Lévy's characterization of Brownian motion and some continued fractions

based on joint work with

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0.1. Outline

1. A process that lives on an arbitrary closed subset of $\mathbb{R}$
   (a) "Lévy's characterization"
   (b) Construction via Brownian motion
   (c) Uniqueness
2. Explicit computations, when $\mathbb{T} = \pm q^{\mathbb{Z}} \cup \{0\}$.
   (a) Recurrences for Laplace transforms of hitting times
   (b) Recurrences and continued fractions
   (c) Explicit solutions with q-hypergeometric functions
   (d) As $q \downarrow 1$
   (e) Inversion of some Laplace transforms

1. The general process

1.1. Lévy's characterization

Brownian motion is the unique $\mathbb{R}$-valued stochastic process $(\xi_t)_{t\in\mathbb{R}_+}$ such that:

(I) ξ has continuous sample paths,

(II) ξ is a martingale,

(III) $(\xi_t^2 - t)_{t\in\mathbb{R}_+}$ is a martingale.

The fact that the process is a Feller-Dynkin Markov process comes free.
Note that even a compensated Poisson process $N_t - t$ satisfies (II) and (III), whereas Dubins and Schwartz proved that anything satisfying (I) and (II) is a time change of Brownian motion.

1.2. Extension

Extend the state space: let $\mathbb{T} \subset \mathbb{R}$ be closed, and unbounded in both directions. Let $(\xi_t)_{t\in\mathbb{R}_+}$ be a $\mathbb{T}$-valued stochastic process $(\xi_t)_{t\in\mathbb{R}_+}$ that satisfies:

(I') if $x < y < z$ are in $\mathbb{T}$, and $\xi_r = x$ and $\xi_t = z$, then $\xi_s = y$ for some $r < s < t$, (ξ does not "skip points")

(II) ξ is a martingale,

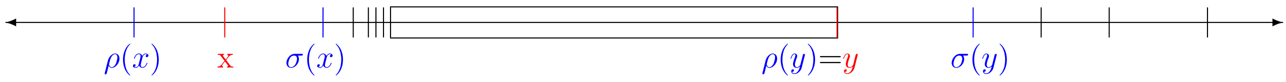
(III) $(\xi_t^2 - t)_{t\in\mathbb{R}_+}$ is a martingale.

Theorem 1 *[BEPR] For any closed, unbounded* $\mathbb{T} \subset \mathbb{R}$*, there is a unique* $\mathbb{T}$*-valued martingale* $(\xi_t)_{t\in\mathbb{R}^+}$ *that does not skip points and for which* $(\xi_t^2 - t)_{t\in\mathbb{R}^+}$ *is also a martingale.* ξ *is further a Feller-Dynkin Markov process.*

Corollary 1 *Any* $\mathbb{T}$*-valued martingale that does not skip points is a time change of* $(\xi_t)_{t\in\mathbb{R}^+}$*.*

1.3. Construction

We know it exists because we can construct it.



For a point $x \in \mathbb{T}$ set

$$\rho(x) := \sup\{y \in \mathbb{T} \ : \ y < x\} \qquad \sigma(x) := \inf\{y \in \mathbb{T} \ : \ y > x\}.$$

Let m be Lesbegue measure and define the measure μ on $\mathbb{T}$ by

$$\mu := \mathbf{1}_{\mathbb{T}} \cdot m + \sum_{x \in \mathbb{T}} \frac{\sigma(x) - \rho(x)}{2} \delta_x.$$

Let ℓ_t^a be local time for Brownian motion B_t at a, and define the continuous additive functional

