Symmetric Chain Decomposition for Cyclic Quotients of Boolean Algebras and Relation to Cyclic Crystals

Patricia Hersh
North Carolina State University

Anne Schilling
University of California, Davis
Background

- The rank generating function $a_0 + a_1 t + a_2 t^2 + \ldots + a_r t^r$ of a graded poset $P$ is unimodal if $a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_p \geq \ldots \geq a_r$ for some $p$.

It is symmetric if $a_i = a_{r-i}$ for all $i$.

Example: $P =$ Boolean algebra $B_n$, i.e. poset of subsets of $\{1, \ldots, n\}$

$$B_3 = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}$$

$$1 + 3t + 3t^2 + t^3$$

rank generating function
Thm (Stanley, Proctor, & others): \( \binom{n}{k} \) is symmetric & unimodal, i.e. the polynomial counting partitions in a \( k \times (n-k) \) rectangle by \( \# \) boxes is symmetric & unimodal.

e.g.

\[
1 + 8 + 2 \varphi^2 \\
1 + 3 \varphi + 2 \varphi^2 + 2 \varphi^3 + 2 \varphi^4 + \varphi^5 + \varphi^6 = \binom{5}{2} \\
1 \leq 1 \leq 2 \leq 2 \geq 2 \geq 1 \geq 1
\]
Idea: Construct vector spaces $V_1, V_2, \ldots, V_k(n-k)$ which are the weight spaces of an $\mathfrak{sl}_2$-representation with $\dim(V_i) = \# \text{ partitions of "area" i}$ within a rectangle.

Deduce unimodality from decomposition into irreducible repn's + nature of $\mathfrak{sl}_2$ irreducible representations.

<table>
<thead>
<tr>
<th>weights</th>
<th>(4)</th>
<th>(2)</th>
<th>(0)</th>
<th>(-2)</th>
<th>(-4)</th>
<th>dimensions of weight spaces</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2 (\Lambda_1)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3 (V_1)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4 (V_1)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3 (V_1)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2 (V_1)</td>
</tr>
</tbody>
</table>
**Calculation:** Let \( V_i = \langle e_S | S \subseteq \{1, 2, \ldots, k(n+1)\} \rangle \)

i.e. vector spaces from ranks of Boolean algebras.

\[
(UD-DU)(e_S) = \left( \text{\# elements covered by } S \right) e_S
- \left( \text{\# elements covering } S \right) e_S
= (1S_1 - (n - 1S_1)) e_S
= (21S_1 - n) e_S
\]

- \( U = \text{"up operator"} \)
  - sends \( e_S \) to formal sum of poset elements covering \( S \)
- \( D = \text{"down operator"} \)

Since \( \{1, 2, 3\} \)

\[ \overset{U}{\Rightarrow} \overset{D}{\Rightarrow} \]

\( \{1, 2, 3\} = S \)

\( \overset{D}{\Rightarrow} \overset{U}{\Rightarrow} \)

\( \{1, 3\} \)

\( \overset{D}{\Rightarrow} \)

\( \{1\} \)

and

\[ \overset{D}{\Rightarrow} \overset{U}{\Rightarrow} \]

\( \overset{D}{\Rightarrow} \overset{U}{\Rightarrow} \)

\( \overset{U}{\Rightarrow} \)

\( \overset{D}{\Rightarrow} \)

\( \overset{U}{\Rightarrow} \)

\( n - 1S_1 \) ways for \( D \circ U \)

\( 1S_1 \) ways for \( U \circ D \)

Consider subspaces \( V_i \) whose dimensions count partitions in \( k \times (n-k) \) box by area. Use that \( D \circ U \) commute with group action to get \( sl_2 \)-rep
**Definition:** A poset $P$ has a symmetric chain decomposition (SCD) if it decomposes into nonoverlapping, saturated chains symmetric about the middle rank(s).

**Example:**

**Easy Fact:** Symmetric chain decomposition $\Rightarrow$ rank generating function is symmetric $\Rightarrow$ unimodal; largest antichain middle rank
Thin (Greene-Kleitman) Any Boolean algebra (more generally any product of chains) has a symmetric chain decomposition.

