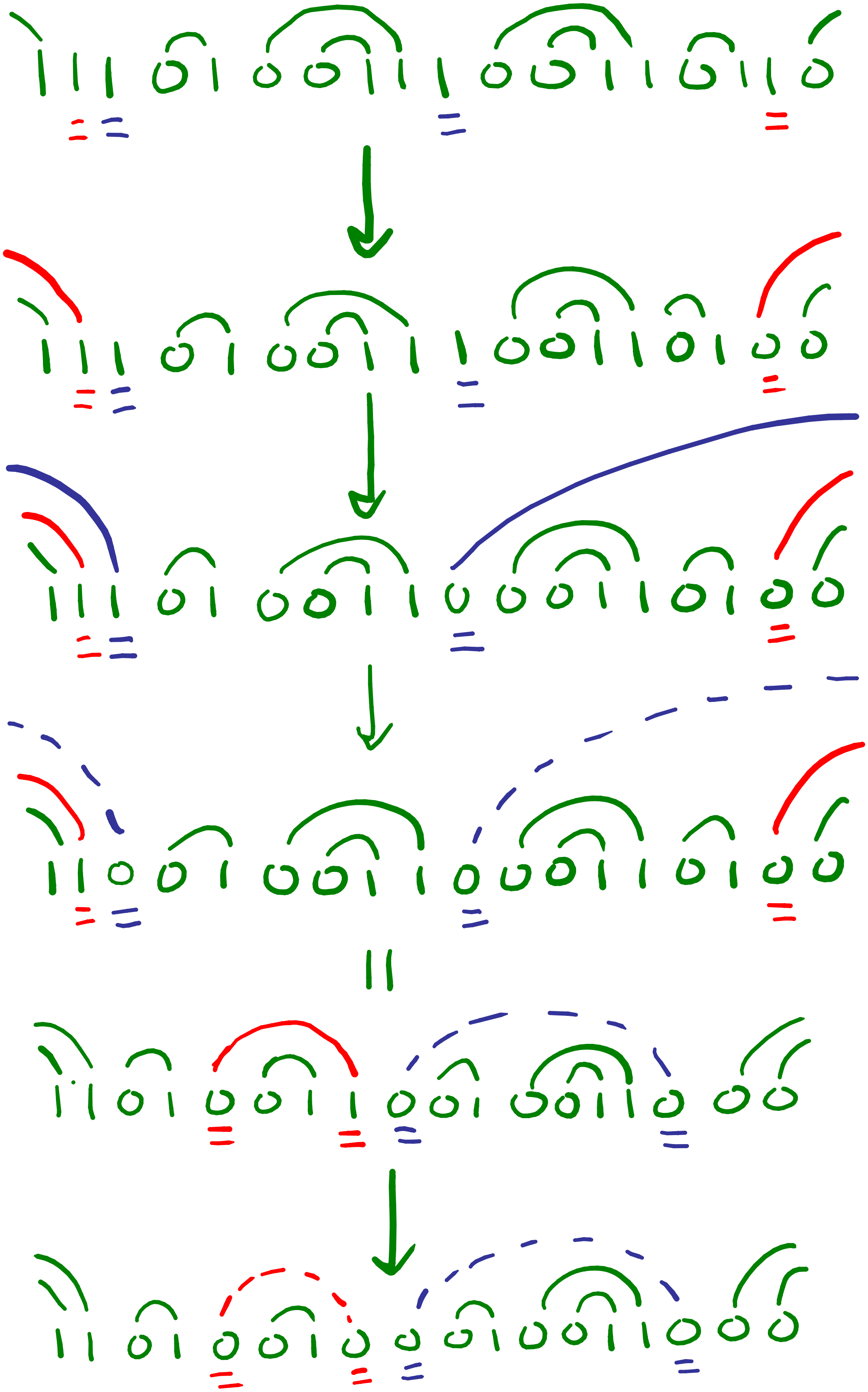
"Down Map" in lower half: "Undo" most recently created 01-pair by turning its remaining 1 into a 0

e.g.