$$A_u^\mu := \int_{\mathbb{R}} \ell_u^a \, \mu(da)$$

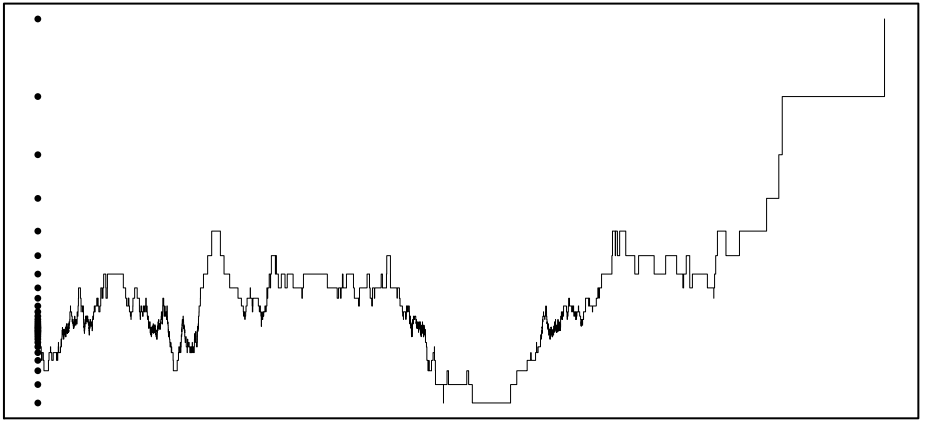
and let θ_t^μ be the right continuous inverse of A^μ

$$\theta_t^\mu := \inf\{u \ : \ A_u^\mu > t\}.$$

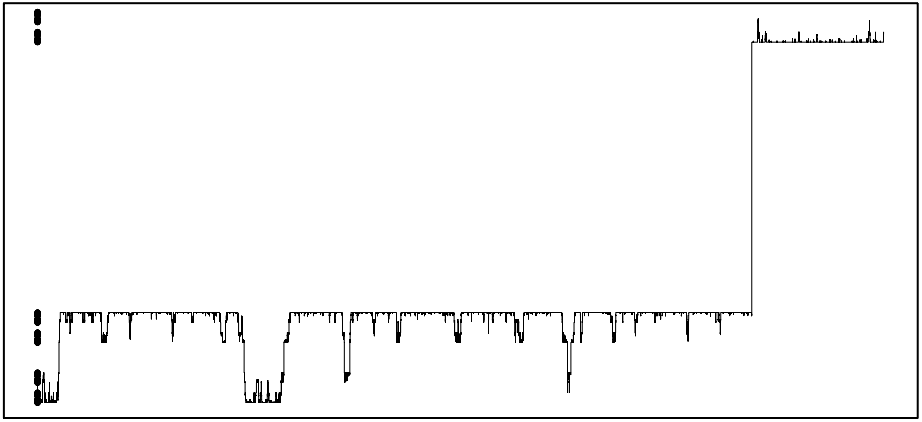
By the time change of B_t with respect μ we mean the process

$$\xi_t := B_{\theta_t^\mu}.$$

Note: ξ is also reversible with respect to μ.



A sample path on $\pm\left(\frac{4}{3}\right)^{\mathbb{Z}} \cup \{0\}$.



A sample path on the Cantor set.

Claim 1 $(\xi_t)_{t\in\mathbb{R}_+}$ *satisfies properties (I'), (II), and (III).*

Proof: θ_t^μ is a stopping time, so ξ is a martingale that does not skip points. This only leaves property (III).

(III) Check in the case that $\rho(x) < x < \sigma(x)$, and $T = \inf\{t : \xi_t \neq x\}$. Then

$$\begin{aligned}\mathbb{E}^x \xi_T^2 &= \frac{(\sigma(x) - x)}{(\sigma(x) - \rho(x))}\rho(x)^2 + \frac{(x - \rho(x))}{(\sigma(x) - \rho(x))}\sigma(x)^2 \\ &= x(\sigma(x) + \rho(x)) - \sigma(x)\rho(x) \\ &= (\sigma(x) - x)(x - \rho(x)) + x^2.\end{aligned}$$

We need to show that $\mathbb{E}^x T = (\sigma(x) - x)(x - \rho(x))$.

Since $|B_t - x| - \ell_t^x$ is an $\mathbb{E}^x$-martingale,

$$\mathbb{E}^x \ell_T^x = \frac{x - \rho(x)}{\sigma(x) - \rho(x)}(\sigma(x) - x) + \frac{\sigma(x) - x}{\sigma(x) - \rho(x)}(x - \rho(x)) = 2\frac{(\sigma(x) - x)(x - \rho(x))}{(\sigma(x) - \rho(x))},$$

and therefore,

$$\mathbb{E}^x T = \mu(x)\, \mathbb{E}^x \ell_T^x = (\sigma(x) - x)(x - \rho(x)).$$

□

1.4. Uniqueness

Theorem 2 (thanks to Pat Fitzsimmons) *Let $x \in \mathbb{T}$, and suppose ζ is a càdlàg $\mathbb{T}$-valued martingale with $\zeta_0 = x$ that does not skip points, and for which $(\zeta_t^2 - t)$ is a martingale. Then ζ has the same distribution as ξ under $\mathbb{P}^x$.*

proof sketch: Let $A \subset \mathbb{T}$, and let $\tau = \inf\{t \geq S : \xi_t \in A\}$, for some stopping time S for the canonical filtration.