Idea: Elements of $B_n$ can be represented as sequences in $\{0,1,3\}^n$

* e.g. $[1, 3, 4, 7, 8] \in B_9 \rightarrow 1011000110$

  positions 1, 3, 4, 7, 8

  - Parenthesize consecutive 01 pairs, removing pairs from further consideration, continuing

  * e.g. $[1, (01), 16, (01), 10], i.e., \overbrace{1011101101011010000000000000}$

  - Unmatched part of form $1^r$'s
  - Symmetric chain obtained by letting $r$ range from 0 to $n+5$

  * e.g. $1011101101011010000000000000 \cdots$
Remark: Boolean algebras also have symmetrical unimodal rank generating function since

\[
\binom{n}{3} = \binom{n}{n} \leq \binom{n}{2} = \binom{n}{n-1} \leq \ldots \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lfloor \frac{n}{2} \rfloor}
\]

Qn: What about quasilex posets of \( B_n \)?
Thm (J. Griggs): Let $P$ be a ranked poset of rank $n$ with $N_k$ elements of rank $k$ such that:

1. $N_0 = N_n \leq N_1 = N_{n-1} \leq ... \leq N_{\frac{n}{2}} = N_{\frac{n}{2}}$

2. $P$ has the LYM property

Then $P$ has a symmetric chain decomposition.

**What is the LYM property?**

**Ans:** For every antichain $F$ (i.e., every collection of incomparable elements of $P$), $\sum_{x \in F} \frac{1}{N_{\text{rank}(x)}} \leq 1$

**Note:** $F = \{\text{elements of rank } k\} \implies \sum_{x \in F} \frac{1}{N_k} = \frac{N_k}{N_k} = 1$
Symmetric Venn Diagrams \& the Quotient of $\mathbb{R}_n$ by Cyclic Group (work of Griggs-Killam-Savage)

A Venn diagram is a collection of simple closed curves s.t. each subset of $\{1, 2, \ldots, n\}$ is represented by distinct region.

\[ \text{e.g.} \]

\[
\begin{array}{c}
\emptyset \\
\{1, 3\} \\
\{1, 2, 3\} \\
\{1, 2, 3, 1\} \\
\{2, 3\} \\
\{1, 3\} \\
\{2, 3\} \\
\{1, 2, 3\} \\
\end{array}
\]

Qn: For which $n$ is there a Venn diagram of subsets of $\{1, \ldots, n\}$ with $C_n$ symmetry?
Thm (Griggs-Killiam-Savage): For $n$ prime, these do exist. These are constructed from a symmetric chain decomposition for $B_n/C_n$.

**Idea of GKS Sym. Chain Decomp.**

Associate cyclic composition to each element of $B_n/C_n$

**E.g.** \(110 \overline{10000000} 110 \overline{1100} \rightarrow (3,7,3,4)\)

Rotate to get lexicographically smallest composition

**E.g.** \(110 \overline{1100} \overline{1101000000} \rightarrow (3,4,3,7)\)

Bracket consecutive 01 pairs leaving 1's

**E.g.** \(1101001101000000\) unpaired

Map to \(1^{05+1}\) **E.g.** \(1101001101000000\)
Idea to Obtain Cyclically Symmetric Venn Diagram

- Draw SCD as planar graph, adding edges from top & bottom of symmetric chain to elements covering & covered by them in longer symmetric chains.

E.g.:

```
    11111
     |
1110  \
     |
1100  \
     |
1000  \
     |
0000
```

- Cycle diagram around, filling in other orbit elements & take dual graph.
Other Related Work:

Thm (Kelly Kross Jordan, 2010): There exists an SCD for $B_n/C_n$ for every $n$.

Note: One big difference between our approach & other papers— they work on subposet of $B_n$ comprised of orbit representatives, while we work directly on quotient poset. Our approach is also completely explicit.
Theorem (H.-Schilling): There is an explicit symmetric chain decomposition for $B_n/lc_n$ for all $n$ via a cyclic analogue of Kashiwara's $\mathfrak{sl}_2$-lowering operator from the theory of crystal bases.

