Question: *Form a random graph G on N nodes by connecting each pair of nodes independently with probability p. Let T be the number of triangles in G, and show that T is approximately Poisson distributed for appropriate scaling of p as N becomes large.*

Solution: For each of the $\binom{N}{3}$ triples of vertices (i, j, k), let

$$X_{i,j,k} = \begin{cases} 1 & \text{if } i,j,k \text{ connected,} \\ 0 & \text{otherwise.} \end{cases}$$

so that $\mathbb{E}(X_{i,j,k}) = \mathbb{P}(X_{i,j,k} = 1) = p^3$.

Thus $\mathbb{E}(T) = \binom{N}{3} p^3 \to \lambda$ if $p \sim \frac{(6\lambda)^{1/3}}{N}$.

Let

$$V_{i,j,k} = \begin{cases} T - 1 & \text{if } X_{i,j,k} = 1, \\ T + \text{number of edges added by including edges } (i,j),(j,k),(k,i) & \text{otherwise.} \end{cases}$$

so that as required we have

$$\begin{aligned} \mathbb{P}(V_{i,j,k} = l) &= \mathbb{P}(X_{i,j,k} = 1 \,\&\, S = l+1) + \mathbb{P}(X_{i,j,k} = 0)\mathbb{P}(S = l+1 \mid X_{i,j,k} = 1) \\ &= \mathbb{P}(X_{i,j,k} = 1)\mathbb{P}(S = l+1 \mid X_{i,j,k} = 1) + \mathbb{P}(X_{i,j,k} = 0)\mathbb{P}(S = l+1 \mid X_{i,j,k} = 1) \\ &= \mathbb{P}(S = l+1 \mid X_{i,j,k} = 1) \end{aligned}$$

Since we do not have $\mathbb{P}(V_{i,j,k} \geq T) = 1$, we must find $\mathbb{E}(|T - V_{i,j,k}|)$. We do this by consider the three pairs $(i, j,)$, (j, k) and (k, i). If all three are connected, which happens with probability p^3, then $X_{i,j,k} = 1$ and hence $T - V_{i,j,k} = 1$.

If one pair, say (i, j), is not connected, then we will have $X_{i,j,k} = 0$. To find $V_{i,j,k}$ we will add in the edge connecting that pair. The number of new triangles added will be Binomial$(N-3, p^2)$, since for each of the other $N-3$ vertices a new triangle will be added if the two other required edges exist, and moreover each of those pairs of edges will exist independently. Thus the expected number of new triangles given (i, j) is not connected is $(N-3)p^2$. Since (i, j) is not connected with probability $1-p$, the expected number of new triangles is $(1-p)(N-3)p^2$. Since there are three pairs of vertices, $(i, j,)$, (j, k) and (k, i), by linearity of expectation we have

$$\mathbb{E}(|T - V_{i,j,k}|) = p^3 \cdot 1 + (1-p)(N-3)p^2 + (1-p)(N-3)p^2 + (1-p)(N-3)p^2 = (10-3N)p^3 + 3(N-3)p^2$$

Thus, using the Stein-Chen Theorem, we have

$$
\begin{aligned}
d_{TV}(T, \text{Pois}(\lambda)) &\leq 2(1 \wedge \frac{1}{\lambda}) \sum_{i,j,k} \mathbb{P}(X_{i,j,k} = 1)\mathbb{E}(|T - V_{i,j,k}|) \\
&= 2(1 \wedge \frac{1}{\lambda})\binom{N}{3} p^3 \left((10 - 3N)p^3 + 3(N-3)p^2\right) \\
&\to 2(1 \wedge \frac{1}{\lambda})\binom{N}{3} p^3 \left(3(N-3)p^2\right) \\
&\to C\frac{1}{N}
\end{aligned}
$$

where C is some constant and where we have used the fact that $p \sim O(N^{-1})$.