- (I'), (II) $\Rightarrow \zeta_\tau \stackrel{d}{=} \xi_\tau$.
- The Chacon-Jamison theorem (extended by Walsh) $\Rightarrow \zeta$ is a time-change of ξ.
- (III) $\Rightarrow$ The time-change is trivial.

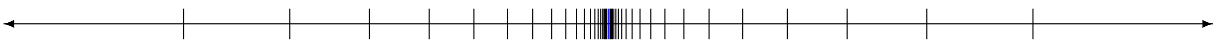
…(plus the proof that ζ holds for exponential times at isolated points — Chacon-Jamison only holds if there are no holding points)

2. Hitting times and continued fractions:

2.1. The q-timescale

Fix $q > 1$, and suppose the state space is

$$\mathbb{T}_q = \{\ldots, -q^2, -q, -1, -q^{-1}, -q^{-2}, \ldots, 0, \ldots, q^{-2}, q^{-1}, 1, q, q^2, \ldots\} = (-q^{\mathbb{Z}}) \cup \{0\} \cup (q^{\mathbb{Z}}).$$



The process ξ on $\mathbb{T}_q$ has jump rates

$$q^n \longrightarrow q^{n+1} \quad \text{at rate } \frac{q^{-2n}}{(q-1)(q-1/q)}$$

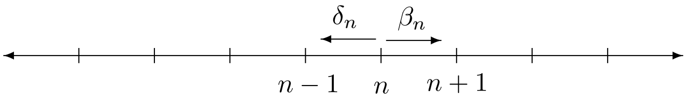
$$q^n \longrightarrow q^{n-1} \quad \text{at rate } \frac{q^{-2n+1}}{(q-1)(q-1/q)}.$$

note: To avoid factors of $c_q := (q-1)(q-1/q)$, we immediately define $X_t := \xi_{c_q t}$, and do most computations for X.

What can we compute explicitly? How about hitting times? ξ_t started on $\mathbb{R}^+$ and stopped upon hitting zero is a birth-death chain, for which there exists a large body of theory.

2.2. The hitting time recurrence

Suppose Z is a bilateral birth-and-death process (on $\mathbb{Z}$). For $n \in \mathbb{Z}$, write β_n for the birth rate at n, and δ_n for the death rate at n.



Let $\tau_n = \inf\{t \geq 0 : Z_t = n\}$ be the hitting time of n, and for $m > n$

$$H_n^{\downarrow}(\lambda) := \mathbb{E}^n[e^{-\lambda\tau_{n-1}}] \qquad H_n^{\uparrow}(\lambda) := \mathbb{E}^n[e^{-\lambda\tau_{n+1}}]$$
$$H_{n,m}(\lambda) := \mathbb{E}^n[e^{-\lambda\tau_m}] = H_n^{\uparrow}(\lambda)H_{n+1}^{\uparrow}(\lambda)\cdots H_{m-1}^{\uparrow}(\lambda),$$

By conditioning on the direction of the first jump,

$$\begin{aligned} H_n^{\downarrow}(\lambda) &= \mathbb{E}^n\left[e^{-\lambda\tau_{n-1}}\mathbf{1}_{\tau_{n-1}<\tau_{n+1}} + e^{-\lambda\tau_{n+1}}\mathbf{1}_{\tau_{n+1}<\tau_{n-1}}\mathbb{E}^{n+1}\left[e^{-\lambda\tau_{n-1}}\right]\right] \\ &= \frac{\delta_n}{\delta_n+\beta_n+\lambda} + \frac{\beta_n}{\delta_n+\beta_n+\lambda}H_{n+1}^{\downarrow}(\lambda)H_n^{\downarrow}(\lambda), \end{aligned}$$

