Question: *Let $S(t)$ be a continuous-time Markov process that jumps at rate 1; if $S(t) = z$ then the jump is to $z+1$ with probability $\frac{1}{2}\left(1+\frac{a}{\sqrt{N}}\right)$ and to $z-1$ with probability $\frac{1}{2}\left(1-\frac{a}{\sqrt{N}}\right)$ for fixed (large) N. Let*

$$X(t) = N^{-1/2}S(Nt).$$

Find the generator of the diffusion approximation for X and identify the diffusion process.

Solution: Since S jumps at rate 1, X jumps at rate N, and if S moves by 1, then X moves by $1/\sqrt{N}$. Therefore, the generator G_N of X is

$$G_N f(x) = N\frac{1}{2}\left(1+\frac{a}{\sqrt{N}}\right)\left(f(x+\frac{1}{\sqrt{N}}) - f(x)\right) + N\frac{1}{2}\left(1-\frac{a}{\sqrt{N}}\right)\left(f(x-\frac{1}{\sqrt{N}}) - f(x)\right)$$

and Taylor expanding f about x this is

$$\begin{aligned}
G_N f(x) &= N\frac{1}{2}\left(1+\frac{a}{\sqrt{N}}\right)\left(\frac{1}{\sqrt{N}}f'(x) + \frac{1}{2}\frac{1}{N}f''(x) + O(N^{-3/2})\right) \\
&\quad + N\frac{1}{2}\left(1-\frac{a}{\sqrt{N}}\right)\left(-\frac{1}{\sqrt{N}}f'(x) + \frac{1}{2}\frac{1}{N}f''(x) + O(N^{-3/2})\right) \\
&= N\frac{a}{\sqrt{N}}\frac{1}{\sqrt{N}}f'(x) + N\frac{1}{2}\frac{1}{N}f''(x) + O(N^{-3/2}) \\
&\approx af'(x) + \frac{1}{2}f''(x),
\end{aligned}$$

which is the generator for Brownian motion with (upwards) drift a.

The scaling is important: If we take the bias to be of larger order then $1/\sqrt{N}$, then we get a deterministic process, $X(t) \approx at$, with generator $a\frac{d}{dx}$, in accordance with the Law of Large Numbers. If we take the bias to of smaller order, then the drift dissappears in the fluctuations, and we are left with standard Brownian motion, without drift.