Question: *Describe explicitly the limiting procedure to obtain the diffusion process of Example* 13.3.12 *("Diffusion approximation to the branching process."). Find hitting probabilities for the resulting diffusion, namely, the probability that the diffusion process hits a before hitting b if begun at x, for each a, b, and x.*

Solution: Let $(Z(t))_{t\geq 0}$ be the continuous time version of the branching process. That is, a branching process in which each individual gives birth with rate 1. Let μ be the mean number of offspring per birth event and let σ^2 be the variance of the number of offspring per birth event.

To find the appropriate scaling for the diffusion limit, set $X(t) = \frac{1}{N^\beta}Z(N^\gamma t)$. Since we will be sending N to infinity, all that matters is the ratio of β to γ, so without loss of generality we may set $\gamma = 1$, so that

$$X(t) = \frac{1}{N^\beta}Z(Nt).$$

We first find the infinitesimal mean,

$$\mu_X(x,t) = \lim_{h\downarrow 0}\frac{1}{h}\mathbb{E}(X(t+h)) - X(t)|X(t) = x)$$

Recall that if there are m individuals alive in a branching process then the next birth event happens with rate m, so given $X(t) = x$, i.e. given $Z(Nt) = xN^\beta$, the birth rate before time has been scaled would be xN^β, but the scaling of time by N means that births happen at rate $xN^{1+\beta}$. Using the fact that the expected change in the original population given one birth event has occured is $\mu - 1$, so that the change in the scaled population is $\frac{\mu-1}{N^\beta}$, we have:

$$\begin{aligned}
&\mu_X(x,t)\\
&= \lim_{h\downarrow 0}\tfrac{1}{h}\mathbb{E}(X(t+h)) - X(t)|X(t) = x)\\
&= \lim_{h\downarrow 0}\tfrac{1}{h}\mathbb{E}(N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt)|N^{-\beta}Z(Nt) = x)\\
&= xN^{1+\beta}\left(\tfrac{\mu-1}{N^\beta}\right)\\
&= xN(\mu-1)
\end{aligned}$$

To get a non-trivial limit for large N (i.e. not zero or infinity), we must have $\mu - 1 = O(N^{-1})$. Set $\mu = 1 + \alpha N^{-1}$, so that $\mu_X(x,t) = \alpha x$.

We now find the infinitesimal variance:

$$\begin{aligned}
&\sigma_X^2(x,t)\\
&= \lim_{h\downarrow 0}\tfrac{1}{h}\mathbb{E}((X(t+h)) - X(t))^2|X(t) = x)\\
&= \lim_{h\downarrow 0}\tfrac{1}{h}\mathbb{E}((N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt))^2|N^{-\beta}Z(Nt) = x)\\
&= xN^{1+\beta}\left(\tfrac{\sigma^2+(\mu-1)^2}{N^{2\beta}}\right)\\
&= xN^{1-\beta}(\sigma^2 + (\mu-1)^2)\\
&= xN^{1-\beta}(\sigma^2 + \alpha^2 N^{-2})\\
&\approx xN^{1-\beta}\sigma^2
\end{aligned}$$

To get a non-trivial limit for large N (i.e. not zero or infinity), whilst keeping σ^2 greater than zero and finite, we must have $1-\beta = 0$, i.e. $\beta = 1$. Then $\sigma_X^2(x,t) = x\sigma^2$.

The calculations above can be expanded so that it is clear what is happening in the limit as $h \to 0$:

$$
\begin{aligned}
&\mu_X(x,t) \\
&= \lim_{h\downarrow 0} \tfrac{1}{h}\mathbb{E}(X(t+h)) - X(t)|X(t)=x) \\
&= \lim_{h\downarrow 0} \tfrac{1}{h}\mathbb{E}(N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt)|N^{-\beta}Z(Nt)=x) \\
&= \lim_{h\downarrow 0} \tfrac{1}{h}\mathbb{P}(S \text{ jumps once during } (Nt, Nt+Nh)|Z(Nt)=xN^{\beta}) \\
&\qquad\qquad \times\ \mathbb{E}(N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt)|N^{-\beta}Z(Nt)=x, Z \text{ jumps once during } (Nt, Nt+Nh)) \\
&\quad +\ \lim_{h\downarrow 0} \tfrac{1}{h}\mathbb{P}(Z \text{ jumps} > \text{once during } (Nt, Nt+Nh)|Z(Nt)=xN^{\beta}) \\
&\qquad\qquad \times\ \mathbb{E}(N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt)|N^{-\beta}Z(Nt)=x, Z \text{ jumps} > \text{once during } (Nt, Nt+Nh)) \\
&= \lim_{h\downarrow 0} \tfrac{1}{h}(xN^{\beta}Nhe^{-xN^{\beta}Nh}) \\
&\qquad\qquad \times\ \mathbb{E}(N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt)|N^{-\beta}Z(Nt)=x, Z \text{ jumps once during } (Nt, Nt+Nh)) \\
&\quad +\ \lim_{h\downarrow 0} \tfrac{1}{h}O(h^2) \\
&\qquad\qquad \times\ \mathbb{E}(N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt)|N^{-\beta}Z(Nt)=x, Z \text{ jumps} > \text{once during } (Nt, Nt+Nh)) \\
&= \lim_{h\downarrow 0} xN^{\beta+1}\mathbb{E}(N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt)|N^{-\beta}Z(Nt)=x, Z \text{ jumps once during } (Nt, Nt+Nh)) \\
&\quad +0 \\
&= xN\mathbb{E}(Z(Nt+Nh) - Z(Nt)|N^{-\beta}Z(Nt)=x, Z \text{ jumps once during } (Nt, Nt+Nh)) \\
&= xN(\mu-1)
\end{aligned}
$$