or if $\rho_n := \frac{\delta_n}{\beta_n}$,

$$
\begin{aligned}
H_n^{\downarrow}(\lambda) &= 1 + \rho_n + \frac{\lambda}{\beta_n} - \frac{\rho_n}{H_{n+1}^{\downarrow}(\lambda)} \\
&= 1 + \rho_n + \frac{\lambda}{\beta_n} - \cfrac{\rho_n}{1 + \rho_{n+1} + \frac{\lambda}{\beta_{n+1}} - \frac{\rho_{n+1}}{H_{n+2}^{\downarrow}(\lambda)}} \\
&= 1 + \rho_n + \frac{\lambda}{\beta_n} - \cfrac{\rho_n}{1 + \rho_{n+1} + \frac{\lambda}{\beta_{n+1}} - \cfrac{\rho_{n+1}}{1 + \rho_{n+2} + \frac{\lambda}{\beta_{n+2}} - \ddots - \frac{\rho_{n+m}}{H_{n+m}^{\downarrow}}}} \\
&\stackrel{?}{=} 1 + \rho_n + \frac{\lambda}{\beta_n} - \cfrac{\rho_n}{1 + \rho_{n+1} + \frac{\lambda}{\beta_{n+1}} - \cfrac{\rho_{n+1}}{1 + \rho_{n+2} + \frac{\lambda}{\beta_{n+2}} - \cfrac{\rho_{n+2}}{1 + \rho_{n+3} + \ddots}}}
\end{aligned}
$$

2.3. Continued fractions and three-term recurrences

We are led to a continued fraction. But consider: both $\phi = \frac{1+\sqrt{5}}{2}$ and $\phi = \frac{1-\sqrt{5}}{2}$ solve

$$\phi^2 - \phi - 1 = 0 \quad \Rightarrow \quad \phi = 1 + \frac{1}{\phi} = 1 + \cfrac{1}{1 + \cfrac{1}{\phi}} = \cdots$$

so what value are we to assign the infinite continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}} = ?$$

The building block of a continued fraction is a Möebius map, say with fixed points a and b

$$s_{a,b}(z) := \frac{ab}{a+b-z} \qquad : a \mapsto a, \quad b \mapsto b,$$
$$\text{and} \quad s'(a) = \frac{1}{s'(b)} = \frac{a}{b}$$

so if $a, b \in \mathbb{R}$ and $|a| < |b|$, then $|s'(a)| < 1$ and $|s'(b)| > 1$, so for any $x \neq b$,

$$s^n_{a,b}(x) = \cfrac{ab}{a+b-\cfrac{ab}{a+b-\cfrac{ab}{a+b-\cfrac{ab}{a+b-x}}}} \quad \longrightarrow \quad a \quad \text{as } n \to \infty.$$

Our hitting time recurrences are of the form

$$w_{n+1} = s_{a_n,b_n} \circ s_{a_{n-1},b_{n-1}} \circ \cdots \circ s_{a_{n-m},b_{n-m}}(w_{n-m}).$$

Claim 2 *For each $n \geq 0$, let*

$$S_n(z) = \cfrac{u_1}{v_1 + \cfrac{u_2}{v_2 + \ddots \cfrac{u_n}{v_n + z}}}.$$

Then

$$S_n(z) = \frac{P_n + zP_{n-1}}{Q_n + zQ_{n-1}}$$

where $\{P_n\}_{n \geq -1}$ and $\{Q_n\}_{n \geq -1}$ solve the three-term recurrence

$$\begin{aligned} P_n &= u_n P_{n-1} + v_n P_{n-2} & \qquad P_{-1} = 1 \quad P_0 = 0 \\ Q_n &= u_n Q_{n-1} + v_n Q_{n-2} & \qquad Q_{-1} = 0 \quad Q_0 = 1 \end{aligned}$$

Proof: By induction, $S_0 = z$, and

$$\begin{aligned} S_{n+1}(z) &= S_n(\frac{u_n}{v_n + z}) \\ &= \frac{P_n + (\frac{u_n}{v_n+z})P_{n-1}}{Q_n + (\frac{u_n}{v_n+z})Q_{n-1}} \\ &= \frac{(v_n P_n + u_n P_{n-1}) + zP_n}{(v_n Q_n + u_n Q_{n-1}) + zQ_n}. \end{aligned}$$

$\square$

Definition 1 $\{\tilde{X}_n\}$ *is a minimal solution to the three-term recurrence*

$$X_n = u_n X_{n-1} + v_n X_{n-2} \tag{1}$$

if for all linearly independent solutions $\{X_n\}$*,*

$$\lim_{n\to\infty} \frac{\tilde{X}_n}{X_n} = 0.$$

$\{\tilde{X}_n\}$, if it exists, is unique up to constant multiples.