Remark: To be well-defined, need unique symmetric chain leading down to element from above, which will follow from injectivity of down map at higher ranks

Example:



(Lyndon word)

Provably

Equivalent Description: proceed downward in a symmetric chain by turning unmatched 1's to 0's from right to left in Lyndon word

Idea for Injectivity:

1. **Top Half:** Suppose two cyclic words A & A' map to same B

$$A = a_1 \cdots \underbrace{a_r}_{1} \cdots \underbrace{a_s}_{0} \cdots a_t \qquad A' = a'_1 \cdots \underbrace{a'_r}_{0} \cdots \underbrace{a'_s}_{1} \cdots a'_t$$

$a_i = a'_i = b_i$ for $i \neq r, s$

$$B = b_1 \cdots \underbrace{b_r}_{0} \cdots \underbrace{b_s}_{0} \cdots b_t$$

- May assume A or A' is Lyndon, but not both
- Both A & A' have identical matching arcs between pairs of letters both in $a_{r+1} \cdots a_{s-1}$.
- Will prove $a'_r \frown a'_s$ in A', so a'_s already matched, a contradiction.

To this end:

Suppose A is Lyndon: Then...

- no unmatched 1's in $a_{r+1} \cdots a_{s-1}$
- all 0's in $a'_{r+1} \cdots a'_{s-1}$ matched with 1's in $a'_{r+1} \cdots a'_{s-1}$
- a'_r not matched with letter in $a'_{r+1} \cdots a'_s$ so all 0's in $a'_{r+1} \cdots a'_{s-1}$ matched with 1's in $a'_{r+1} \cdots a'_{s-1}$

$\Rightarrow a'_r$ matched with a'_s $\Rightarrow\Leftarrow$

Suppose A' is Lyndon: Then...

- again, each 0 in $a'_{r+1} \cdots a'_{s-1}$ matched with 1 in $a'_{r+1} \cdots a'_{s-1}$
- if there is 1 in $a'_{r+1} \cdots a'_{s-1}$ not matched with 0 in $a'_{r+1} \cdots a'_{s-1}$ then...

$$a_s \cdots a_j \leq_{Lyn} a_1 \cdots a_{j-s+1} \leq_{Lyn} a'_1 \cdots a'_{s-j+1}$$
$$\leq_{Lyn} a'_s \ldots a'_j <_{Lyn} a_s \cdots a_j \Rightarrow\Leftarrow$$

2. Injectivity in Bottom Half

Inverse map by (1) changing all unmatched 0's into 1's to obtain element of same symmetric chain in top half. (If originally element is m ranks below middle rank, get element m ranks above middle). (2) Apply down map 2m-1 times.