And the expanded calculations for the infinitesimal variance are:

$\sigma_X^2(x,t)$

$= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}((X(t+h)) - X(t))^2|X(t) = x)$

$= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{E}((N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt))^2|N^{-\beta}Z(Nt) = x)$

$= \lim_{h\downarrow 0} \frac{1}{h}\mathbb{P}(S \text{ jumps once during } (Nt, Nt+Nh)|Z(Nt) = xN^\beta)$

$\quad\times\ \mathbb{E}((N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt))^2|N^{-\beta}Z(Nt) = x, Z \text{ jumps once during } (Nt, Nt+Nh))$

$\quad+\ \lim_{h\downarrow 0} \frac{1}{h}\mathbb{P}(Z \text{ jumps} > \text{once during } (Nt, Nt+Nh)|Z(Nt) = xN^\beta)$

$\quad\times\ \mathbb{E}((N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt))^2|N^{-\beta}Z(Nt) = x, Z \text{ jumps} > \text{once during } (Nt, Nt+Nh))$

$= \lim_{h\downarrow 0} \frac{1}{h}(xN^\beta Nhe^{-xN^\beta Nh})$

$\quad\times\ \mathbb{E}((N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt))^2|N^{-\beta}Z(Nt) = x, Z \text{ jumps once during } (Nt, Nt+Nh))$

$\quad+\ \lim_{h\downarrow 0} \frac{1}{h}O(h^2)$

$\quad\times\ \mathbb{E}((N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt))^2|N^{-\beta}Z(Nt) = x, Z \text{ jumps} > \text{once during } (Nt, Nt+Nh))$

$= \lim_{h\downarrow 0} xN^{\beta+1}\mathbb{E}((N^{-\beta}Z(Nt+Nh) - N^{-\beta}Z(Nt))^2|N^{-\beta}Z(Nt) = x, Z \text{ jumps once during } (Nt, Nt+Nh))$

$\quad+0$

$= xN^{1-\beta}\mathbb{E}((Z(Nt+Nh) - Z(Nt))^2|N^{-\beta}Z(Nt) = x, Z \text{ jumps once during } (Nt, Nt+Nh))$

$= xN^{1-\beta}(\sigma^2 + (\mu-1)^2)$

$= xN^{1-\beta}(\sigma^2 + \alpha^2N^{-2})$

$\approx xN^{1-\beta}\sigma^2$

To find hitting probabilities, we look for martingales associated with our diffusion X. X has generator

$$Gf(x) = \alpha x f'(x) + \frac{1}{2}\sigma^2 x f''(x)$$

If we find h such that $Gh(x) = 0$ for all x, then $h(X(t))$ is a martingale. Solutions to

$$\alpha x h'(x) + \frac{1}{2}\sigma^2 x h''(x) = 0$$

are found by cancelling the x, then integrating to get

$$\alpha h(x) + \frac{1}{2}\sigma^2 h'(x) = c$$

for some constant c. i.e. $h'(x) = \frac{2\alpha}{\sigma^2}h(x) + \frac{2c}{\sigma^2}$, which has solution

$$h(x) = Ae^{\frac{2\alpha}{\sigma^2}x} + B$$

for some constants A and B. We choose to set $A = 1$ and $B = 0$ to see that

$$M(t) := e^{\frac{2\alpha}{\sigma^2}X(t)}$$

is a martingale. Suppose $X_0 = x$ with $0 < a < x < b$ and let $T = \inf\{t > 0 : X(t) \in \{a, b\}\}$. Let

$$p_a = \mathbb{P}(X_T = a) = 1 - p_b = 1 - \mathbb{P}(X_T = b)$$

Then by applying the optional stopping theorem, which applies since $\mathbb{P}(T < \infty) = 1$ and because $M(t)$ is uniformly bounded for $t < T$, we have

$$e^{\frac{2\alpha}{\sigma^2}x} = \mathbb{E}(M(0)) = \mathbb{E}(M(T)) = p_a e^{\frac{2\alpha}{\sigma^2}a} + p_b e^{\frac{2\alpha}{\sigma^2}b}$$

Using $p_a = 1 - p_b$ we can solve this to get

$$p_a = \frac{e^{\frac{2\alpha}{\sigma^2}x} - e^{\frac{2\alpha}{\sigma^2}b}}{e^{\frac{2\alpha}{\sigma^2}a} - e^{\frac{2\alpha}{\sigma^2}b}}$$

and

$$p_a = \frac{e^{\frac{2\alpha}{\sigma^2}a} - e^{\frac{2\alpha}{\sigma^2}x}}{e^{\frac{2\alpha}{\sigma^2}a} - e^{\frac{2\alpha}{\sigma^2}b}}$$