Theorem 3 (Pincherle) $S_n(0)$ *converges to a finite value if and only if there is a minimal solution* $\{\tilde{X}_n\}$ *to* (1)*, in which case as* $n \to \infty$*,*

$$S_n(0) = \cfrac{u_1}{v_1 + \cfrac{u_2}{v_2 + \ddots \, \frac{u_n}{v_n}}} \longrightarrow -\frac{\tilde{X}_0}{\tilde{X}_{-1}}.$$

In fact, $S_n(z) \to -\frac{\tilde{X}_0}{\tilde{X}_{-1}}$ for any z that "stays away from the bad point(s).

2.4. q-background

We'll need to solve recurrence relations with lots of q's in them, or rather, q-difference equations. Analogous to how hypergeometric functions solve second-order differential equations, q-hypergeometric functions (or *basic* hypergeometric functions) solve second-order q-difference equations.

Fix $q > 1$, and define the q-shifted factorial:

$$(z;q)_n := \prod_{k=0}^{n-1}(1 - zq^k) \quad \text{for } n \in \mathbb{N},\ z \in \mathbb{C},$$

$$(z;q)_\infty := \prod_{k=0}^{\infty}(1 - zq^k) \quad \text{for } |z| < 1.$$

For convenience, let $(a_1, a_2, \ldots, a_n; q)_n = \prod_k (a_k; q)_n$. Define the q-hypergeometric series

$${}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q; z) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k ((-1)^k q^{\frac{k(k-1)}{2}})^{1+s-r} z^k}{(b_1, \ldots, b_s, q; q)_k}.$$

We'll be using, for instance

$$ {}_1\phi_1(a;b;q;z) := \sum_{k=0}^{\infty} \frac{\left(\prod_{l=0}^{k-1}(1-aq^l)\right)(-1)^k q^{\frac{k(k-1)}{2}}}{\left(\prod_{l=0}^{k-1}(1-bq^l)(1-q^{l+1})\right)} z^k $$

$$ {}_1\phi_0(a;-;q;z) := \sum_{k=0}^{\infty} \frac{\left(\prod_{l=0}^{k-1}(1-aq^l)\right)}{\left(\prod_{l=0}^{k-1}(1-q^{l+1})\right)} z^k. $$

Some of these have nice product formulæ, like

Theorem 4 (The q–binomial theorem)

$$ {}_1\phi_0(a;-;q;z) = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad \text{if } |z|<1, |q|<1, a \in \mathbb{C}. $$

One commonly used q–analogue of the exponential that appears in our work is

$$ e_q(z) := {}_1\phi_0(0;-;q;z) = \frac{1}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k}, \qquad \text{for } |z|<1, $$

3. Return to the q-timescale

Let τ_x be the hitting time of x, and denote the Laplace transform of the "time to go down from q^n" by

$$H_n^{\downarrow}(\lambda) := \mathbb{E}^{q^n} \exp(-\lambda \tau_{q^{n-1}}).$$

From the discussion above,

$$\begin{aligned} H_0^{\downarrow}(\lambda) &= \frac{q}{1 + q + \lambda - H_1^{\downarrow}(\lambda)} \\ &= \cfrac{q}{1 + q + \lambda - \cfrac{q}{1 + q + q^2\lambda - H_2^{\downarrow}(\lambda)}} \\ &= \cfrac{q}{1 + q + \lambda - \cfrac{q}{1 + q + q^2\lambda - \cfrac{q}{1 + q + q^4\lambda - \ddots - H_n^{\downarrow}(\lambda)}}}. \end{aligned}$$

Closed-form expressions for continued fractions of this form are listed in Ramanujan's "lost" notebook, and evaluations for various ranges of the parameters can be found (e.g. Hirschhorn; Gupta, Ismail, & Masson), giving