e.g. 1110100010000011000

↓ 0's to 1's

1110111011110011000

Lyndon word start

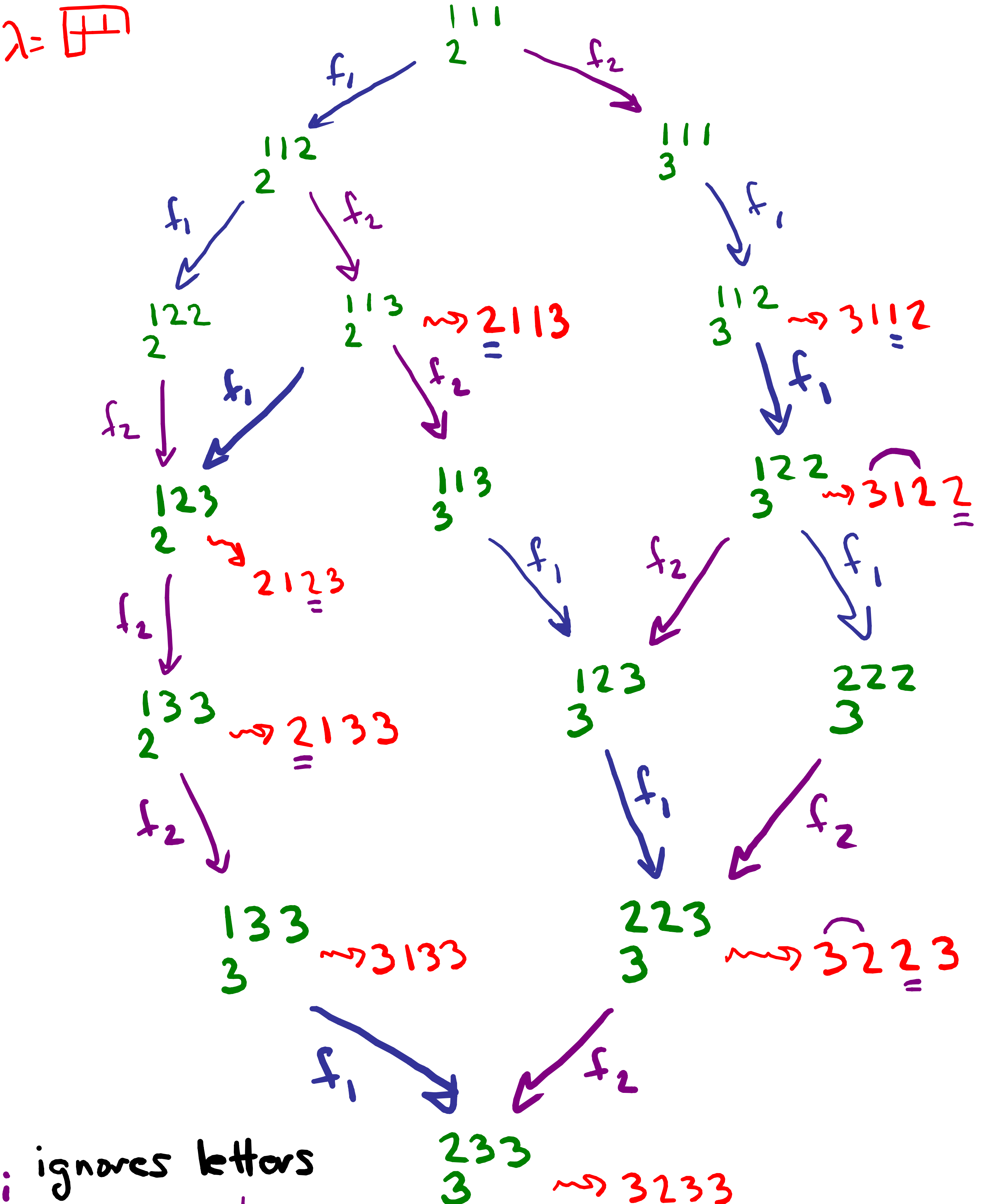
↓ Apply Down Map 4 Times

1100001100011101000

(Type A) Crystals of Highest Weight Representations & their Kashiwara Lowering Operators

e.g. $\lambda = \begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$

$$\begin{tikzcd}
 & \begin{smallmatrix}1&1\\2&\end{smallmatrix} \rightsquigarrow 211 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
\begin{smallmatrix}1&2\\2&\end{smallmatrix} \rightsquigarrow 212 \arrow[d, "f_2"] & & \begin{smallmatrix}1&1\\3&\end{smallmatrix} \rightsquigarrow 311 \arrow[d, "f_1"] \\
\begin{smallmatrix}1&3\\2&\end{smallmatrix} \rightsquigarrow 213 \arrow[d, "f_2"] & & \begin{smallmatrix}1&2\\3&\end{smallmatrix} \rightsquigarrow 312 \arrow[d, "f_1"] \\
\begin{smallmatrix}1&3\\3&\end{smallmatrix} \rightsquigarrow 313 \arrow[dr, "f_1"'] & & \begin{smallmatrix}2&2\\3&\end{smallmatrix} \rightsquigarrow 322 \arrow[dl, "f_2"] \\
 & \begin{smallmatrix}2&3\\3&\end{smallmatrix} \rightsquigarrow 323 &
\end{tikzcd}$$



f_i ignores letters other than i & $i+1$, pairs $i+1$ followed by i, then $f_i : i^r (i+1)^s \mapsto i^{r-1} (i+1)^{s+1}$

Reference: H.- Schilling, IMRN, 2013

Key Observation (Reiterated):

Our "down map" giving symmetric chain decomposition for B_n / C_n is exactly the Keshlwara lowering operator on a circle.

Further Questions:

1. Symmetric chain decomposition (SCD) for other posets by interpreting via crystals / crystal operators?
2. SCD for B_{2n} / D_n i.e. quotient by dihedral group?
3. SCD for $L(m,n)$ i.e. poset of partitions in a rectangle?

Thanks!