Idea: • bracket consecutive 01-pairs cyclically

e.g. 110111010000000000

• take the lexicographically earliest cyclic rearrangement of word with alphabet order 1 < 0, i.e. take the Lyndon word

e.g. 1110100000000001110
"Down map" in top half of quotient poset:

- Turn rightmost unmatched 1 in Lyndon word to 0
- This new 0 belongs to a new 01 pair unless it was the only unmatched 1

E.g. \[ \begin{array}{c}
\text{Lyndon word} \\
\text{rewriting}
\end{array} \]
"Down Map" in lower half: "Undo" most recently created 01-pair by turning its remaining 1 into a 0

e.g.

\[
\begin{array}{c}
\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}
\end{array}
\]

\[\text{Down Map}\]

\[
\begin{array}{c}
\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}
\end{array}
\]

Remark: To be well-defined, need unique symmetric chain leading down to element from above, which will follow from injectivity of down map at higher ranks
Example:  

(Lyndon word)  

Equivalently Description: proceed downward in a symmetric chain by turning unmatched 1's to 0's from right to left in Lyndon word.
Idea for Injectivity:

1. Top Half: Suppose two cyclic words $A^t \ A'$ map to same $B$

$A'' = \overrightarrow{\overset{a_r}{a} \ a_5 \ a_t} \overset{0}{\sim} \overset{1}{\sim}$
$A' = \overrightarrow{\overset{a'_r}{a'} \ a'_5 \ a'_t} \overset{0}{\sim} \overset{1}{\sim}$

$a_i = a'_i = b_i$

for $i \neq r, s$

$B = \overrightarrow{\overset{b_r}{b} \ b_5 \ b_t} \overset{0}{\sim} \overset{0}{\sim} \overset{0}{\sim}$

- May assume $A$ or $A'$ is Lyndon, but not both

- Both $A \ A'$ have identical matching arcs between pairs of letters both in $a_{r:5} \ a_{s:1}$.

- Will prove $a'_r \ a'_s$ in $A'$, so $a'_s$ already matched, a contradiction.
To this end:

Suppose A is Lyndon: Then...

- no unmatched 1's in $a_{r+1-\epsilon s}$
- all 0's in $a_{r+1-\epsilon s}$, matched with 1's in $a_{r+1-\epsilon s}$
- $a_r'$ not matched with letter in $a_{r+1-\epsilon s}$
  so all 0's in $a_{r+1-\epsilon s}$, matched with 1's in $a_{r+1-\epsilon s}$

$\Rightarrow a_r'$ matched with $a_s' \iff$

Suppose $A'$ is Lyndon: Then...

- again, each 0 in $a_{r+1-\epsilon s}$, matched with 1 in $a_{r+1-\epsilon s}$
- if there is 1 in $a_{r+1-\epsilon s}$, not matched with 0 in $a_{r+1-\epsilon s}$, then...

\[
\begin{align*}
a_s' & \leq_{\text{Lyn}} a_r' \leq_{\text{Lyn}} a_s' \leq_{\text{Lyn}} a_{r+1} \leq_{\text{Lyn}} a_{s-\epsilon j} \leq_{\text{Lyn}} a_{s-\epsilon j} \\
& \leq_{\text{Lyn}} a_s' \leq_{\text{Lyn}} a_j \leq_{\text{Lyn}} a_s' \leq_{\text{Lyn}} a_j
\end{align*}
\]
2. Injectivity in Bottom Half

Inverse map by (1) changing all unmatched 0's into 1's to obtain element of same symmetric chain in top half. (If originally element is in ranks below middle rank, get element in ranks above middle). (2) Apply down map 2m-1 times.

E.g. \[ \text{original string} \rightarrow \text{modified string} \]

\[ \text{Apply Down Map 4 Times} \]

\[ \text{0's to 1's} \]
(Type A) Crystals of Highest Weight Representations & their Kashiwara Lowering Operators

e.g. $\lambda = \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$
\[ \lambda = \begin{array}{c} \Box \end{array} \]

- \( f_1 \) ignores letters other than \( i \) or \( i+1 \),
- Pairs \( i+1 \) followed by \( i \), then \( f_i : i^{r(i+1)} \mapsto i^{r-1}(i+1)^{s+1} \)

\( \Rightarrow 2113 \)
\( \Rightarrow 3112 \)
\( \Rightarrow 3222 \)
\( \Rightarrow 3133 \)
\( \Rightarrow 3233 \)
Reference: H.-Schilling, IMRN, 2013

Key Observation (Reiterated):

Our “down map” giving symmetric chain decomposition for $B_n/C_n$ is exactly the Kleshlava lowering operator on a circle.

Further Questions:

1. Symmetric chain decomposition (SCD) for other posets by interpreting via crystals/crystal operators?

2. SCD for $B_{2n}/D_n$ i.e. quotient by dihedral group?

3. SCD for $L(m,n)$ i.e. poset of partitions in a rectangle?

Thanks!