Theorem 5 *[BEPR] The Laplace transform of an the adjacent hitting time is*

$$\mathbb{E}^1 e^{-\lambda\tau_{1/q}} = \frac{q}{\lambda} \frac{{}_0\phi_1(-;0;q^{-1};\frac{1}{\lambda q})}{{}_0\phi_1(-;0;q^{-1};\frac{1}{\lambda q^{-1}})},$$

or by another evalutation of the continued fraction,

$$\mathbb{E}^1 e^{-\lambda\tau_{1/q}} = \frac{1}{(\lambda q^{-1}+1)} \frac{{}_1\phi_1(0;-\frac{1}{\lambda q};q^{-2};-\frac{1}{\lambda q^2})}{{}_1\phi_1(0;-\frac{1}{\lambda q^{-1}};q^{-2};-\frac{1}{\lambda q})}.$$

The Laplace transform of a downwards hitting time is

$$\begin{aligned}\mathbb{E}^1 \exp(-\lambda\tau_{q^{-m}}) &= \frac{q^{m^2-2m}}{\lambda^m} \frac{{}_0\phi_1(-;0;q^{-1};\frac{1}{\lambda q})}{{}_0\phi_1(-;0;q^{-1};\frac{1}{\lambda q^{-2m-1}})} \\ &= \frac{1}{(-\lambda q^{-1};q^{-2})_m} \frac{{}_1\phi_1(0;-\frac{1}{\lambda q};q^{-2};-\frac{1}{\lambda q^2})}{{}_1\phi_1(0;-\frac{1}{\lambda q^{-1}}q^{2m};q^{-2};-\frac{1}{\lambda}q^{2m})}.\end{aligned}$$

The Laplace transform of the hitting time of zero is

$$\mathbb{E}^1 e^{-\lambda\tau_0} = {}_1\phi_1(0;-\frac{1}{\lambda q^1};q^{-2};-\frac{1}{\lambda q^2}) \frac{e_{q^{-2}}(-\lambda q^{-1})}{e_{q^{-2}}(\frac{1}{q})}.$$

Proof: To use the continued fraction, we need check that the Laplace transforms $H_n^{\downarrow}(\lambda)$ asymptotically stay away from the unstable fixed points in the transformation. This is important — we could have just as easily expanded the fraction in the other direction, which continued fraction is not equal to the Laplace transform.

Once we know it applies, the continued fraction leads us to the three-term recurrence

$$U_{n+1}(\lambda) = (1 + q + q^{2n}\lambda)U_n(\lambda) - qU_{n-1}(\lambda),$$

which has an explicit minimal solution $\tilde{U}$. The Laplace transforms of adjacent hitting times are given by the ratio of adjacent terms of $\{\tilde{U}_n\}$,

$$\mathbb{E}^{q^n} \exp -(\lambda \tau_{q^{n-1}}) = \frac{\tilde{U}_n}{\tilde{U}_{n-1}},$$

and non-adjacent hitting times telescope

$$\mathbb{E}^{q^n} \exp -(\lambda \tau_{q^{n-m}}) = \prod_{k=0}^{m+1} \frac{\tilde{U}_{n-k}}{\tilde{U}_{n-k-1}} = \frac{\tilde{U}_n}{\tilde{U}_{n-m}}.$$

Taking limits and using q-transformation formulae gives the expression for $m = \infty$.
□

3.1. The distribution of the hitting time to zero

Theorem 6 *[BEPR] For the process ξ begun at q^n, the hitting time to zero is distributed as*

$$q^{2n+2N-1}\sum_{i=0}^{\infty} q^{-2i}T_i,$$

where the T_i are iid rate 1 exponentials, and N is distributed according to a q-analogue of the Poisson distribution,

$$\mathbb{P}\{N=k\} = \frac{1}{e_{q^{-2}}(\frac{1}{q})}\frac{q^{-k}}{(q^{-2};q^{-2})_k}, \quad k \geq 0.$$

Corollary 2 *As $q \downarrow 1$, $(q-1)q^{2N+1}$ converges to the stable($\frac{1}{2}$) distribution with density*

$$\frac{1}{\sqrt{2\pi t^3}}\exp\left(-\frac{1}{2t}\right), \quad t>0.$$

Proof(of Theorem): We showed that

$$\mathbb{E}^1 e^{-\lambda\tau_0} = {}_1\phi_1(0; -\frac{1}{\lambda q^1}; q^{-2}; -\frac{1}{\lambda q^2}) \frac{e_{q^{-2}}(-\lambda q^{-1})}{e_{q^{-2}}(\frac{1}{q})}$$

$$= \frac{1}{e_{q^{-2}}(\frac{1}{q})} \left(\prod_{i=0}^{\infty} \frac{q^{2i+1}}{q^{2i+1}+\lambda} \right) \left(\sum_{k=0}^{\infty} \prod_{l=0}^{k-1} \left[\frac{q^{-2l-1}}{(q^{-2l-1}+\lambda)} \right] \frac{q^{-k}}{(q^{-2};q^{-2})_k} \right).$$

This is the Laplace transform of the random variable

$$\left(\sum_{i=-\infty}^{0} q^{2i-1} T_i + \sum_{i=1}^{N} q^{2i-1} T_i \right) = \sum_{i=-\infty}^{N} q^{2i-1} T_i \stackrel{d}{=} q^{2N-1} \sum_{i=0}^{\infty} q^{-2i} T_i.$$

□

Note: If N_t is a standard Poisson process, then [Bertoin, Biane, & Yor]

$$\sum_{i=0}^{\infty} q^{-2i} T_i \stackrel{d}{=} \int_0^{\infty} q^{N_t} dt.$$

Lemma 1 *If $x_q \in \mathbb{T}^q$, $x_q \to x$ as $q \downarrow 1$, then ξ started at x_q converges to Brownian motion begun at x (in the usual Skorhod topology) as $q \downarrow 1$.*

Proof(of Corollary): For Brownian motion started at 1, the hitting time of 0 has the stable($\frac{1}{2}$) with this density. By the lemma, $c_q q^{2N} \sum_{i=0}^{\infty} q^{-2i} T_i$ converges to this same stable distribution as $q \downarrow 1$. (recall $c_q = (q-1)(q-1/q)$, a time-scaling) From Lai's strong law of large numbers for Abelian summation we have that

$$\lim_{q \downarrow 1} \sum_{i=0}^{\infty} (1-q^{-2}) q^{-2i} T_i = \mathbb{E}[T_0] = 1, \quad \text{a.s.}$$

and so $c_q(1-q^{-2})^{-1} q^{2N} = (q-1)q^{2N+1}$ also converges to the same stable distribution. $\square$

3.2. The rest of the process?

We can compute the moments $\mathbb{E}^x \xi_t^m$ of the process explicitly, but these grow like $\exp(m^2/2)$, so the moment problem is not well–posed.
We do have formulæ for the Laplace transforms of hitting times. In principle, then, we know everything about the process, since these are the building blocks for explicit formulæ for

The Laplace transform of the local time at zero.

$\Downarrow$

The entrance law for excursions from zero.

$\Downarrow$

The resolvent of the process.

(Laplace transforms of the transition probabilities)

. . . so "all" that stands in the way of explicit transition probabilities is to invert some Laplace transforms.

3.3. Aside on orthogonal polynomials

Recall our hitting time recurrence

$$H_n^{\downarrow}(\lambda) = \frac{\delta_n}{\delta_n + \beta_n + \lambda} + \frac{\beta_n}{\delta_n + \beta_n + \lambda} H_{n+1}^{\downarrow}(\lambda) H_n^{\downarrow}(\lambda),$$

which leads to a continued fraction, whose approximants are given by solutions to

$$P_{n+1} = (\delta_n + \beta_n + \lambda) P_n - \delta_n P_{n-1}.$$

The initial conditions for P imply that they are polynomials in λ — and they are the unique system of polynomials orthogonalized by the spectral measure of the birth-death chain, killed upon reaching 1.

For the birth-death chain on $\mathbb{T}_q \cap [1, \infty)$, the spectral measure is known explicitly, and orthogonalizes some associated continuous dual q-Hahn polynomials. Can this tell us the spectral measure of our process?

4. Miscellania

4.1. Proof of reversibility

Claim 3 *ξ is reversible with respect to μ.*

It suffices to show for all bounded sets $U, V \subset \mathbb{T}$, and $t \geq 0$, that

$$\int_U \mathbb{P}^x(\xi_t \in V)\mu(dx) = \int_V \mathbb{P}^y(\xi_t \in U)\mu(dy),$$

or, taking the Laplace transform, for all $\lambda > 0$,

$$\int_0^\infty \int_U e^{-\lambda t}\mathbb{P}^x(\xi_t \in V)\mu(dx)dt = \int_0^\infty \int_V e^{-\lambda t}\mathbb{P}^y(\xi_t \in U)\mu(dy)dt.$$

Since $\xi_t = B_{\theta^\mu_t}$, and θ^μ_t is the inverse of $A^\mu_t = \int_{\mathbb{R}} \ell^x_t \mu(dx)$, by a change of variables

$$\begin{aligned}
\int_0^\infty \int_U e^{-\lambda t}\mathbb{P}^x(\xi_t \in V)\mu(dx)dt &= \int_U \mathbb{E}^x \left[\int_0^\infty e^{-\lambda t}\mathbf{1}_{\{\xi_t\in V\}}dt\right]\mu(dx) \\
&= \int_U \mathbb{E}^x \left[\int_0^\infty e^{-\lambda t}\mathbf{1}_{\{B_{\theta^\mu_t}\in V\}}dt\right]\mu(dx) \\
&= \int_U \mathbb{E}^x \left[\int_0^\infty e^{-\lambda A^\mu_t}\mathbf{1}_{\{B_t\in V\}}dA_t\right]\mu(dx) \\
&= \int_U \mathbb{E}^x \left[\int_0^\infty e^{-\lambda A^\mu_t}\mathbf{1}_{\{B_t\in V\}}\left(\int_{\mathbb{R}} d\ell^y_t \mu(dy)\right)\right]\mu(dx),
\end{aligned}$$

and since $d\ell^y_t$ is nonzero if and only if $B_t = y$,

$$\begin{aligned}
\int_0^\infty \int_U e^{-\lambda t}\mathbb{P}^x(\xi_t \in V)\mu(dx)dt &= \int_U \int_{\mathbb{R}} \mathbb{E}^x \left[\int_0^\infty e^{-\lambda A^\mu_t}\mathbf{1}_{\{B_t\in V\}}d\ell^y_t\right]\mu(dy)\mu(dx) \\
&= \int_U \int_V \mathbb{E}^x \left[\int_0^\infty e^{-\lambda A^\mu_t}d\ell^y_t\right]\mu(dy)\mu(dx),
\end{aligned}$$

which is symmetric in U and V, by reversibility of B_t.

4.2. Proof of Pincherle's theorem

Theorem 7 (Pincherle) *$S_n(0)$ converges to a finite value if and only if there is a minimal solution $\{\tilde{X}_n\}$ to (1), in which case as $n \to \infty$,*

$$S_n(0) = \cfrac{u_1}{v_1 + \cfrac{u_2}{v_2 + \ddots \, \frac{u_n}{v_n}}} \longrightarrow -\frac{\tilde{X}_0}{\tilde{X}_{-1}}.$$

Proof: Let $\{Y_n\}$ be a solution to (1), linearly independent of $\{\tilde{X}_n\}$. The three-term recurrence is linear, so there are $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ such that

$$P = \alpha\tilde{X} + \beta Y$$
$$Q = \delta\tilde{X} + \gamma Y.$$

Then

$$S_n(0) = \frac{\alpha\tilde{X}_n + \beta Y_n}{\delta\tilde{X}_n + \gamma Y_n} = \frac{\alpha\tilde{X}_n/Y_n + \beta}{\delta\tilde{X}_n/Y_n + \gamma} \to \frac{\beta}{\gamma} \quad \text{as } n \to \infty.$$

The initial conditions are

$$\begin{pmatrix} P_{-1} & Q_{-1} \\ P_0 & Q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{X}_{-1} & Y_{-1} \\ \tilde{X}_0 & Y_0 \end{pmatrix} \begin{pmatrix} \alpha & \delta \\ \beta & \gamma \end{pmatrix}$$

which imply that

$$\frac{\beta}{\gamma} = \frac{-\tilde{X}_0}{\tilde{X}_{-1}}$$

. $\square$